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INSTITUTIONUM CALCULI INTEGRALIS

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SUPPLEMENTA
ET
ADDITIONES
AD
INSTITUTIONES CALCULI INTEGRALIS.

SUPPLEMENTUM I.

AD TOM. I. CAP. II.

DE

INTEGRATIONE FORMULARUM DIFFERENTIALIUM IRRATIONALIUM.

1.) De integratione formularum differentialium irrationalium. *Acta Academiae Scientiar. Petropolitanae. Tom. IV. Pars I. Pag.* 4 — 31.

Problema 1.

§. 1. *Si functio X praeter ipsam variabilem x etiam formulam irrationalem $s = \sqrt{(a + bx)}$ involvat: ita tamen, ut X sit functio rationalis binarum quantitatum x et s, formulam differentialem $X\partial x$ ab irrationalitate liberare.*

Solutio.

Cum irrationalitas tantum in formula $s = \sqrt{(a + bx)}$ insit, hanc tantum ita per idoneam substitutionem tolli oportet, ut inde valor ipsius x non fiat irrationalis. Hoc autem praestabitur, ponendo $a + bx = zz$, ut fiat $s = z$ et $x = \frac{zz - a}{b}$, hincque $\partial x = \frac{2}{b} z\partial z$; quibus valoribus substitutis, tota formula differentialis $X\partial x$ ad rationalem, novam variabilem z complectens, perducitur.

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Exemplum 1.

§. 2. Si fuerit $\partial y = \frac{\partial x}{\sqrt{(a+bx)}}$, seu $\partial y = \frac{\partial x}{s}$, posito $\sqrt{(a+bx)} = z$, fiet $\partial y = \frac{2}{b}\partial z$, et integrando $y = \frac{2z}{b}$, unde facta substitutione colligitur $y = \frac{2}{b}\sqrt{(a+bx)} + C$.

Exemplum 2.

§. 3. Si fuerit $\partial y = \partial x \sqrt{(a+bx)} = s\partial x$, sumto $\sqrt{(a+bx)} = z$, erit $\partial y = z\partial x = \frac{2}{b} zz\partial z$, unde integrando fit $y = \frac{2}{3b} z^3$, et facta substitutione prodit

$$y = \frac{2}{3b}(a+bx)^{\frac{3}{2}} + C.$$

Quod integrale si debeat evanescere facto $x = 0$, fiet

$$C = -\frac{2a\sqrt{a}}{3b},$$

ideoque

$$y = \frac{2(a+bx)^{\frac{3}{2}} - 2a\sqrt{a}}{3b}.$$

Exemplum 3.

§. 4. Si fuerit $\partial y = \frac{x\partial x}{\sqrt{(a+bx)}}$, facta substitutione $\sqrt{(a+bx)} = z$, erit

$$\partial y = \frac{2(zz-a)\partial z}{bb} = \frac{2zz\partial z - 2a\partial z}{bb},$$

unde fit integrando

$$y = \frac{2}{3bb}z^3 - \frac{2a}{bb}z + C,$$

et facta restitutione

$$y = \frac{2}{3bb}(a+bx)^{\frac{3}{2}} - \frac{2a}{bb}\sqrt{(a+bx)} + C$$
$$= \frac{2\sqrt{(a+bx)}}{bb}(\tfrac{1}{3}bx - \tfrac{2}{3}a) + C.$$

Exemplum 4.

§. 5. Si fuerit $\partial y = \frac{\partial x}{(a+bx)^{\frac{3}{2}}}$, facta substitutione $\sqrt{(a+bx)} = z$, erit $\partial y = \frac{\partial x}{z^3}$; quae formula porro ob $\partial x = \frac{2z\partial z}{b}$ abit in $\partial y = \frac{2\,\partial z}{b\,zz}$, qua integrata fit $y = -\frac{2}{bz}$, seu facta restitutione, $y = \frac{-2}{b\sqrt{(a+bx)}} + C$. Ubi notetur, pro C sumi debere $\frac{2}{b\sqrt{a}}$, casu quo integrale evanescere debeat facto $x = 0$.

Problema 2.

§. 6. *Si fuerit X functio quaecunque rationalis binarum quantitatum x et s, existente $s = \sqrt[3]{(a+bx)}$, formulam differentialem $X\partial x$ ab irrationalitate liberare.*

Solutio.

Ponatur $\sqrt[3]{(a+bx)} = z$, ut sit $s = z$, erit $a + bx = z^3$, hincque $x = \frac{z^3 - a}{b}$, et $\partial x = \frac{3zz\partial z}{b}$; quibus valoribus substitutis tota formula fiet rationalis.

Exemplum 1.

§. 7. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt[3]{(a+bx)}} = \frac{\partial x}{s},$$

posito $\sqrt[3]{(a+bx)} = z$ et substituto valore hinc nato

$$\partial x = \frac{3zz\partial z}{b}, \quad \text{erit } \partial y = \frac{3z\partial z}{b},$$

unde integrando fit

$$y = \frac{3}{2b}zz = \frac{3}{2b}\sqrt[3]{(a+bx)^2} + C.$$

Exemplum 2.

§. 8. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt[3]{(a+bx)^2}} = \frac{\partial x}{ss},$$

posito $\sqrt[3]{(a+bx)} = z$ fiet $\partial y = \frac{3\partial z}{b}$, hinc integrando

$$y = \tfrac{3}{b} z = \tfrac{3}{b} \sqrt[3]{(a+bx)} + C.$$

Exemplum 3.

§. 9. Si fuerit $\partial y = \partial x \sqrt[3]{(a+bx)} = s\partial x$, facta substitutione fit $\partial y = \frac{3z^3 \partial z}{b}$, hinc integrando

$$y = \tfrac{3}{4b} z^4 = \tfrac{3}{4b} (a+bx) \sqrt[3]{(a+bx)} + C.$$

Problema 3.

§. 10. *Si fuerit X functio rationalis binarum quantitatum x et s, existente* $s = \sqrt[n]{(a+bx)}$, *formulam differentialem* $X\partial x$ *ab irrationalitate liberare.*

Solutio.

Ponatur $\sqrt[n]{(a+bx)} = z$, ut sit $s = z$, erit $a + bx = z^n$, hinc

$$x = \frac{z^n - a}{b} \text{ et } \partial x = \frac{nz^{n-1}\partial z}{b};$$

quibus valoribus substitutis formula proposita $X\partial x$ certe fiet rationalis, si modo numerus exponentialis n fuerit integer.

Exemplum 1.

§. 11. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt[n]{(a + bx)}} = \frac{\partial x}{s},$$

posito $\sqrt{(a + bx)} = z$, ob valorem inde natum

$$\partial x = \frac{nz^{n-1}}{b} \partial z$$

habebitur

$$\partial y = \frac{nz^{n-2}}{b} \partial z;$$

unde integrando colligimus

$$y = \frac{n}{b(n-1)} z^{n-1} + C.$$

sive restitutis valoribus

$$y = \frac{n}{b(n-1)} (a + bx)^{\frac{n-1}{n}} + C = \frac{n}{b(n-1)} \cdot \frac{a + bx}{\sqrt[n]{(a + bx)}} + C.$$

Exemplum 2.

§. 12. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt[n]{(a + bx)^\lambda}} = \frac{\partial x}{s^\lambda},$$

posito $\sqrt[n]{(a + bx)} = z$, et substituto valore

$$\partial x = \frac{nz^{n-1} \partial z}{b}, \text{ fiet}$$

$$\partial y = \frac{nz^{n-1} \partial z}{bz^\lambda} = \frac{n}{b} z^{n-\lambda-1} \partial z,$$

cujus integrale dat

$$y = \frac{n}{b(n-\lambda)}(a+bx)^{\frac{n-\lambda}{n}} + C, \text{ sive}$$

$$y = \frac{n}{b(n-\lambda)} \cdot \frac{a+bx}{\sqrt[n]{(a+bx)^\lambda}}.$$

Ex his autem exemplis jam apparet, integrationem non impediri, etiamsi exponentes n et λ non fuerint numeri integri.

Problema 4.

§. 13. *Si fuerit X functio rationalis binarum quantitatum x et s, existente $s = \sqrt{[a + b\sqrt{(f+gx)}]}$, quae formula ergo duplicem irrationalitatem involvit, formulam differentialem $X\partial x$ ab hac duplici irrationalitate liberare.*

Solutio.

Ponatur iterum $\sqrt{[a+b\sqrt{(f+gx)}]} = z$, ut sit $s = z$, erit sumtis quadratis $a + b\sqrt{(f+gx)} = zz$, hinc

$$b\sqrt{(f+gx)} = zz - a:$$

ac sumtis denuo quadratis

$$bb(f+gx) = (zz-a)^2,$$

unde colligitur

$$x = \frac{(zz-a)^2}{bbg} - \frac{f}{g}, \text{ hincque}$$

$$\partial x = \frac{4z\partial z(zz-a)}{bbg}.$$

Quibus valoribus substitutis tota formula reddetur rationalis.

Corollarium.

§. 14. Perspicuum est, eodem modo irrationalitatem tolli posse, si fuerit multo generalius

$$s = \sqrt[n]{[a + b\sqrt[m]{(f+gx)}]}.$$

Posita enim hac formula $= z$, fiet

$$a + b\sqrt[m]{(f+gx)} = z^n \text{ et } b\sqrt[m]{(f+gx)} = z^n - a.$$

Porro $b^m(f+gx) = (z^n - a)^m$, et hinc colligitur

$$x = \frac{(z^n - a)^m}{b^m g} - \frac{f}{g}, \text{ ideoque}$$

$$\partial x = \frac{mnz^{n-1}\partial z\,(z^n - a)^{m-1}}{b^m g}.$$

Sicque etiam hoc modo tota formula rationalis evadet.

Problema 5.

§. 15. *Si fuerit X functio rationalis binarum quantitatum s et x, existente $s = \sqrt{\frac{a+bx}{f+gx}}$, formulam differentialem $X\partial x$ ab irrationalitate liberare.*

Solutio.

Ponatur $\sqrt{\frac{a+bx}{f+gx}} = z$, et sumtis quadratis erit

$$\frac{a+bx}{f+gx} = zz, \text{ hincque } x = \frac{fzz - a}{b - gzz},$$

unde differentiando colligitur

$$\partial x = \frac{2bfz\partial z - 2agz\partial z}{(b - gzz)^2}.$$

Hisque valoribus substitutis formula proposita $X\partial x$ ad rationalitatem erit perducta.

Exemplum 1.

§. 16. Si fuerit $\partial y = \frac{\partial x}{s} = \frac{\partial x\sqrt{(f+gx)}}{\sqrt{(a+bx)}}$, posito

$$\sqrt{\frac{a+bx}{f+gx}} = z \text{ erit } \partial y = \frac{\partial x}{z},$$

et substituto loco ∂x valore supra invento colligitur

$$\partial y = \frac{2(bf-ag)\partial z}{(b - gzz)^2};$$

quae formula, uti jam satis constat, reduci potest ad talem $\int\frac{\partial z}{b-gzz}$, cujus autem integratio vel per logarithmos vel per arcus circulares expedietur.

Exemplum 2.

§. 17. Sit specialius $\partial y = \frac{\partial x \sqrt{(1-x)}}{\sqrt{(1+x)}}$, ubi $f = 1$, $g = -1$, $a = 1$ et $b = 1$, ideoque

$$z = \frac{\sqrt{(1+x)}}{\sqrt{(1-x)}}, \quad \text{et} \quad \partial x = \frac{4 z \partial z}{(1+zz)^2};$$

quibus valoribus substitutis fiet $\partial y = \frac{4 \partial z}{(1+zz)^2}$. Statuatur ergo

$$\int \frac{4 \partial z}{(1+zz)^2} = \frac{Az}{1+zz} + B \int \frac{\partial z}{1+zz} = y,$$

unde sumtis differentialibus fiet

$$\frac{4}{(1+zz)^2} = \frac{A - Azz}{(1+zz)^2} + \frac{B}{1+zz} = \frac{A + B + (B - A) zz}{(1+zz)^2}.$$

Oportet igitur sit $A + B = 4$ et $B - A = 0$, ideoque $A = 2$ et $B = 2$; et quia $\int \frac{\partial z}{1+zz} =$ Arc. tang. z, adipiscimur

$$y = \frac{2z}{1+zz} + 2 \text{ Arc. tang.} z;$$

quocirca facta restitutione, ob $1 + zz = \frac{2}{1-x}$, obtinebitur

$$y = \sqrt{(1-xx)} + 2 \text{ Arc. tang.} \sqrt{\frac{1+x}{1-x}}.$$

Cum igitur hujus arcus tangens sit $\sqrt{\frac{1+x}{1-x}}$, erit ejus sinus $= \sqrt{\frac{1+x}{2}}$ et cosinus $= \sqrt{\frac{1-x}{2}}$; anguli vero dupli sinus erit $\sqrt{(1-xx)}$ et cosinus $= -x$, unde fiet

$$2 \text{ Arc. tang.} \sqrt{\frac{1+x}{1-x}} = \text{Arc. cos.} -x = \frac{\pi}{2} + \text{Arc. sin.} x;$$

quocirca integrale quaesitum erit

$$y = \sqrt{(1-xx)} + \frac{\pi}{2} + \text{Arc. sin.} x + C,$$

quod si ita capi debeat, ut evanescat posito $x = 0$, erit

$$C = -1 - \frac{\pi}{2}, \text{ ideoque}$$

$$y = \sqrt{(1-xx)} - 1 + \text{Arc. sin.} x.$$

Tum igitur, si sumatur $x = 1$, fiet $y = \frac{\pi}{2} - 1$, qui valor in fractionibus decimalibus dat 0,5707963.

Problema 6.

§. 18. *Si fuerit X functio rationalis binarum variabilium x et s, existente* $s = \sqrt[n]{\frac{a+bx}{f+gx}}$, *formulam differentialem* $X\partial x$ *ad rationalitatem perducere.*

Solutio.

Posito $s = \sqrt[n]{\frac{a+bx}{f+gx}} = z$, erit $\frac{a+bx}{f+gx} = z^n$, hincque

$$x = \frac{fz^n - a}{b - gz^n}, \quad \text{consequenter} \quad \partial x = \frac{n(bf - ag)z^{n-1}\partial z}{(b - gz)^{2n}};$$

hisque valoribus substitutis tota formula proposita $X\partial x$ ad rationalitatem erit perducta.

Problema 7.

§. 19. *Si fuerit X functio binarum quantitatum x et s, existente* $s = \sqrt{(a + bxx)}$, *formulam differentialem* $\frac{X\partial x}{x}$ *ab irrationalitate liberare.*

Solutio.

Ponamus $s = \sqrt{(a + bxx)} = z$, erit $a + bxx = zz$, hinc $xx = \frac{zz - a}{b}$, et quia in functione X tantum quadratum xx, ejusque ergo potestates pares occurrunt: hac substitutione jam functio X evadet rationalis. Sumtis vero logarithmis

$$2lx = l(zz - a) - lb,$$

differentiando fit

$$\frac{2\partial x}{x} = \frac{2z\partial z}{zz - a}, \quad \text{ideoque} \quad \frac{\partial x}{x} = \frac{z\partial z}{zz - a}.$$

Hoc ergo modo formula proposita $X \cdot \frac{\partial x}{x}$ prorsus reddetur rationalis.

Exemplum 1.

§. 20. Si fuerit

$$\partial y = \frac{x\partial x}{\sqrt{(a+bxx)}}, \text{ erit } \partial y = \frac{\partial x}{x} \cdot \frac{xx}{\sqrt{(a+bxx)}} = \frac{xx}{s} \cdot \frac{\partial x}{x}.$$

Posito ergo $\sqrt{(a+bxx)} = z$ erit $\partial y = \frac{\partial z}{b}$, unde colligitur integrando $y = \frac{z}{b} = \frac{\sqrt{(a+bxx)}}{b}$.

Exemplum 2.

§. 21. Si fuerit

$$\partial y = \frac{x^3\partial x}{\sqrt{(a+bxx)}} = \frac{\partial x}{x} \cdot \frac{x^4}{s},$$

ponendo $\sqrt{(a+bxx)} = z$, ut sit

$$xx = \frac{zz-a}{b} \text{ et } \frac{\partial x}{x} = \frac{z\partial z}{zz-a},$$

erit $\partial y = \frac{1}{bb}\partial z(zz-a)$, hincque integrando adipiscimur $y = \frac{z}{3bb}(zz-3a)$; unde facta restitutione prodibit integrale quaesitum $y = \frac{bxx-2a}{3bb}\sqrt{(a+bxx)} + C$.

Exemplum 3.

§. 22. Si fuerit

$$\partial y = \frac{x^3\partial x}{\sqrt{(a+bxx)^3}}, \text{ erit } \partial y = \frac{\partial x}{x} \cdot \frac{x^4}{s^3};$$

hinc posito

$$\sqrt{(a+bxx)} = s = z \text{ fiet } \partial y = \frac{\partial z}{bb}\left(\frac{zz-a}{zz}\right),$$

unde sumto integrali fiet $y = \frac{1}{bb}\left(\frac{zz+a}{z}\right)$, quocirca facta restitutione resultat $y = \frac{2a+bxx}{bb\sqrt{(a+bxx)}} + C$.

Problema 8.

§. 23. *Si fuerit X functio rationalis binarum quantita-*

tum x^n *et* s, *existente* $s = \sqrt[m]{(a + bx^n)}$, *formulam differentialem* $X \frac{\partial x}{x}$ *ad rationalitatem perducere.*

Solutio.

Posito $s = \sqrt[m]{(a + bx^n)} = z$, fiet $a + bx^n = z^m$ et $x^n = \frac{z^m - a}{b}$. Quia igitur in functione X tantum potestas x^n occurrit, ea rationalis reddetur, si hi valores substituantur. Tum vero sumtis logarithmis hábebitur

$$n\,l\,x = l\,(z^m - a) - l\,b,$$

et differentiando

$$\frac{\partial x}{x} = \frac{m z^{m-1}\,\partial z}{n\,(z^m - a)},$$

sicque tota formula proposita fiet rationalis.

Exemplum.

§. 24. Sit

$$\partial y = \frac{x^{n-1}\,\partial x}{\sqrt[m]{(a + bx^n)}} = \frac{\partial x}{x} \cdot \frac{x^n}{s},$$

factaque substitutione orietur haec aequatio

$$\partial y = \frac{m z^{m-2}\,\partial z}{n\,b},$$

qua integrata prodibit

$$y = \frac{m z^{m-1}}{nb\,(m-1)} = \frac{m}{nb\,(m-1)} \sqrt[m]{(a + bx^n)^{m-1}} + \mathrm{C}, \text{ sive}$$

$$y = \frac{m}{nb\,(m-1)} \cdot \frac{a + bx^n}{\sqrt[m]{(a + bx^n)}} + \mathrm{C}.$$

Problema 9.

§. 25. *Si fuerit X functio rationalis quantitatum xx et s, existente* $s = \sqrt{\frac{a + bxx}{f + gxx}}$, *formulam differentialem* $X \frac{\partial x}{x}$ *ab irrationalitate liberare.*

Solutio.

Ponatur $s = \sqrt{\frac{a + bxx}{f + gxx}} = z$, eritque $\frac{a + bxx}{f + gxx} = zz$, hinc $xx = \frac{fzz - a}{b - gzz}$, unde functio X penitus fit rationalis. Porro sumtis logarithmis

$$2lx = l(fzz - a) - l(b - gzz),$$

differentietur, ut prodeat

$$\frac{2\partial x}{x} = \frac{2fz\partial z}{fzz - a} + \frac{2gz\partial z}{b - gzz} = \frac{2(bf - ag)z\partial z}{(fzz - a)(b - gzz)},$$

unde fit

$$\frac{\partial x}{x} = \frac{(bf - ag)z\partial z}{(fzz - a)(b - gzz)};$$

sicque tota formula differentialis fiet rationalis.

Exemplum.

§. 26. Si fuerit $\partial y = \frac{\partial x}{\sqrt{(f + gxx)}}$, repraesentemus hanc formulam ita

$$\partial y = \frac{\partial x}{x} \cdot \frac{x}{\sqrt{(f + gxx)}} = \frac{\partial x}{x} \sqrt{\frac{xx}{f + gxx}}.$$

Hic ergo erit $a = 0$, $b = 1$, et

$$z = \frac{x}{\sqrt{f + gxx}}, \text{ ita ut } \partial y = \frac{z\partial x}{x};$$

erit autem

$$\frac{\partial x}{x} = \frac{\partial z}{z(1 - gzz)}, \text{ unde fit } \partial y = \frac{\partial z}{1 - gzz},$$

cujus formulae integratio per logarithmos expedietur, si fuerit g numerus positivus: sin autem fuerit negativus per arcus circulares

absolvetur. Sit igitur 1°.) $g = + hh$, erit

$$\partial y = \frac{\partial z}{1 - hhzz}, \quad \text{ideoque}$$

$$y = \frac{1}{2h} \, l \, \frac{1 + hz}{1 - hz};$$

et restitutis valoribus supra indicatis, erit

$$y = \frac{1}{2h} \, l \left(\frac{\sqrt{(f + hhxx)} + hx}{\sqrt{(f + hhxx)} - hx}\right) = \frac{1}{h} \, l \, \frac{\sqrt{(f + hhxx)} + hx}{\sqrt{f}}.$$

Sit 2°.) g quantitas negativa, puta $g = - hh$, erit

$$\partial y = \frac{\partial z}{1 + hhzz} = \frac{1}{h} \cdot \frac{h\partial z}{1 + hhzz},$$

unde colligitur

$$y = \frac{1}{h} \text{ Arc. tang. } hz = \frac{1}{h} \text{ Arc. tang. } \frac{hx}{\sqrt{(f - hhxx)}}.$$

Ubi manifestum est, f esse debere quantitatem positivam, quia alioquin formula differentialis esset imaginaria.

Corollarium.

§. 27. Hinc ergo si proponatur formula

$$\partial y = \sqrt{(1 + xx)}, \quad \text{ubi } f = 1 \text{ et } g = 1,$$

ex casu priore ob $h = + 1$ erit

$$\int \frac{\partial x}{\sqrt{(1 + xx)}} = l \, [\sqrt{(1 + xx)} + x].$$

At si fuerit

$$\partial y = \frac{\partial x}{\sqrt{(1 - xx)}}, \quad \text{ubi } f = 1 \text{ et } g = - 1,$$

colligitur ex casu posteriore $x =$ Arc. tang. $\frac{x}{\sqrt{(1 - xx)}}$, unde concluditur

$$\int \frac{\partial x}{\sqrt{(1 - xx)}} = \text{Arc. sin. } x = \text{Arc. cos. } \sqrt{(1 - xx)}.$$

Problema 10.

§. 28. *Si fuerit X functio rationalis quantitatum x^n et s, existente $s = \sqrt[n]{\left(\frac{a+bx^n}{f+gx^n}\right)}$, formulam differentialem $X\frac{\partial x}{x}$ rationalem efficere.*

Solutio.

Ponatur $s = \sqrt[n]{\left(\frac{a+bx^n}{f+gx^n}\right)} = z$, eritque

$$\frac{a+bx^n}{f+gx^n} = z^n, \text{ hinc } x^n = \frac{fz^n - a}{b - gz^n},$$

tum autem sumtis logarithmis, erit

$$nlx = l(fz^n - a) - l(b - gz^n),$$

et differentiando

$$\frac{\partial x}{x} = \frac{fz^{n-1}\partial z}{fz^n - a} + \frac{gz^{n-1}\partial z}{b - gz^n} = \frac{(bf - ag)z^{n-1}\partial z}{(fz^n - a)(b - gz^n)};$$

quibus valoribus substitutis formula proposita fit rationalis.

Problema 11.

§. 29. *Si fuerit X functio rationalis binarum quantitatum x^n et s, existente $s = \sqrt[m]{\left(\frac{a+bx^n}{f+gx^n}\right)}$, formulam differentialem $X\frac{\partial x}{x}$ ab omni irrationalitate liberare.*

Solutio.

Statuatur $s = \sqrt[m]{\left(\frac{a+bx^n}{f+gx^n}\right)} = z$, eritque

$$\frac{a+bx^n}{f+gx^n} = z^m, \text{ unde fit } x^n = \frac{fz^m - a}{b - gz^m};$$

hinc sumtis logarithmis erit

$$nlx = l(fz^m - a) - l(b - gz^m),$$

hinc differentiando

$$\frac{n\partial x}{x} = \frac{m(bf - ag)\, z^{m-1}\, \partial z}{(fz^m - a)(b - gz^m)},$$

ideoque

$$\frac{\partial x}{x} = \frac{m(bf - ag)\, z^{m-1}\, \partial z}{n(fz^m - a)(b - gz^m)},$$

quibus valoribus substitutis irrationalitas formulae propositae penitus tollitur.

Problema 12.

§. 30. *Si fuerit X functio rationalis quaecunque binarum quantitatum x et s, existente $s = \sqrt{(\alpha + \beta x + \gamma xx)}$, formulam differentialem $X\partial x$ ad rationalitatem perducere.*

Solutio.

Hic duos casus a se invicem distingui convenit, prout γ fuerit vel quantitas positiva vel negativa.

I. Sit γ quantitas positiva, ac ponatur $\gamma = cc$ et $\beta = 2bc$, ut habeatur

$$s = \sqrt{(\alpha + 2bcx + ccxx)} = \sqrt{[\alpha - bb + (b + cx)^2]}$$

ubi loco $\alpha - bb$ brevitatis ergo scribatur e, ut sit

$$s = \sqrt{[e + (b + cx)^2]}.$$

Jam statuatur $s = b + cx + z$, eritque

$$ss = e + (b + cx)^2 = (b + cx)^2 + 2(b + cx)z + zz,$$

unde sequitur

$$e - zz = 2z(b + cx), \quad \text{sive} \quad b + cx = \frac{e - zz}{2z};$$

hincque colligitur

$$x = \frac{e - zz}{2cz} - \frac{b}{c}, \quad \text{seu} \quad x = \frac{e - 2bz - zz}{2cz}.$$

Aequatio autem $b + cx = \frac{e - zz}{2z}$ differentiata praebet

$$c\partial x = -\frac{e\partial z}{2zz} - \frac{\partial z}{2} = -\frac{e\partial z - zz\partial z}{2zz},$$

unde deducitur

$$\partial x = -\frac{\partial z(e + zz)}{2czz}, \quad \text{at ob}$$

$$b + cx = \frac{e - zz}{2z} \quad \text{fiet} \quad s = \frac{e + zz}{2z}.$$

His ergo valoribus substitutis formula nostra $X\partial x$ reddetur rationalis. Postquam igitur ejus integrale fuerit inventum, loco z valor ante inventus $\sqrt{[e + (b + cx)^2]} - b - cx$ erit substituendus.

II. Sin autem γ fuerit quantitas negativa, ponatur

$$\gamma = -cc \quad \text{et} \quad \beta = -2bc,$$

ut habeatur

$$s = \sqrt{(\alpha - 2bcx - ccxx)} = \sqrt{[\alpha + bb - (b + cx)^2]},$$

ubi evidens est, quantitatem $\alpha + bb$ necessario esse debere positivam, quia alioquin s evaderet imaginarium. Quamobrem ponamus brevitatis gratia $\alpha + bb = aa$, ut fiat

$$s = \sqrt{[aa - (b + cx)^2]},$$

ad quam formam rationalem efficiendam statuamus

$$\sqrt{[aa - (b + cx)^2]} = a - (b + cx)z,$$

unde sumtis quadratis erit

$$aa - (b + cx)^2 = aa - 2az(b + cx) + (b + cx)^2 zz$$

quae aequatio reducitur ad hanc:

$$-(b + cx) = -2az + (b + cx)zz,$$

unde reperitur

$$b + cx = \frac{2az}{1+zz}, \quad \text{ideoque}$$

$$x = \frac{2az - b - bzz}{c(1+zz)}.$$

Illa autem aequatio differentiata dat

$$c\partial x = \frac{2a\partial z(1+zz) - 4azz\partial z}{(1+zz)^2} = \frac{2a\partial z(1-zz)}{(1+zz)^2};$$

unde fit

$$\partial x = \frac{2a\partial z(1-zz)}{c(1+zz)^2}.$$

Porro autem, cum sit

$$s = a - (b+cx)z, \quad \text{ob } b + cx = \frac{2az}{1+zz}$$

erit $s = \frac{a(1-zz)}{1+zz}$, quocirca, si loco x, s et ∂x inventi hi valores substituantur, formula proposita differentialis $X\partial x$ evadet rationalis, et per variabilem z exprimetur, cujus integrale postquam fuerit inventum, loco z ubique ejus restituatur valor assumtus

$$z = a - \sqrt{[aa - (b+cx)^2]},$$

et integrale obtinebitur per solam variabilem x expressum.

Exemplum 1.

§. 31. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt{[e + (b+cx)^2]}},$$

quae formula ad casum priorem pertinet, erit

$$\partial y = \frac{\partial x}{s} = -\frac{\partial z}{cz}, \text{ ob } \partial x = -\frac{\partial z(e+zz)}{2ezz} \text{ et } s = \frac{e+zz}{2z},$$

cujus integrale est $y = -\frac{1}{c} lz$; restituto ergo valore

$$z = \sqrt{[e + (b+cx)^2]} - b - cx, \quad \text{erit}$$

$$y = -\frac{1}{c} l[\sqrt{[e + (b+cx)^2]} - b - cx] + C,$$

quod integrale si evanescere debeat posito $x = 0$, fiet

$$C = \frac{1}{c} l[\sqrt{(e+bb)} - b].$$

Corollarium.

§. 32. Si ponatur $b = 0$ et $c = 1$, sive

$$\partial y = \frac{\partial x}{\sqrt{(e + xx)}}, \text{ erit integrale}$$

$$y = -l[\sqrt{(e + xx)} - x] + l\sqrt{e} = l\frac{\sqrt{e}}{\sqrt{(e + xx)} - x},$$

quae formula reducitur ad hanc

$$y = l\frac{\sqrt{(e + xx)} + x}{\sqrt{e}}.$$

Cum vero porro sit

$$\partial \,.\, \sqrt{(e + xx)} = \frac{x\partial x}{\sqrt{(e + xx)}}, \text{ erit}$$

$$\int \frac{x\partial x}{\sqrt{(e + xx)}} = \sqrt{(e + xx)}.$$

Si igitur hae duae formulae combinentur, habebitur ista integratio notatu digna

$$\int \frac{A\partial x + Bx\partial x}{\sqrt{(e + xx)}} = A\,l\frac{\sqrt{(e + xx)} + x}{\sqrt{e}} + B\sqrt{(e + xx)}.$$

Exemplum 2.

§. 33. Sit $\partial y = \frac{\partial x}{\sqrt{[aa - (b + cx)^2]}}$, quae formula ad casum secundum est referenda, ita ut sit $\partial y = \frac{\partial x}{s}$. Cum igitur sit

$$\partial x = \frac{2a\partial z\,(1 - zz)}{c\,(1 + zz)^2} \text{ et } s = \frac{a\,(1 - zz)}{1 + zz}, \text{ erit}$$

$$y = \frac{\partial x}{s} = \frac{2}{c} \cdot \frac{\partial z}{1 + zz},$$

unde fit integrando $y = \frac{2}{c}$ Arc. tang. z. Quia igitur est

$$z = \frac{a - \sqrt{[aa - (b + cx)^2]}}{b + cx}, \text{ erit}$$

$$y = \frac{2}{c} \text{ Arc. tang. } \frac{a - \sqrt{[aa - (b + cx)^2]}}{b + cx} + C.$$

Corollarium.

§. 34. Sit igitur $b = 0$ et $c = 1$, seu formula differen-

tialis proposita $\partial y = \frac{\partial x}{\sqrt{(aa - xx)}}$, reperieturque

$$y = 2 \text{ Arc. tang. } \frac{a - \sqrt{(aa - xx)}}{x} + C.$$

Quia igitur tangens hujus arcus est $\frac{a - \sqrt{(aa - xx)}}{x}$; tangens dupli arcus erit $= \frac{x}{\sqrt{(aa - xx)}}$, ita ut sit

$$y = \text{Arc. tang. } \frac{x}{\sqrt{(aa - xx)}}:$$

hujus autem arcus sinus erit $\frac{x}{a}$, sicque integrale quaesitum

$$\int \frac{\partial x}{\sqrt{(aa - xx)}} = \text{Arc. sin. } \frac{x}{a}.$$

Quia porro

$$\partial \,.\, \sqrt{(aa - xx)} = - \frac{x \partial x}{\sqrt{(aa - xx)}}, \text{ erit}$$

$$\int \frac{x \partial x}{\sqrt{(aa - xx)}} = - \sqrt{(aa - xx)}:$$

quocirca ista generalior conficitur integratio

$$\int \frac{A \partial x + B x \partial x}{\sqrt{(aa - xx)}} = A \,.\, \text{Arc. sin. } \frac{x}{a} - B \sqrt{(aa - xx)}.$$

Problema 13.

§. 35. *Si fuerit V functio rationalis binarum quantitatum v^n et s, existente*

$$s = \sqrt{(\alpha + \beta v^n + \gamma v^{2n})},$$

formulam differentialem $V v^{n-1} \partial v$ ab irrationalitate liberare.

Solutio.

Ponatur $v^n = x$, erit

$$s = \sqrt{(\alpha + \beta x + \gamma xx)} \text{ et } v^{n-1} \partial v = \frac{\partial x}{n};$$

hic ergo jam erit V functio rationalis binarum quantitatum x et s, existente

$$s = \sqrt{(\alpha + \beta x + \gamma xx)}$$

et formula ab irrationalitate liberanda erit $\frac{V\partial x}{n}$; qui casus prorsus convenit cum problemate praecedente, ideoque eandem habebit solutionem.

Scholion.

§. 36. Praecepta hactenus tradita ad omnes fere formulas differentiales, quae quidem adhuc tractari potuerunt, extenduntur. Interim tamen ejusmodi casus occurrere possunt, quibus idonea substitutio, ad irrationalitatem tollendam necessaria, non tam facile perspicitur, sed acri judicio demum investigare licet, in quo negotio cum praecepta generalia tradere nondum liceat, exempla quaedam particularia speciminis loco in medium afferamus.

Exemplum 1.

§. 37. *Si proposita fuerit haec formula irrationalis*

$$\partial P = \frac{\partial x\,(1 + xx)}{(1 - xx)\sqrt{(1 + x^4)}},$$

ejus integrale P investigare.

Si quis hic ejusmodi uti vellet substitutione, qua formula $\sqrt{(1 + x^4)}$ ad rationalitatem perduceretur, oleum et operam esset perditurus, interim tamen singulari artificio sequens substitutio negotium conficere poterit. Statuatur

$$\frac{x\sqrt{2}}{1 - xx} = p, \text{ eritque}$$

$$1 + pp = \frac{1 + x^4}{(1 - xx)^2}, \text{ hinc}$$

$$\sqrt{(1 + pp)} = \frac{\sqrt{(1 + x^4)}}{1 - xx};$$

tum vero erit differentiando

$$\partial p = \frac{\partial x \sqrt{2}\,(1 + xx)}{(1 - xx)^2},$$

ex quibus valoribus colligitur

$$\frac{\partial p}{\sqrt{(1+pp)}} = \frac{\partial x \sqrt{2}(1+xx)}{(1-xx)\sqrt{(1+x^4)}},$$

quae feliciter cum formula ipsa proposita convenit, ita ut sit

$$\frac{\partial p}{\sqrt{(1+pp)}} = \partial P \sqrt{2}, \quad \text{sive } \partial P = \frac{1}{\sqrt{2}} \cdot \frac{\partial p}{\sqrt{(1+pp)}},$$

unde colligitur integrando

$$P = \frac{1}{\sqrt{2}}\, l\,[\sqrt{(1+pp)} + p].$$

Quare si loco p et $\sqrt{(1+pp)}$ valores dati substituantur, haec obtinetur integratio satis memorabilis

$$P = \int \frac{\partial x(1+xx)}{(1-xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}}\, l\, \frac{\sqrt{(1+x^4)}+\sqrt{2}}{1+xx}.$$

Exemplum 2.

§. 38. *Si proposita fuerit haec formula irrationalis* $\partial Q = \frac{\partial x(1-xx)}{(1+xx)\sqrt{(1+x^4)}}$, *ejus integrale* Q *investigare.*

Ad hoc praestandum fiat $\frac{x\sqrt{2}}{1+xx} = q$, eritque

$$\sqrt{(1-qq)} = \frac{\sqrt{(1+x^4)}}{1+xx};$$

tum vero erit $\partial q = \frac{\partial x(1-xx)\sqrt{2}}{(1+xx)^2}$, atque hinc colligitur

$$\frac{\partial q}{\sqrt{(1-qq)}} = \frac{\partial x(1-xx)\sqrt{2}}{(1+xx)\sqrt{(1+x^4)}} = \partial Q\sqrt{2},$$

unde fit

$$Q = \frac{1}{\sqrt{2}} \int \frac{\partial q}{\sqrt{(1-qq)}} = \frac{1}{\sqrt{2}} \text{Arc. sin. } q.$$

Restituto ergo pro q valore assumto, ista obtinebitur integratio

$$Q = \int \frac{\partial x(1-xx)}{(1+xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} \text{ Arc. sin. } \frac{x\sqrt{2}}{1+xx}.$$

Scholion.

§. 39. Cum istae duae formulae

$$\frac{\partial x(1+xx)\sqrt{2}}{(1-xx)\sqrt{(1+x^4)}} \text{ et } \frac{\partial x(1-xx)\sqrt{2}}{(1+xx)\sqrt{(1+x^4)}}$$

perductae sint ad has simplices

$$\frac{\partial p}{\sqrt{(1+pp)}} \text{ et } \frac{\partial q}{\sqrt{(1-qq)}},$$

quarum utraque facile ab irrationalitate liberatur, istae ipsae formulae propositae ope idoneae substitutionis ab irrationalitate liberari possunt; unde mirum non est, earum integralia sive per logarithmum sive per arcum circularem exhiberi potuisse. Satis enim jam est ostensum, omnium formularum differentialium rationalium integralia semper vel per logarithmos et arcus circulares, vel adeo algebraice exhiberi posse; quod igitur etiam de illis formulis irrationalibus est tenendum, quas certae substitutionis ope ad rationalitatem perducere licet. Unde vicissim plures Geometrae concluserunt: si quae formula differentialis nullo plane modo ab irrationalitate liberari queat, tum ejus integrale etiam neque per logarithmos nec arcus circulares, multo minus algebraice exprimi posse, sed ad aliud genus quantitatum transcendentium referri oportere. Caeterum combinatio duorum praecedentium exemplorum manuducit ad solutionem sequentium.

Exemplum 3.

§. 40. *Si proposita fuerit haec formula differentialis*

$$\partial y = \frac{\partial x \sqrt{(1+x^4)}}{1-x^4},$$

ejus integrale invenire.

Hanc formulam per neutram substitutionem ante usurpatam rationalem reddere licet: utraque tamen juncta negotium confici poterit, namque ejus integrale per logarithmos et arcus circulares sequenti artificio expedietur. Formula enim proposita in binas sequentes partes discerpi potest, quae sunt

$$\partial y = \frac{\frac{1}{2}\partial x(1+xx)}{(1-xx)\sqrt{(1+x^4)}} + \frac{\frac{1}{2}\partial x}{(1+xx)\sqrt{(1+x^4)}},$$

quippe quarum summa ipsam formulam nostram propositam producit; prodit enim

$$\partial y = \frac{\frac{1}{2}\partial x(1+xx)^2 + \frac{1}{2}\partial x(1-xx)^2}{(1-x^4)\sqrt{(1+x^4)}} = \frac{\partial x(1+x^4)}{(1-x^4)\sqrt{(1+x^4)}}$$
$$= \frac{\partial x\sqrt{(1+x^4)}}{1-x^4}.$$

Quod si ergo duo praecedentia exempla in subsidium vocentur, manifesto fiet $\partial y = \frac{1}{2}\partial P + \frac{1}{2}\partial Q$, consequenter integrale quaesitum erit $y = \frac{1}{2}P + \frac{1}{2}Q$, quod sequenti modo exprimere licebit

$$\int \frac{\partial x\sqrt{(1+x^4)}}{1-x^4} = \frac{1}{2\sqrt{2}}\, l\, \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx} + \frac{1}{2\sqrt{2}}\,\text{Arc. sin.}\,\frac{x\sqrt{2}}{1+xx}.$$

Exemplum 4.

§. 41. *Si proposita fuerit haec formula differentialis* $\partial y = \frac{xx\,\partial x}{(1-x^4)\sqrt{(1+x^4)}}$, *ejus integrale investigare.*

Haec formula simili modo ac praecedens tractari potest; discerpatur enim in sequentes duas partes:

$$\frac{\frac{1}{4}\partial x(1+xx)}{(1-xx)\sqrt{(1+x^4)}} - \frac{\frac{1}{4}\partial x(1-xx)}{(1+xx)\sqrt{(1+x^4)}},$$

quippe quae conjunctae producunt

$$\partial y = \frac{\frac{1}{4}\partial x(1+xx)^2 - \frac{1}{4}\partial x(1-xx)^2}{(1-x^4)\sqrt{(1+x^4)}}$$
$$= \frac{\frac{1}{4}\partial x\cdot 4xx}{(1+x^4)\sqrt{(1+x^4)}} = \frac{xx\,\partial x}{(1-x^4)\sqrt{(1+x^4)}},$$

quae cum sit ipsa formula proposita, erit ex praecedentibus exemplis $\partial y = \frac{1}{4}\partial P - \frac{1}{4}\partial Q$, consequenter $y = \frac{1}{4}P - \frac{1}{4}Q$, hinc integrale quaesitum ita reperietur expressum

$$\int \frac{xx\,\partial x}{(1-x^4)\sqrt{(1+x^4)}} = \frac{1}{4\sqrt{2}}\, l\, \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx} - \frac{1}{4\sqrt{2}}\ \text{Arc. sin.}\ \frac{x\sqrt{2}}{1+xx}.$$

Scholion.

§. 42. Haec duo postrema exempla si nullo plane modo ope cujuspiam substitutionis ad rationalitatem perduci possent, insigne praeberent documentum, quod conclusio supra memorata quandoque fallere possit: Re autem attentius perpensa inveni omnia haec quatuor exempla ope unicae substitutionis immediate ad rationalitatem perduci ideoque integrari posse; id quod ostendisse utique operae erit pretium.

Alia resolutio quatuor postremorum exemplorum.

§. 43. Statuatur pro primo exemplo

$$v = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}, \quad \text{eritque} \quad \sqrt{(1+vv)} = \frac{1+xx}{\sqrt{(1+x^4)}};$$

tum vero

$$\sqrt{(1-vv)} = \frac{1-xx}{\sqrt{(1+x^4)}},$$

unde fit

$$\sqrt{\frac{1+vv}{1-vv}} = \frac{1+xx}{1-xx} \quad \text{et} \quad \sqrt{(1-v^4)} = \frac{1-x^4}{1+x^4}.$$

At differentiando adipiscimur

$$\partial v = \frac{\partial x\,(1-x^4)\sqrt{2}}{(1+x^4)\sqrt{(1+x^4)}}.$$

Cum nunc sit $\frac{1-x^4}{1+x^4} = \sqrt{(1-v^4)}$, erit

$$\partial v = \frac{\partial x\sqrt{2}.\sqrt{(1-v^4)}}{\sqrt{(1+x^4)}}, \quad \text{sive} \quad \frac{\partial v}{\sqrt{(1-v^4)}} = \frac{\partial x\sqrt{2}}{\sqrt{(1+x^4)}};$$

quae aequalitas maxime est notatu digna. Quod si jam haec ae-

quatio multiplicetur per $\sqrt{\frac{1+vv}{1-vv}} = \frac{1+xx}{1-xx}$, nascetur haec aequatio

$$\frac{\partial v}{1-vv} = \frac{\partial x(1+xx)\sqrt{2}}{(1-xx)\sqrt{(1+x^4)}},$$

sicque erit

$$\int\frac{\partial x(1+xx)}{(1-xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}}\int\frac{\partial v}{1-vv} = \frac{1}{2\sqrt{2}}\, l\,\frac{1+v}{1-v}.$$

Deinde aequatio

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{\sqrt{(1-v^4)}} = \frac{\partial x}{\sqrt{(1+x^4)}}$$

multiplicetur per

$$\sqrt{\frac{1-vv}{1+vv}} = \frac{1-xx}{1+xx},$$

ac prodibit formula exempli secundi

$$\int\frac{\partial x(1-xx)}{(1+xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}}\int\frac{\partial v}{1+vv} = \frac{1}{\sqrt{2}}\,\text{Arc. tang.}\, v.$$

Porro eadem aequatio

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{\sqrt{(1-v^4)}} = \frac{\partial x}{\sqrt{(1+x^4)}}$$

dividatur per

$$\sqrt{(1-v^4)} = \frac{1-x^4}{1+x^4},$$ et prodibit

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{1-v^4} = \frac{\partial x\sqrt{(1+x^4)}}{1-x^4};$$

quae est ipsa formula exempli tertii, ita ut jam sit

$$\int\frac{\partial x\sqrt{(1+x^4)}}{1-x^4} = \frac{1}{\sqrt{2}}\int\frac{\partial v}{1-v^4} = \frac{1}{2\sqrt{2}}\int\frac{\partial v}{1+vv} + \frac{1}{2\sqrt{2}}\int\frac{\partial v}{1-vv},$$

quod integrale cum ante invento egregie convenit. Tandem postrema aequatio hic inventa

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{1-v^4} = \frac{\partial x\sqrt{(1+x^4)}}{1-x^4}$$

ducatur in $vv = \frac{2xx}{1+x^4}$, ut prodeat

$$\frac{1}{\sqrt{2}}\cdot\frac{vv\partial v}{1-v^4} = \frac{2xx\partial x\sqrt{(1+x^4)}}{(1-x^4)(1+x^4)} = \frac{2xx\partial x}{(1-x^4)\sqrt{(1+x^4)}},$$

unde pro exemplo quarto colligitur

$$\int\frac{xx\,\partial x}{(1-x^4)\sqrt{(1+x^4)}} = \frac{1}{2\sqrt{2}}\int\frac{vv\,\partial v}{1-v^4} = -\frac{1}{4\sqrt{2}}\int\frac{\partial v}{1+vv} + \frac{1}{4\sqrt{2}}\int\frac{\partial v}{1-vv},$$

unde cum sit $v = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, erit

$$\int \frac{\partial v}{1-vv} = \tfrac{1}{2} l \frac{1+v}{1-v} = \tfrac{1}{2} l \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{\sqrt{(1+x^4)}-x\sqrt{2}}$$
$$= \tfrac{1}{2} l \frac{[\sqrt{(1+x^4)}+x\sqrt{2}]^2}{(1-xx)^2} = l \frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx}.$$

Deinde vero est

$$\int \frac{\partial v}{1+vv} = \text{Arc. tang. } v = \text{Arc. sin. } \frac{v}{\sqrt{(1+vv)}} = \text{Arc. sin. } \frac{x\sqrt{2}}{1+xx}.$$

Scholion.

§. 44. Quanquam autem haec quatuor exempla ad rationalitatem reducere licuit, tamen conclusio supra memorata, quod omnes formulae integrales, quae nullo modo rationales effici queant, ad aliud pertineant transcendentium genus, neque per solos logarithmos et arcus circulares expediri possint, non solum manet suspecta, sed etiam falsitas ejus evidenter ob oculos poni potest. Sit enim functio

$$X = \frac{a}{\sqrt{(1+xx)}} + \frac{b}{\sqrt[3]{(1+x^3)}} + \frac{c}{\sqrt[4]{(1+x^4)}};$$

tum certe formula differentialis $X\partial x$ nullo modo ad rationalitatem perduci poterit; interim tamen singulos ejus partes

$$\frac{a\partial x}{\sqrt{(1+xx)}}, \quad \frac{b\partial x}{\sqrt[3]{(1+x^3)}} \quad \text{et} \quad \frac{c\partial x}{\sqrt[4]{(1+x^4)}}$$

seorsim rationales effici et integralia per logarithmos et arcus circulares exhiberi possunt. Corodinis loco hic sequens problema notatu dignum adjungamus.

Problema 14.

§. 45. *Formularum integralium* $\int \frac{\partial x}{\sqrt{(1+x^4)}}$ *et* $\int \frac{\partial v}{\sqrt{(1-v^4)}}$ *valores per series investigare, pro casibus, quibus ponitur tam* $v = 1$ *quam* $x = 1$.

Solutio.

Cum posito $v = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, ut supra fecimus, evidens sit, sumto $x = 0$ fore etiam $v = 0$, et sumto $x = 1$ fore $v = 1$, ita ut hae duae quantitates x et v simul evanescant et simul unitati aequentur: hinc deducimus istam aequationem differentialem attentione dignissimam

$$\frac{1}{\sqrt{2}} \cdot \frac{\partial v}{\sqrt{(1-v^4)}} = \frac{\partial x}{\sqrt{(1+x^4)}},$$

quas ergo ambas formulas in series converti oportet; erit autem

$$\frac{1}{\sqrt{(1-v^4)}} = (1-v^4)^{-\frac{1}{2}} = 1 + \tfrac{1}{2} v^4 + \frac{1.3}{2.4} v^8 + \frac{1.3.5}{2.4.6} v^{12} + \text{etc. et}$$

$$\frac{1}{\sqrt{(1+x^4)}} = (1+x^4)^{-\frac{1}{2}} = 1 - \tfrac{1}{2} x^4 + \frac{1.3}{2.4} x^8 - \frac{1.3.5}{2.4.6} x^{12} + \text{etc.}$$

Illa jam per ∂v multiplicata et integrata praebet

$$\int \frac{\partial v}{\sqrt{(1-v^4)}} = v + \frac{1}{2.5} v^5 + \frac{1.3}{2.4.9} v^9 + \frac{1.3.5}{2.4.6.13} v^{13} + \text{etc.}$$

unde posito $v = 1$, valor hujus integralis erit

$$1 + \frac{1}{2.5} + \frac{1.3}{2.4.9} + \frac{1.3.5}{2.4.6.13} + \frac{1.3.5.7}{2.4.6.8.17} + \text{etc.}$$

quam seriem littera A indicemus. Simili modo altera series in ∂x ducta et integrata producit

$$\int \frac{\partial x}{\sqrt{(1+x^4)}} = x - \frac{1}{2.5} x^5 + \frac{1.3}{2.4.9} x^9 - \frac{1.3.5}{2.4.6.13} x^{13} + \text{etc.}$$

cujus valor facto $x = 1$ erit

$$1 - \frac{1}{2.5} + \frac{1.3}{2.4.9} - \frac{1.3.5}{2.4.6.13} + \frac{1.3.5.7}{2.4.6.8.17} - \text{etc.}$$

quem littera B designemus, ita ut sit $B = \frac{A}{\sqrt{2}}$, sive $A = B\sqrt{2}$; unde patet, priorem seriem se habere ad posteriorem ut $\sqrt{2} : 1$.

Scholion.

§. 46. Valor formulae integralis $\int \frac{\partial v}{\sqrt{(1-v^4)}}$ etiam hoc modo per seriem investigari potest. Cum sit

$$\frac{1}{\sqrt{(1-v^4)}} = \frac{(1+vv)^{-\frac{1}{2}}}{\sqrt{(1-vv)}}, \quad \text{et}$$

$$(1+vv)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}vv + \tfrac{1.3}{2.4}v^4 - \tfrac{1.3.5}{2.4.6}v^6 + \text{etc.}$$

notetur esse $\int \frac{\partial v}{\sqrt{(1-vv)}} = \frac{\pi}{2}$. Deinde pro integratione reliquorum terminorum ponatur

$$\int \frac{v^{n+2}\partial v}{\sqrt{(1-vv)}} = Av^{n+1}\sqrt{(1-vv)} + B\int \frac{v^n \partial v}{\sqrt{(1-vv)}},$$

quae aequatio differentiata dat

$$\frac{v^{n+2}}{\sqrt{(1-vv)}} = (n+1)Av^n\sqrt{(1-vv)} - \frac{Av^{n+2}}{\sqrt{(1-vv)}} + \frac{Bv^n}{\sqrt{(1-vv)}},$$

unde per $\sqrt{(1-vv)}$ multiplicando prodit

$$v^{n+2} = (n+1)Av^n - (n+1)Av^{n+2} - Av^{n+2} + Bv^n.$$

Hinc termini in quibus inest v^{n+2}, inter se aequati praebent $1 = -(n+2)A$, ideoque $A = -\frac{1}{n+2}$; termini vero v^n continentes praebent $0 = (n+1)A + B$, unde fit $B = \frac{n+1}{n+2}$, ita ut in genere sit

$$\int \frac{v^{n+2}\partial v}{\sqrt{(1-vv)}} = -\frac{1}{n+2}v^{n+1}\sqrt{(1-vv)} + \frac{n+1}{n+2}\int \frac{v^n \partial v}{\sqrt{(1-vv)}},$$

quod integrale uti requiritur evanescit posito $v = 0$. Ponatur nunc $v = 1$, eritque

$$\int \frac{v^{n+2}\partial v}{\sqrt{(1-vv)}} = \frac{n+1}{n+2}\int \frac{v^n \partial v}{\sqrt{(1-vv)}};$$

hinc ergo pro n scribendo successive valores 0, 2, 4, 6, 8, etc. erit

$$\text{I.} \quad \int \frac{vv\partial v}{\sqrt{(1-vv)}} = \tfrac{1}{2}\cdot\tfrac{\pi}{2};$$

$$\text{II.} \quad \int \frac{v^4\partial v}{\sqrt{(1-vv)}} = \tfrac{3}{4}\cdot\tfrac{1}{2}\cdot\tfrac{\pi}{2};$$

$$\text{III.} \quad \int \frac{v^6 \partial v}{\sqrt{(1-vv)}} = \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2};$$

etc. etc.

quibus valoribus adhibitis, erit casu $v = 1$

$$\int \frac{\partial v}{\sqrt{(1-v^4)}} = \frac{\pi}{2} - \frac{1}{2^2} \cdot \frac{\pi}{2} + \frac{1^2 . 3^2}{2^2 . 4^2} \cdot \frac{\pi}{2} - \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} \cdot \frac{\pi}{2} + \text{etc.}$$

$$= \frac{\pi}{2} \left(1 - \frac{1^2}{2^2} + \frac{1^2 . 3^2}{2^2 . 4^2} - \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} + \frac{1^2 . 3^2 . 5^2 . 7^2}{2^2 . 4^2 . 6^2 . 8^2} - \text{etc.}\right)$$

ita ut sit ex problemate praecedente

$$1 - \frac{1}{2.5} + \frac{1.3}{2.4.9} - \frac{1.3.5}{2.4.9.13} + \text{etc.}$$

$$= \frac{\pi}{2} \left(1 - \frac{1^2}{2^2} + \frac{1^2 . 3^2}{2^2 . 4^2} - \frac{1^2 3^2 . 5^2}{2^2 . 4^2 . 6^2} + \text{etc.}\right)$$

unde fit

$$\frac{\pi}{2} = \frac{1 - \frac{1}{2.5} + \frac{1.3}{2.4.9} - \frac{1.3.5}{2.4.6.13} + \text{etc.}}{1 - \frac{1^2}{2^2} + \frac{1^2 . 3^2}{2^2 . 4^2} - \frac{1^2 . 3^2 . 5^2}{2^2 . 4^2 . 6^2} + \text{etc.}}$$

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2) De integratione formulae irrationalis

$$\int \frac{x^n \partial x}{\sqrt{(aa - 2bx + cxx)}}.$$

Acta Academiae Scientiar. Petropolitanae. Tom. VI. Pars II. Pag. 62 — 67.

Problema 15.

Invenire integrale hujus formulae irrationalis

$$\int \frac{x^n \partial x}{\sqrt{(aa - 2bx + cxx)}}.$$

Solutio.

§. 47. Incipiamus a casu simplicissimo, quo $n = 0$, et quaeramus integrale formulae $\frac{\partial x}{\sqrt{(aa - 2bx + cxx)}}$, quae posito

$x = \frac{b+z}{c}$ transit in hanc $\frac{\partial z}{\sqrt{(aacc - bbc + czz)}}$, ubi duo casus distingui convenit, prout c fuerit vel quantitas positiva vel negativa. Sit igitur primo $c = +ff$, et formula nostra fiet $\frac{\partial z}{f\sqrt{(aaff - bb + zz)}}$, cujus integrale est $\frac{1}{f} l \frac{z + \sqrt{(aaff - bb + zz)}}{C}$, ideoque erit nostrum integrale

$$\frac{1}{\sqrt{c}} l \frac{cx - b + \sqrt{(aac - 2bcx + cc\,xx)}}{C},$$

quod ergo ita sumtum, ut evanescat posito $x = 0$, evadet

$$\frac{1}{\sqrt{c}} l \frac{cx - b + \sqrt{c}\,(aa - 2bx + cxx)}{-b + a\sqrt{c}}.$$

At vero si c fuerit quantitas negativa, puta $c = -gg$, formula differentialis per z expressa erit $\frac{\partial z}{g\sqrt{(aagg + bb - zz)}}$, cujus integrale est $\frac{1}{g}$ Arc. sin. $\frac{z}{\sqrt{(aagg + bb)}} + C$; quare integrale ita sumtum, ut evanescat posito $x = 0$, fiet

$$-\frac{1}{g}\text{ Arc. sin. }\frac{cx - b}{\sqrt{(aagg + bb)}} + \frac{1}{g}\text{ Arc. sin. }\frac{b}{\sqrt{(aagg + bb)}}.$$

§. 48. Denotet nunc Π valorem formulae integralis $\int \frac{\partial x}{\sqrt{(aa - 2bx + cxx)}}$ ita sumtum, ut evanescat posito $x = 0$, sive c fuerit quantitas positiva sive negativa; ac si sit $c = +ff$ erit uti vidimus

$$\Pi = \frac{1}{f} l \frac{ffx - b + f\sqrt{(aa - 2bx + ffxx)}}{af - b};$$

altero vero casu, quo $c = -gg$, erit

$$\Pi = -\frac{1}{g}\text{ Arc. sin. }\frac{ggx + b}{\sqrt{(aagg + bb)}} + \frac{1}{g}\text{ Arc. sin. }\frac{b}{\sqrt{(aagg + bb)}},$$

sive ambobus arcubus contractis habebimus

$$\Pi = \frac{1}{g}\text{ Arc. sin. }\frac{bg\sqrt{(aa - 2bx - ggxx)} - abg - ag^3x}{aagg + bb}.$$

Quoniam igitur mox ostendemus, integrationem formulae generalis $\int \frac{x^n \partial x}{\sqrt{(aa - 2bx + cxx)}}$ semper reduci posse ad casum $n = 0$, si modo fuerit n numerus integer positivus, omnia haec integralia per istum valorem Π exprimi poterunt.

§. 49. Jam post integrationem quantitati variabili x ejusmodi valorem constantem tribuamus, quo formula irrationalis

$$\sqrt{(aa - 2bx + cxx)}$$

ad nihilum redigatur, id quod fit, si sumatur $x = \frac{b \pm \sqrt{(bb - aac)}}{c}$, ideoque duobus casibus. Ponamus pro utroque casu functionem Π abire in Δ, ita ut casu $c = ff$ sit

$$\Delta = \frac{1}{f} l \frac{\sqrt{(bb - aaff)}}{af - b} = \frac{1}{f} l \sqrt{\frac{b + af}{b - af}};$$

pro altero autem casu, quo $c = -gg$

$$\Delta = \frac{1}{g} \text{Arc. sin.} \frac{\pm ag\sqrt{(bb + aagg)}}{aagg + bb} = \frac{1}{g} \text{Arc. sin.} \frac{ag}{\sqrt{(bb + aagg)}}.$$

Hos autem valores Δ in sequentibus casibus, quibus ipsa formula radicalis $\sqrt{(aa - 2bx + cxx)}$ evanescit, potissimum sumus contemplaturi.

§. 50. Nunc ad sequentem casum progressuri, consideremus formulam $s = \sqrt{(aa - 2bx + cxx)} - a$, ut scilicet evanescat facto $x = 0$, et quoniam est

$$\partial s = \frac{-b\partial x + cx\partial x}{\sqrt{(aa - 2bx + cxx)}},$$

erit vicissim integrando

$$c\int \frac{x\partial x}{\sqrt{(aa - 2bx + cxx)}} = b\int \frac{\partial x}{\sqrt{(aa - 2bx + cxx)}} + s,$$

unde colligimus

$$\int \frac{x\partial x}{\sqrt{(aa + 2bx + cxx)}} = \frac{b}{c}\Pi + \frac{\sqrt{(aa - 2bx + xx)} - a}{c};$$

quare si post integrationem statuamus $x = \frac{b \pm \sqrt{(bb - aac)}}{c}$, quippe quibus casibus fit $\sqrt{(aa - 2bx + cxx)} = 0$ et $\Pi = \Delta$ fiet

$$\int \frac{x\partial x}{\sqrt{(aa - 2bx + cxx)}} = \frac{b}{c}\Delta - \frac{a}{c}.$$

§. 51. Sumamus porro $s = x\sqrt{(aa - 2bx + cxx)}$, fiet $\partial s = \frac{aa\partial x - 3bx\partial x + 2cxx\partial x}{\sqrt{(aa - 2bx + cxx)}}$, unde vicissim integrando colligitur

$$2c\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}=3b\int\frac{x\partial x}{\sqrt{(aa-2bx+cxx)}}-aa\int\frac{\partial x}{\sqrt{(aa-2bx+cxx)}}+s,$$

unde statim pro casu $\sqrt{(aa-2bx+cxx)}=0$ deducimus

$$\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}=\frac{(3bb-aac)}{2cc}\Delta-\frac{3ab}{2cc}.$$

§. 52. Jam ad altiores potestates ascensuri statuamus $s=xx\sqrt{(aa-2bx+cxx)}$, et quia hinc fit

$$\partial s=\frac{2aax\partial x-5bxx\partial x+3cx^3\partial x}{\sqrt{(aa-2bx+cxx)}},\ \text{erit}$$

$$3c\int\frac{x^3\partial x}{\sqrt{(aa-2bx+cxx)}}=5b\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}$$
$$-2aa\int\frac{x\partial x}{\sqrt{(aa-2bx+cxx)}}+s,$$

hincque porro pro casu quo post integrationem statuitur

$$x=\frac{b\pm\sqrt{(bb-aac)}}{c},\ \text{habebitur}$$

$$\int\frac{x^3\partial x}{\sqrt{(aa-2bx+cxx)}}=\left(\frac{5b^3-3aabc}{2c^3}\right)\Delta-\frac{15abb}{6c^3}+\frac{2a^3}{3cc}$$
$$\text{vel}=\left(\frac{5b^3}{2c^3}-\frac{3aab}{2cc}\right)\Delta-\frac{5abb}{2c^3}+\frac{2a^3}{3cc}.$$

§. 53. Simili modo sit $s=x^3\sqrt{(aa-2bx+cxx)}$, et quia hinc fit

$$\partial s=\frac{3aaxx\partial x-7bx^3\partial x+4c^4\partial x}{\sqrt{(aa-2bx+cxx)}},$$

erit vicissim integrando

$$4c\int\frac{x^4\partial x}{\sqrt{(aa-2bx+cxx)}}=7b\int\frac{x^3\partial x}{\sqrt{(aa-2bx+cxx)}}$$
$$-3aa\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}+s;$$

tum igitur pro casu quo fit $\sqrt{(aa-2bx+cxx)}=0$, habebimus

$$\int\frac{x^4\partial x}{\sqrt{(aa-2bx+cxx)}}=\left(\frac{35b^4}{8c^4}-\frac{15aabb}{4c^3}+\frac{3a^4}{8cc}\right)\Delta-\frac{35ab^3}{8c^4}+\frac{55a^3b}{24c^3}.$$

§. 54. Quo autem ordo in his formulis melius explorari possit, singulas exhibeamus per factores, quemadmodum ordine oriuntur, sine ulla abbreviatione, atque hoc modo formulae integrales inventae ita repraesententur

$$\int \frac{\partial x}{\sqrt{(aa - 2bx + cxx)}} = \Delta,$$

$$\int \frac{x \partial x}{\sqrt{(aa - 2bx + cxx)}} = \frac{b}{c} \Delta - \frac{a}{c},$$

$$\int \frac{xx \partial x}{\sqrt{(aa - 2bx + cxx)}} = \left(\frac{1.3bb}{1.2cc} - \frac{aa}{1.2c}\right) \Delta - \frac{1.3.ab}{1.2.cc},$$

$$\int \frac{x^3 \partial x}{\sqrt{(aa - 2bx + cxx)}} = \left(\frac{1.3.5b^3}{1.2.3c^3} - \frac{1.3.5aab}{1.2.3cc}\right) \Delta - \frac{1.3.5abb}{1.2.3c^3} + \frac{1.2.2a^3}{1.2.3cc},$$

$$\int \frac{x^4 \partial x}{\sqrt{(aa - 2bx + cxx)}} = \left(\frac{1.3.5.7b^4}{1.2.2.4c^4} - \frac{1.3.5.6aabb}{1.2.3.4c^4} + \frac{1.3.3a^4}{1.2.3.4cc}\right)\Delta$$
$$- \frac{1.3.5.7ab^3}{1.2.3.4c^4} + \frac{1.5.11a^3b}{1.2.3.4c^3}.$$

§. 55. Instituamus nunc in genere istam evolutionem, sumendo $s = x^n \sqrt{(aa - 2bx + cxx)}$, et quia hinc fit

$$\partial s = \frac{naax^{n-1}\partial x - (2n+1)bx^n \partial x + (n+1)cx^{n+1}\partial x}{\sqrt{(aa - 2bx + cxx)}},$$

inde vicissim integrando colligitur

$$(n+1)c\int \frac{x^{n+1}\partial x}{\sqrt{(aa - 2bx + cxx)}} = (2n+1)b\int \frac{x^n \partial x}{\sqrt{(aa - 2bx + cxx)}},$$
$$- naa\int \frac{x^{n-1}\partial x}{\sqrt{(aa - 2bx - cxx)}} + x^n \sqrt{(aa - 2bx + cxx)}.$$

Quod si vero jam ante elicuerimus

$$\int \frac{x^{n-1}\partial x}{\sqrt{(aa - 2bx + cxx)}} = \mathrm{M}\Delta - \mathfrak{M} \quad \text{et}$$

$$\int \frac{x^n \partial x}{\sqrt{(aa - 2bx + cxx)}} = \mathrm{N}\Delta - \mathfrak{N},$$

ita ut hae duae formulae sint cognitae, sequens ex iis ita determinabitur, ut sit

$$\int \frac{x^{n+1}\partial x}{\sqrt{(aa-2bx+cxx)}} = \left[\frac{(2n+1)bN}{(n+1)c} - \frac{naaM}{(n+1)c}\right]\Delta$$
$$- \frac{(2n+1)b\mathfrak{N}}{(n+1)c} + \frac{naa\mathfrak{M}}{(n+1)c}.$$

Hoc igitur modo has integrationes, quousque libuerit, continuare licet, dum ex binis quibusque sequens ope hujus regulae formatur, ita ut omnia haec integralia vel a logarithmis vel ab arcubus circularibus pendeant, prouti coëfficiens c fuerit vel positivus vel negativus. Manifestum autem est istos valores assignari non posse, nisi exponens n fuerit numerus integer positivus.

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3) De integratione formulae $\int \frac{\partial x \sqrt{(1+x^4)}}{1-x^4}$, aliarumque ejusdem generis, per logarithmos et arcus circulares. *M. S. Academiae exhib. die* 16 *Sept.* 1776.

§. 56. Cum mihi non ita pridem contigisset, integrale hujus formulae $\int \frac{\partial x \sqrt{(1+x^4)}}{1-x^4}$ per arcum circularem et logarithmum exprimere, haec integratio eo magis mihi visa est notatu digna, quod nullo modo perspiciebam, eam ad rationalitatem perduci posse, quandoquidem certum est, istam formulam, quae simplicior videatur, $\int \partial x \sqrt{(1+x^4)}$, neutiquam ad rationalitatem revocari posse, neque enim videbam, accessionem denominatoris $1-x^4$ hanc reductionem promovere posse, hincque concludebam dari ejusmodi formulas differentiales irrationales, quarum integralia per logarithmos et arcus circulares exhibere liceat, etiamsi nulla substitutione ab irrationalitate liberari queant: quaequidem conclusio utique valet pro formulis compositis, quanquam enim istae formulae

$$\int \frac{\partial x}{\sqrt[3]{(1 + x^3)}} \text{ et } \int \frac{\partial x}{\sqrt[4]{(1 + x^4)}}$$

ad rationalitatem reduci possunt, tamen formula ex iis composita

$$\int \partial x \left[\frac{A}{\sqrt[3]{(1 + x^3)}} + \frac{B}{\sqrt[4]{(1 + x^4)}}\right]$$

per nullam plane substitutionem ad aliam formulam rationalem reduci potest; propterea quod utraque pars peculiarem substitutionem postulat.

§. 57. Interim tamen cum formulam propositam

$$\int \frac{\partial x \sqrt{(1 + x^4)}}{1 - x^4} = S$$

attentius essem contemplatus, inveni, eam ab irrationalitate liberari posse, ope hujus substitutionis prorsus singularis

$$x = \frac{\sqrt{(1 + tt)} + \sqrt{(1 - tt)}}{t\sqrt{2}}.$$

Hinc enim fit

$$\partial x = -\frac{\partial t}{tt\sqrt{2(1 + tt)}} - \frac{\partial t}{tt\sqrt{2(1 - tt)}},$$

quae duae partes ad eundem denominatorem reductae dant

$$\partial x = -\frac{\partial t}{tt\sqrt{2(1 - t^4)}} [\sqrt{(1 - tt)} + \sqrt{(1 + tt)}].$$

Cum igitur sit

$$\sqrt{(1 + tt)} + \sqrt{(1 - tt)} = tx\sqrt{2},$$

hoc valore substituto fiet

$$\partial x = -\frac{x\,\partial t}{t\sqrt{(1 - t^4)}},$$

ita ut sit

$$\partial S = -\frac{x\,\partial t \sqrt{(1 + x^4)}}{t(1 - x^4)\sqrt{(1 - t^4)}}.$$

§. 68. Porro autem sumtis quadratis erit

$$xx = \frac{1 + \sqrt{(1 - t^4)}}{tt},$$

unde colligimus

$$1 + xx = \frac{1 + tt + \sqrt{(1 - t^4)}}{tt} = \frac{\sqrt{(1 + tt)}}{tt}[\sqrt{(1 + tt)} + \sqrt{(1 - tt)}],$$

sicque ob

$$\sqrt{(1 + tt)} + \sqrt{(1 - tt)} = tx\sqrt{2}, \text{ erit}$$

$$1 + xx = \frac{x\sqrt{2(1 + tt)}}{t}.$$

Simili modo erit

$$1 - xx = -\left(\frac{1 - tt + \sqrt{(1 - t^4)}}{tt}\right)$$

$$= -\frac{\sqrt{(1 - tt)}}{tt}[\sqrt{(1 - tt)} + \sqrt{(1 + tt)}] = -\frac{x\sqrt{2(1 - tt)}}{t}.$$

Hinc igitur sequitur fore

$$1 - x^4 = -\frac{2xx\sqrt{(1 - t4)}}{tt},$$

qui valor in nostra formula substitutus praebet

$$\partial S = + \frac{t\partial t\sqrt{(1 + x^4)}}{2x(1 - t^4)}.$$

§. 69. Deinde sumtis quadratis habebimus

$$(1 + xx)^2 = \frac{2xx(1 + tt)}{tt} \text{ et}$$

$$(1 - xx)^2 = \frac{2xx(1 - tt)}{tt},$$

quibus additis prodibit

$$(1 + xx)^2 + (1 - xx)^2 = 2(1 + x^4) = \frac{4xx}{tt},$$

unde fit

$$\sqrt{(1 + x^4)} = \frac{x\sqrt{2}}{t};$$

quo valore substituto nostra formula abit in hanc

$$\partial S = \frac{1}{\sqrt{2}} \cdot \frac{\partial t}{1 - t4};$$

quae ergo formula est rationalis et solam variabilem t complectitur.

§. 60. Cum igitur porro sit

$$\frac{1}{1-t^4} = \tfrac{1}{2} \cdot \frac{1}{1+tt} + \tfrac{1}{2} \cdot \frac{1}{1-tt},$$

tum vero integrando reperiatur

$$\int \frac{\partial t}{1+tt} = \text{Arc. tang. } t, \quad \text{et}$$

$$\int \frac{\partial t}{1-tt} = \tfrac{1}{2} l \frac{1+t}{1-t} = l \frac{1+t}{\sqrt{1-tt}},$$

quibus valoribus substitutis reperietur

$$S = \frac{1}{2\sqrt{2}} \text{Arc. tang. } t + \frac{1}{2\sqrt{2}} l \frac{1+t}{\sqrt{(1-tt)}}.$$

Quare cum regrediendo sit $t = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, supra autem invenerimus

$$1 + x^4 = \frac{2xx}{tt}, \quad \text{erit} \quad tt = \frac{2xx}{1+x^4},$$

hincque

$$1 - tt = \frac{(1-xx)^2}{1+x^4}, \quad \text{ideoque} \quad \sqrt{(1-tt)} = \frac{1-xx}{\sqrt{(1+x^4)}},$$

his valoribus substitutis, integrale quaesitum per ipsam variabilem x sequenti modo exprimetur

$$\int \frac{\partial x \sqrt{(1+x^4)}}{1-x^4} = \frac{1}{2\sqrt{2}} \text{Arc. tang.} \frac{x\sqrt{2}}{\sqrt{(1+x^4)}} + \frac{1}{2\sqrt{2}} l \frac{x\sqrt{2}+\sqrt{(1+x^4)}}{1-xx}.$$

§. 61. Hic autem merito quaeretur, quonam artificio ad substitutionem illam, quae primo intuitu a scopo prorsus aliena videtur pertigerim? quandoquidem nemo certe in eam incidisset, neque etiam ipse memini, quanam ratione ad eam sim perductus. Verum postquam omnia momenta accuratius perpendissem, methodum multo planiorem detexi, qua istud negotium sine tot ambagibus absolvi potest, quam igitur hic perspicue proponi conveniet.

Methodus planior et magis naturalis, formulam integralem propositam tractandi.

§. 62. Quo ex formula $\partial S = \frac{\partial x \surd (1 + x^4)}{1 - x^4}$ irrationalitatem saltem apparenter tollamur, ponamus $\surd (1 + x^4) = px$, ut fiat $\partial S = \frac{px\,\partial x}{1 - x^4}$. Cum igitur sit $1 + x^4 = pp\,xx$, erit radicem extrahendo

$$xx = \tfrac{1}{2} pp + \surd (\tfrac{1}{4} p^4 - 1).$$

Ponatur hic $\frac{1}{2} pp = q$, ut habeamus

$$xx = q + \surd (qq - 1), \text{ et}$$
$$2lx = l\,[q + \surd (qq - 1)],$$

hincque differentiando $\frac{2\,\partial x}{x} = \frac{\partial q}{\surd (qq - 1)}$: ergo loco q restituto valore $\frac{1}{2} pp$, erit $\frac{2\,\partial x}{x} = \frac{2p\,\partial p}{\surd (p^4 - 4)}$, sicque fiet $\partial x = \frac{xp\,\partial p}{\surd (p^4 - 4)}$, quo valore substituto fit $\partial S = \frac{p^2 x^2 \partial p}{(1 - x^4) \surd (p^4 - 4)}$.

§. 63. Ut nunc hinc quantitatem x penitus ejiciamus, quoniam invenimus

$$xx = \frac{pp + \surd (p^4 - 4)}{2}, \text{ erit}$$
$$x^4 = \frac{p^4 - 2 + pp \surd (p^4 - 4)}{2}, \text{ hincque}$$
$$1 - x^4 = \frac{4 - p^4 - pp \surd (p^4 - 4)}{2} = - \frac{\surd (p^4 - 4)\,[pp + \surd (p^4 - 4)]}{2}.$$

Unde colligitur fore $\frac{xx}{1 - x^4} = - \frac{1}{\surd (p^4 - 4)}$, quo valore substituto impetramus formulam differentialem rationalem per novam variabilem p expressam, quae est

$$\partial S = - \frac{pp\,\partial p}{p^4 - 4}, \text{ existente } p = \frac{\surd (1 + x^4)}{x};$$

unde idem integrale, quod ante nacti sumus, deducitur. Similis autem substitutio cum successu adhiberi potest in formulis integralibus multo magis generalibus; veluti in sequente problemate ostendemus.

Problema 16.

§. 64. *Propositam formulam integralem* $S = \int \frac{\partial x \sqrt{(a+bxx+cx^4)}}{a-cx^4}$ *ope idoneae substitutionis ab omni irrationalitate liberare.*

Solutio.

Ad speciem saltem irrationalitatis tollendam, ponamus

$$\sqrt{(a + bxx + cx^4)} = px,$$

ut habeamus $S = \int \frac{px\,\partial x}{a - cx^4}$. Cum igitur sit

$$p = \frac{\sqrt{(a + bxx + cx^4)}}{x}, \text{ erit}$$

$$\partial p = - \frac{a\partial x + cx^4\,\partial x}{xx\sqrt{(a+bxx+cx^4)}} = - \frac{a\partial x + cx^4\partial x}{p\,x^3},$$

unde erit

$$\partial x = - \frac{px^3\,\partial p}{a - cx^4},$$

quo valore substituto fiet

$$\partial S = - \frac{ppx^4\,\partial p}{(a - cx^4)^2}.$$

§. 65. Deinde cum sit

$$a + cx^4 = (pp - b)\,xx,$$

hincque porro

$$(a + cx^4)^2 = (pp - b)^2\,x^4,$$

auferatur $4\,acx^4$, ac remanebit

$$(a - cx^4)^2 = [(pp - b)^2 - 4ac]\;x^4,$$

quo substituto formula nostra fiet

$$\partial S = - \frac{pp\,\partial p}{(pp - b)^2 - 4\,ac}.$$

Sicque quantitas variabilis x penitus e calculo est extrusa, ac deducti sumus ad formulam differentialem prorsus rationalem, cujus ergo integratio per logarithmos et arcus circulares nulla amplius

laborat difficultate. Quin etiam formulae adhuc generaliores eodem modo feliciter tractari poterunt.

Problema 17.

§. 66. *Propositam hanc formulam integralem*

$$S = \int \frac{x^{n-2}\,\partial x \sqrt[n]{(a + bx^n + cx^{2n})}}{a - cx^{2n}}$$

ope idoneae substitutionis ab omni irrationalitate liberare.

Solutio.

Utamur igitur hac substitutione

$$\sqrt[n]{(a + bx^n + cx^{2n})} = px,$$

ut formula proposita hanc induat formam

$$\partial S = \frac{px^{n-1}\,\partial x}{a - cx^{2n}};$$

tum vero cum sit

$$p^n = \frac{a + bx^n + cx^{2n}}{x^n},$$

erit differentiando

$$p^{n-1}\,\partial p = -\frac{\partial x\,(a - cx^{2n})}{x^{n+1}},$$

unde fit

$$\partial x = -\frac{p^{n-1}x^{n+1}\,\partial p}{a - cx^{2n}},$$

quo valore substituto formula nostra induet hanc formam

$$\partial S = -\frac{p^n x^{2n}\,\partial p}{(a - cx^{2n})^2}.$$

§. 67. Deinde cum sit

$$a + cx^{2n} = (p^n - b)\, x^n, \text{ erit}$$

$$(a + cx^{2n})^2 = (p^n - b)^2\, x^{2n};$$

hinc subtrahatur $4acx^{2n}$, et remanebit

$$(a - cx^{2n})^2 = [(p^n - b)^2 - 4ac]\, x^{2n},$$

substituto igitur hoc valore fiet

$$\partial S = - \frac{p^n \partial p}{(p^n - b)^2 - 4ac},$$

quae ergo omnino est rationalis, atque adeo integratio per logarithmos et arcus circulares facile expeditur.

Problema 18.

§. 68. *Invenire formulas integrales adhuc generaliores, quae ope substitutionis*

$$\sqrt[n]{(a + bx^n + cx^{2n})} = px$$

ad rationalitatem perduci queant.

Solutio.

Quoniam in praecedente problemate invenimus, hanc formulam differentialem

$$\frac{x^{n-2}\partial x \sqrt[n]{(a + bx^n + cx^{2n})}}{a - cx^{2n}}$$

ope hujus substitutionis reduci ad istam formulam rationalem

$$- \frac{p^n \partial p}{(p^n - b)^2 - 4ac}, \text{ erit}$$

$$\frac{P x^{n-2} \partial x \sqrt[n]{(a + bx^n + cx^{2n})}}{a - cx^{2n}} = - \frac{Pp^n \partial p}{(p^n - b)^2 - 4ac}.$$

ubi loco P functiones quaecunque ipsius x accipi possunt ejusmodi, ut facta substitutione praebeant functiones rationales ipsius p, id quod infinitis modis fieri poterit, quorum praecipuos hic percurramus.

§. 69. Cum vi substitutionis sit

$$\frac{\sqrt[n]{(a + bx^n + cx^{2n})}}{x} = p,$$

loco P potestas quaecunque ipsius p assumi poterit, quae sit p^λ. Sumatur igitur $P = p^\lambda Q$, eritque etiam

$$P = \frac{Q\sqrt{(a + bx^n + cx^{2n})^\lambda}}{x^\lambda};$$

quibus valoribus substitutis prodibit ista aequatio

$$\frac{Qx^{n-\lambda-2}\partial x \sqrt[n]{(a + bx^n + cx^{2n})^{\lambda+1}}}{a - cx^{2n}} = -\frac{Qp^{n+\lambda}\partial p}{(p^n - b)^2 - 4ac}$$

quae posterior formula denuo est rationalis.

§. 70. Deinde in praecedente problemate quoque invenimus esse

$$\frac{(a - cx^{2n})^2}{x^{2n}} = (p^n - b)^2 - 4ac,$$

quam ob rem pro Q sumamus potestatem exponentis i harum quantitatum, vel potius harum quantitatum reciprocam, scilicet capiatur

$$Q = \frac{x^{2in}}{(a - cx^{2n})^{2i}} = \frac{1}{[(p^n - b)^2 - 4ac]^i}.$$

Quibus valoribus substitutis obtinebimus formulam latissime patentem hanc

$$\frac{x^{(2i+1)n-\lambda-2}\partial x \sqrt[n]{(a+bx^n+ex^{2n})^{\lambda+1}}}{(a-cx^{2n})^{2i+1}} = -\frac{p^{n+\lambda}\partial p}{[(p^n-b)^2-4ac]^{i+1}};$$

ubi pro litteris λ et i numeros quoscunque integros sive positivos sive negativos accipere licet, perpetuo enim formula differentialis per p expressa manebit rationalis.

§. 71. Quin etiam haec reductio multo generalior reddi potest, propterea quod necessum non est ut λ sit numerus integer: Quaecunque enim fractio pro λ assumatur, formula per p expressa semper facile ad rationalitatem reduci poterit. Si enim ponamus $\lambda = \frac{\mu}{\nu}$, membrum dextrum fiet

$$-\frac{p^{\frac{\nu n+\mu}{\nu}}\partial p}{[(p^n-b)^2-4ac]^{i+1}}.$$

quae rationalis redditur ponendo $p = q^\nu$, erit enim $\partial p = \nu q^{\nu-1}\partial q$, ideoque hoc membrum

$$-\frac{\nu q^{\mu+\nu n+\nu-1}\partial q}{[(q^{\nu n}-b)^2-4ac]^{i+1}}.$$

Nunc autem uti oportebit hac substitutione

$$\sqrt[n]{(a+bx^n+cx^{2n})} = q^\nu x,$$

atque habebitur ista reductio

$$\frac{x^{(2i+1)n-\frac{\mu}{\nu}-2}\partial x \sqrt[n]{(a+bx^n+cx^{2n})^{\frac{\mu+\nu}{\nu}}}}{(a-cx^{2n})^{2i+1}}$$
$$= -\frac{\nu q^{\mu+\nu n+\nu-1}\partial q}{[(q^{n\nu}-b)^2-4ac]^{i+1}},$$

quae postrema formula utique est rationalis.

§. 72. Ut etiam in membro sinistro exponentes fractos ipsius x tollamus, ponamus $x = y^\nu$, eritque

$$\frac{y^{(2i+1)n\nu-\mu-\nu-1}\partial y \sqrt[n\nu]{(a+by^{n\nu}+cy^{2n\nu})^{\mu+\nu}}}{(a-cy^{2n\nu})^{2i+1}}$$

$$= -\frac{q^{\mu+n\nu+\nu-1}\partial q}{[(q^{n\nu}-b)^2-4ac]^{i+1}},$$

quae expressio autem multo generalior videtur, quam revera est. Si enim loco $n\nu$ ubique scribamus n resultat ista aequatio

$$\frac{y^{(2i+1)n-\mu-\nu-1}\partial y \sqrt[n]{(a+by^n+cy^{2n})^{\mu+\nu}}}{(a-cy^{2n})^{2i+1}}$$

$$= -\frac{q^{\mu+\nu+n-1}\partial q}{[(q^n-b)^2-4ac]^{i+1}};$$

haec autem aequatio manifesto non discrepat ab illa §. 70. allata; si enim hic loco $\mu+\nu-1$ scribamus λ et loco y et q ut ante x et p, ipsa praecedens aequatio reperitur, sicque sufficiet loco λ numeros integros assumere.

Corollarium.

§. 73. Quo clarius indoles harum formularum perspiciatur, sumamus $n = 2$, et formula differentialis variabilem x involvens erit

$$\frac{x^{4i-\lambda}\partial x \sqrt{(a+bxx+cx^4)^{\lambda+1}}}{(a-cx^4)^{2i+1}},$$

quae facta substitutione $\sqrt{(a+bxx+cx^4)} = px$, transmutatur in hanc rationalem

$$-\frac{p^{\lambda+2}\partial p}{[(pp-b)^2-4ac]^{i-1}},$$

unde sumendo $\lambda = 4i$ resultat ista aequatio

$$\frac{\partial x \sqrt{(a + bxx + cx^4)^{4i+1}}}{(a - cx^4)^{ci+1}} = - \frac{p^{4i+2} \partial p}{[(pp - b)^2 - 4ac]^{i+1}},$$

in qua si porro ponatur $i = 0$, fiet

$$\frac{\partial x \sqrt{(a + bxx + cx^4)}}{a - cx^4} = - \frac{pp \partial p}{(pp - b)^2 - 4ac};$$

quae si insuper ponatur $a = 1$, $b = 0$ et $c = 1$, praebet

$$\frac{\partial x \sqrt{(1 + x^4)}}{1 - x^4} = - \frac{pp \partial p}{p^4 - 4},$$

quae est ipsa reductio, quae supra §. 63. fuerat inventa.

Corollarium 2.

§. 74. Si sumamus $n = 3$, prodibit ista reductio generalis

$$\frac{x^{6i-\lambda+1} \partial x \sqrt[3]{(a + bx^3 + cx^6)^{\lambda+1}}}{(a - cx^6)^{2i+1}} = - \frac{p^{\lambda+3} \partial p}{[(p^3 - b)^2 - 4ac]^{i+1}},$$

quae ponendo $i = 0$ migrat in hanc

$$\frac{x^{-\lambda+1} \partial x \sqrt[3]{(a + bx^3 + cx^6)^{\lambda+1}}}{a - cx^6} = - \frac{p^{\lambda+3} \partial p}{(p^3 - b)^2 - 4ac};$$

posito vero $b = 0$, haec prodit formula concinnior

$$\frac{x^{-\lambda+1} \partial x \sqrt[3]{(a+cx^6)^{\lambda+1}}}{a - cx^6} = - \frac{p^{\lambda+3} \partial p}{p^6 - 4ac},$$

cujus duos casus evolvisse juvabit.

I. Sit $\lambda = 0$, eritque

$$\frac{x \partial x \sqrt[3]{(a + cx^6)}}{a - cx^6} = - \frac{p^3 \partial p}{p^6 - 4ac};$$

quae concinnior redditur ponendo $xx = y$, reperietur enim

$$\frac{\partial y \sqrt[3]{(a + cy^3)}}{a - cy^3} = - \frac{2p^3 \partial p}{p^6 - 4ac}.$$

II. Sumto autem $\lambda = 1$, ista prodit expressio

$$\frac{\partial x \sqrt[3]{(a + cx^6)^2}}{a - cx^6} = - \frac{p^4 \partial p}{p^6 - 4ac}.$$

Scholion.

§. 75. Ex his exemplis satis intelligitur, quam egregie reductiones ex nostris formulis generalibus deduci queant, quarum resolutio, nisi methodus nostra adhibeatur, omnes vires analyseos superare videatur.

4.) Memorabile genus formularum differentialium maxime irrationalium, quas tamen ad rationalitatem perducere licet. *M. S. Academiae exhib. d. 15. Maii 1777.*

§. 76. Cum nuper hanc formulam differentialem

$$\frac{\partial x}{(1 - xx) \sqrt[4]{(2xx - 1)}}$$

tractassem eamque singulari modo ad rationalitatem perduxissem, mox vidi eandem methodum succedere in hac generaliori $\frac{\partial x}{(a + bxx) \sqrt[4]{(a + 2bxx)}}$, atque adeo in hac multo generaliori

$\frac{\partial x}{(a+bx^n)\sqrt[2n]{(a+2bx^n)}}$, ubi irrationalitas ad ordinem quantumvis altum assurgere potest; cujus resolutio sequenti modo instituitur.

§. 77. Utor scilicet hac substitutione $\frac{x}{\sqrt[2n]{(a+2bx^n)}} = Z$, ut formula nostra integranda, quam per ∂V indicemus, fiat $\partial V = \frac{\partial x}{x} \cdot \frac{Z}{a+bx^n}$, sumtis ergo logarithmis erit

$$lZ = lx - \frac{1}{2n} l(a + 2bx^n),$$

unde differentiando fit

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{bx^{n-1}\partial x}{a+2bx^n} = \frac{\partial x(a+bx^n)}{x(a+2bx^n)},$$

erit ergo

$$\frac{\partial x}{x} = \frac{\partial Z(a+2bx^n)}{Z(a+bx^n)},$$

hinc ergo nostra formula erit

$$\partial V = \frac{\partial Z(a+2bx^n)}{(a+bx^n)^2}.$$

Cum igitur sit

$$Z^{2n} = \frac{x^{2n}}{a+2bx^n}, \text{ erit } a + 2bx^n = \frac{x^{2n}}{Z^{2n}}.$$

ideoque

$$\partial V = \frac{x^{2n}\partial Z}{Z^{2n}(a+bx^n)^2}.$$

Cum porro sit $aa + 2abx^n = \frac{ax^{2n}}{Z^{2n}}$, addatur utrinque bbx^{2n}, et prodibit

$$(a+bx^n)^2 = \frac{ax^{2n}}{Z^{2n}} + bbx^{2n} = \frac{x^{2n}(a+bbZ^{2n})}{Z^{2n}},$$

quo valore substituto nostra formula evadet

$$\partial V = \frac{\partial Z}{a + bbZ^{2n}},$$

quae ergo formula est rationalis, ideoque per logarithmos et arcus circulares integrari poterit.

§. 78. Observavi porro, cum hic post signum radicale tantum binomium involvatur, ejus loco quoque trinomia, atque adeo polynomia introduci posse. Pro trinomiis autem formula differentialis talem habebit formam

$$\partial V = \frac{\partial x}{(a + bx^n)\sqrt[3n]{(aa + 3abx^n + 3bbx^{2n})}},$$

ubi ergo irrationalitas ad ordinem multo altiorem ascendit. Nihilo vero minus etiam ista formula ab irrationalitate liberari poterit ope similis substitutionis

$$Z = \frac{x}{\sqrt[3n]{(aa + 3abx^n + 3bbx^{2n})}};$$

hinc enim sumtis logarithmis per differentiationem nanciscemur

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{abx^{n-1}\partial x - 2bbx^{2n-1}\partial x}{aa + 3abx^n + 3bbx^{2n}}, \quad \text{seu}$$

$$\frac{\partial Z}{Z} = \frac{\partial x(a + bx^n)^2}{x(aa + 3abx^n + 3bbx^{2n})},$$

ideoque

$$\frac{\partial x}{x} = \frac{\partial Z}{Z} \cdot \frac{aa + 3abx^n + 3bbx^{2n}}{(a + bx^n)^2}.$$

Cum igitur nostra formula jam sit $\partial V = \frac{\partial x}{x} \cdot \frac{Z}{a + bx^n}$, introducto elemento ∂Z, obtinebimus

$$\partial V = \frac{\partial Z(aa + 3abx^n + 3bbx^{2n})}{(a + bx^n)^3}.$$

§. 79. Cum igitur vi substitutionis sit

$$\sqrt[3n]{(aa + 3abx^n + 3bbx^{2n})} = \frac{x}{Z}, \quad \text{erit}$$

$$aa + 3abx^n + 3bbx^{2n} = \frac{x^{3n}}{Z^{3n}}.$$

Multiplicetur utrinque per a, et addatur utrinque $b^3 x^{3n}$, eritque

$$(a + bx^n)^3 = \frac{x^{3n}(a + b^3 Z^{3n})}{Z^{3n}}:$$

hoc igitur valore substituto ex formula nostra littera x penitus excludetur, prodibitque $\partial V = \frac{\partial Z}{a + b^3 Z^n}$. Cujus ergo integrale semper per logarithmos et arcus circulares reperire licebit.

§. 80. Pro quadrinomiis autem ponamus brevitatis gratia

$$\sqrt[4n]{(a^3 + 4aabx^n + 6abbx^{2n} + 4b^3 x^{3n})} = S,$$

ac formula ad rationalitatem reducenda proponatur haec

$$\partial V = \frac{\partial x}{(a + bx^n)S},$$

id quod simili modo succedet ope hujus substitutionis $\frac{x}{S} = Z$, unde formula nostra erit $\partial V = \frac{\partial x}{x} \cdot \frac{Z}{a + bx^n}$. Cum nunc sit

$$\frac{\partial S}{S} = \frac{aabx^{n-1}\partial x + 3abbx^{2n-1}\partial x + 3b^3x^{3n-1}\partial x}{a^3 + 4aabx^n + 6abbx^{2n} + 4b^3x^{3n}},$$

sive

$$\frac{\partial S}{S} = \frac{\partial x}{x} \cdot \frac{bx^n(aa + 3abx^n + 3bbx^{2n})}{S^{4n}},$$

erit $\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{\partial S}{S}$; consequenter

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} \cdot \frac{(a + bx^n)^3}{S^{4n}}, \quad \text{hincque} \quad \frac{\partial x}{x} = \frac{S^{4n}\partial Z}{Z(a + bx^n)^3},$$

quo valore substituto formula nostra erit

$$\partial V = \frac{S^{4n} \partial Z}{(a + bx^n)^4}.$$

§. 81. Cum autem sit

$$S^{4n} = a^3 + 4aabx^n + 6abbx^{2n} + 4b^3x^{3n}, \quad \text{erit}$$

$$aS^{4n} + b^4x^{4n} = (a + bx^n)^4,$$

quo valore substituto erit

$$\partial V = \frac{S^{4n} \partial Z}{aS^{4n} + b^4 x^{4n}}:$$

quia igitur posuimus $Z = \frac{x}{S}$, erit $S = \frac{x}{Z}$, ideoque $S^{4n} = \frac{x^{4n}}{Z^{4n}}$, qui valor surrogatus dabit

$$\partial V = \frac{\partial Z}{a + b^4 Z^{4n}},$$

sicque itidem ad rationalitatem est perducta.

§. 82. Hinc jam facile intelligitur, quo modo pro omnibus polynomiis formulae differentiales comparatae esse debeant, ut tali substitutione ad rationalitatem perduci queant, id quod in sequente problemate expediamus

Problema 19.

§. 83. *Si proposita fuerit haec formula differentialis*

$$\partial V = \frac{\partial x}{(a + bx^n) \sqrt[\lambda n]{[(a + bx^n)^\lambda - b^\lambda x^{\lambda n}]}},$$

eam ad rationalitatem reducere, quantumvis magni numeri pro n et λ accipiantur.

Solutio.

Ponamus etiam hic brevitatis gratia

$$\sqrt[\lambda n]{[(a + bx^n)^\lambda - b^\lambda x^{\lambda n}]} = S,$$

ut formula fiat

$$\partial V = \frac{\partial x}{(a + bx^n)\, S},$$

fiatque insuper $\frac{x}{S} = Z$, ut habeamus

$$\partial V = \frac{\partial x}{x} \cdot \frac{Z}{a + bx^n}.$$

Jam logarithmos differentiando reperietur

$$\frac{\partial S}{S} = \frac{bx^{n-1}\partial x\, (a + bx^n)^{\lambda - 1} - b^\lambda x^{\lambda n - 1}\partial x}{S^{\lambda n}}, \text{ sive}$$

$$\frac{\partial S}{S} = \frac{\partial x}{x} \cdot \frac{bx^n (a + bx^n)^{\lambda - 1} - b^\lambda x^{\lambda n}}{S^{\lambda n}}.$$

Cum igitur sit $\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{\partial S}{S}$, hoc valore substituto erit

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} \cdot \frac{a (a + bx^n)^{\lambda - 1}}{S^{\lambda n}},$$

hincque vicissim erit

$$\frac{\partial x}{x} = \frac{S^{\lambda n}\partial Z}{a Z (a + bx^n)^{\lambda - 1}},$$

quo valore substituto impetramus

$$\partial V = \frac{S^{\lambda n}\partial Z}{a (a + bx^n)^\lambda},$$

quia nunc est $(a + bx^n)^\lambda = S^{\lambda n} + b^\lambda x^{\lambda n}$, erit

$$\partial V = \frac{S^{\lambda n}\partial Z}{a (S^{\lambda n} + b^\lambda x^{\lambda n})}.$$

Denique ob $S = \frac{x}{Z}$, ideoque $S^{\lambda n} = \frac{x^{\lambda n}}{Z^{\lambda n}}$, hoc valore substituto obtinebitur

$$\partial V = \frac{\partial Z}{a(1 + b^{\lambda} Z^{\lambda n})},$$

quae est rationalis unicam variabilem Z involvens, cujus adeo integrale per logarithmos et arcus circulares assignari poterit.

Corollarium 1.

§. 84. Eadem solutio etiam locum habet, si pro λ numeri fracti accipiantur, qua ratione post signum radicale denuo radicalia involvuntur: ita si fuerit $\lambda = \frac{2}{n}$, erit formula radicalis

$$S = \sqrt{[(a + bx^n)^{\frac{2}{n}} - b^{\frac{2}{n}} xx]},$$

et formulae nostrae

$$\partial V = \frac{\partial x}{(a + bx^n) S}$$

integrale erit

$$V = \frac{1}{a} \int \frac{\partial Z}{1 + b^{\frac{2}{n}} ZZ} = \frac{1}{a b^{\frac{1}{n}}} \text{Arc. tang. } b^{\frac{1}{n}} Z.$$

Corollarium 2.

§. 85. Quo haec clariora reddantur, capiamus $a = 1$, $b = 1$, et $n = 4$, ut pro postremo casu sit

$$S = \sqrt{[(1 + x^4)^{\frac{1}{2}} - xx]}, \text{ et } \partial V = \frac{\partial x}{(1 + x^4)\sqrt{[(1 + x^4)^{\frac{1}{2}} - xx]}},$$

cujus integrale posito

$$Z = \frac{x}{\sqrt{[(1 + x^4)^{\frac{1}{2}} - xx]}}, \text{ erit}$$

$$V = \text{Arc. tang. } Z, \text{ sive } V = \text{Arc. tang.} \frac{x}{\sqrt{[(1 + x^4)^{\frac{1}{2}} - xx]}}.$$

Sin autem manente $n = 4$ et $a = 1$, fuerit $b = -1$, ideoque

$$S = \sqrt{[(1 - x^4)^{\frac{1}{2}} - xx\sqrt{-1}]},$$

ipsa formula prodiret imaginaria.

Corollarium 3.

§. 86. Pro eodem casu $\lambda = \frac{2}{n}$, sit $n = 6$, $a = 1$ et $b = 1$, eritque

$$S = \sqrt{[(1 + x^6)^{\frac{1}{3}} - xx]}, \text{ ideoque}$$

$$\partial V = \frac{\partial x}{(1 + x^6)\sqrt{[(1 + x^6)^{\frac{1}{3}} - xx]}}.$$

Cujus integrale posito $\frac{x}{S} = Z$, erit

$$V = \text{Arc. tang. } Z = \text{Arc. tang.} \frac{x}{\sqrt{[(1 + x^6)^{\frac{1}{3}} - xx]}}.$$

Similique modo alia hujus generis exempla pro lubitu formari possunt; verum quamquam formula problematis admodum est generalis, tamen adhuc multo magis generalior fieri potest, uti in sequente problemate sumus ostensuri.

Problema 20.

§. 87. *Si proponatur ista formula differentialis multo generalior, quippe in qua tres occurrunt exponentes indeterminati* λ, n, *et* m,

$$\partial V = \frac{x^{m-1}\,\partial x}{(a+bx^n)\sqrt[\lambda n]{[(a+bx^n)^\lambda - b^\lambda x^{\lambda n})^m}},$$

eam ab irrationalitate liberare.

Solutio.

Ponatur iterum brevitatis gratia

$$\sqrt[\lambda n]{[(a+bx^n)^\lambda - b^\lambda x^{\lambda n}]} = S,$$

ut formula integranda proposita fiat

$$\partial V = \frac{x^{m-1}\,\partial x}{(a+bx^n)\,S^m} = \frac{\partial x}{x} \cdot \frac{x^m}{(a+bx^n)\,S^m},$$

quae ergo si porro ut ante statuamus $\frac{x}{S} = Z$, fiet

$$\partial V = \frac{\partial x}{x} \cdot \frac{Z^m}{a+bx^n},$$

unde variabilem x penitus eliminari oportet. Quoniam nunc ambae litterae S et Z eosdem habent valores, ut in problemate praecedente atque adeo ipsa formula ∂V oriatur, si praecedens per Z^{m-1} multiplicetur, etiam integrale quaesitum obtinebimus, dum superius integrale per Z^{m-1} multiplicabimus, quo facto erit integrale quaesitum

$$V = \frac{1}{a}\int \frac{Z^{m-1}\,\partial Z}{1+b^\lambda Z^{\lambda n}}.$$

Corollarium 1.

§. 88. Si exponentem m negativum capiamus, irrationalitas in numeratorem transferetur, ita posita $m = -1$ habebimus

$$\partial V = \frac{\partial x\sqrt[\lambda n]{[(a+bx^n)^\lambda - b^\lambda x^{\lambda n}]}}{xx\,(a+bx^n)},$$

cujus ergo integrale per Z expressum erit

$$V = \frac{1}{a} \int \frac{\partial Z}{ZZ(1 + b^\lambda Z^{\lambda n})}.$$

Quin etiam per hunc exponentem m irrationalitas simplicior reddi poterit, veluti si sumamus $m = \lambda$, erit

$$\partial V = \frac{x^{\lambda - 1} \partial x}{(a + bx^n) \sqrt[n]{[(a + bx^n)^\lambda - b^\lambda x^{\lambda n}]}}.$$

Cujus integrale posito $Z = \frac{x}{S}$, retinente S superiorem valorem erit

$$V = \frac{1}{a} \int \frac{Z^{\lambda - 1} \partial Z}{1 + b^\lambda Z^{\lambda n}}.$$

Corollarium 2.

§. 89. Deinde vero etiam si pro m fractionem assumamus, irrationalitas adhuc magis complicabitur, veluti si sumamus $m = \frac{1}{2}$, formula differentialis jam erit

$$\partial V = \frac{\partial x}{(a + bx^n)^{2\lambda n} \sqrt{x^{\lambda n} [(a + bx^n)^\lambda - b^\lambda x^{\lambda n}]}}.$$

Verum hic casus facile ad primum problema revocatur statuendo $x = vv$, ita ut sit

$$\partial V = \frac{2 \partial v}{(a + bv^{2n}) \sqrt[2\lambda n]{[(a + bv^{2n})^\lambda - b^\lambda v^{2\lambda n}]}},$$

quae formula a primo problemate aliter non discrepat nisi quod hic exponens n duplo sit major.

Scholion.

§. 90. Quamquam binae litterae a et b pro lubitu tam negative, quam positive accipi possunt, tamen occurrunt casus, qui sub hac generali forma non comprehenduntur: veluti si propona-

tur haec formula $\frac{\partial x}{(1-xx\sqrt{(2xx-1)}}$, haec in problemate primo non continetur, quia fieri deberet $aa=-1$, quod cum in genere evenire posset, etiam problema generale ad hunc casum accommodatum subjungamus.

Problema 21.

§. 91. *Si ponatur ista formula differentialis latissime patens tres exponentes indeterminatos involvens*

$$\partial V = \frac{x^{m-1}\partial x}{(fx^n-g)\sqrt[\lambda n]{[f^\lambda x^{\lambda n}-(fx^n-g)^\lambda]^m}},$$

eam ab omni irrationalitate liberare.

Solutio.

Statúamus ut ante brevitatis gratia

$$\sqrt[\lambda n]{[f^\lambda x^{\lambda n}-(fx^n-g)^\lambda]} = S,$$

tum vero $Z=\frac{x}{S}$, ut formula differentialis fiat

$$\partial V = \frac{\partial x}{x}\cdot\frac{Z^m}{fx^n-g}.$$

Nunc autem sumendo differentialia logarithmica est

$$\frac{\partial S}{S} = \frac{\partial x}{x}\cdot\frac{f^\lambda x^{\lambda n}-fx^n(fx^n-g)^{\lambda-1}}{S^{\lambda n}},$$

atque hinc colligitur fore

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{\partial S}{S} = \frac{\partial x}{x}\cdot\frac{g(fx^n-g)^{\lambda-1}}{S^{\lambda n}},$$

sicque habebitur

$$\frac{\partial x}{x} = \frac{\partial Z}{Z}\cdot\frac{S^{\lambda n}}{g(fx^n-g)^{\lambda-1}},$$

quo valore substituto nanciscimur

$$\partial V = \frac{z^{m-1} \partial z s^{\lambda n}}{g\,(f x^n - g)^\lambda}.$$

Manifesto autem est $(fx^n - g)^\lambda = f^\lambda x^{\lambda n} - S^{\lambda n}$, ideoque

$$\partial V = \frac{z^{m-1} s^{\lambda n} \partial z}{g\,(f^\lambda x^{\lambda n} + s^{\lambda n})};$$

unde postremo ob $S = \frac{x}{z}$ concluditur haec forma

$$\partial V = \frac{z^{m-1} \partial z}{g\,(f^\lambda z^{\lambda n} - 1)},$$

quae formula a praecedentibus tantum signis discrepat.

SUPPLEMENTUM II.

AD TOM. I. CAP. III.

DE

INTEGRATIONE FORMULARUM DIFFERENTIALIUM PER SERIES INFINITAS.

De resolutione formulae integralis, $\int x^{m-1}\partial x(\Delta + x^n)^\lambda$ in seriem semper convergentem. Ubi simul plura insignia artificia circa serierum summationem explicantur. *M. S. Academiae exhib. die* 12 *Aug.* 1779.

§. 1. Obtulit se mihi nuper haec formula integralis $\int \partial x \sqrt{(\Delta + x^4)}$, cujus valor, cum casu quo $\Delta = 0$ sit $\frac{1}{3}x^3$, in mentem mihi venit, eos ejus valores investigare, quos induit, quando Δ est quantitas valde parva. Mox autem vidi, hoc vulgari evolutione praestari neutiquam posse. Cum enim sit

$$\sqrt{(\Delta + x^4)} = \sqrt{\Delta} \times \left(1 + \frac{x^4}{\Delta}\right)^{\frac{1}{2}},$$

ideoque per seriem

$$\sqrt{(\Delta + x^4)} = \sqrt{\Delta} \times \left(1 + \frac{1}{2} \cdot \frac{x^4}{\Delta} - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{x^8}{\Delta^2} + \frac{1 \cdot 1 \cdot 3}{4 \cdot 5 \cdot 6} \cdot \frac{x^{12}}{\Delta^3} - \text{etc.}\right)$$

erit valor formulae hujus integralis

$$\int \partial x \sqrt{(\Delta + x^4)} = x\sqrt{\Delta} \times \left(1 + \frac{1}{2} \cdot \frac{x^4}{5\Delta} - \frac{1 \cdot 1}{2 \cdot 4} \cdot \frac{x^8}{9\Delta^2} + \frac{1 \cdot 1 \cdot 3}{2 \cdot 4 \cdot 6} \cdot \frac{x^{12}}{13\Delta^3} - \text{etc.}\right)$$

quae series ergo manifesto maxime divergit, quoties Δ fuerit quantitas valde parva, atque adeo, quoties fractio $\frac{x^n}{\Delta}$ unitatem superaverit.

§. 2. Ut igitur ad scopum propositum pertingerem, ipsam hanc quaestionem sub hac forma sum contemplatus: *Valorem formulae integralis $\int \partial x \sqrt{(\Delta + x^4)}$ a termino $x = 0$ usque ad terminum $x = a$ extensum per seriem semper convergentem exprimere, quicunque valor litterae Δ tribuatur.* Hunc in finem formulam $\Delta + x^4$ sub hac specie repraesento

$$\Delta + a^4 - (a^4 - x^4),$$

sive hac

$$(\Delta + a^4)\left(1 - \frac{a^4 - x^4}{\Delta + a^4}\right).$$

Hinc igitur erit

$$\sqrt{(\Delta + x^4)} = \sqrt{(\Delta + a^4)} \times \left[1 - \tfrac{1}{2} \cdot \frac{a^4 - x^4}{\Delta + a^4} - \frac{1.1}{2.4}\left(\frac{a^4 - x^4}{\Delta + a^4}\right)^2 - \text{etc.}\right].$$

Sicque totum negotium huc redit ut harum formularum integralium

$$\int \partial x (a^4 - x^4),\ \int \partial x (a^4 - x^4)^2,\ \int \partial x (a^4 - x^4)^3,\ \text{etc.}$$

valores ab $x = 0$ usque ad $x = a$ extensi investigentur, unde primus terminus $\int \partial x$ dabit a.

§. 3. Pro secundo termino habebitur integrando

$$\int \partial x (a^4 - x^4) = a^4 x - \tfrac{1}{5} x^5,$$

cujus valor sumto $x = a$ erit $\frac{4}{5} a^5$. Pro tertio termino erit

$$\int \partial x (a^4 - x^4)^2 = a^8 x - \tfrac{2}{5} a^4 x^5 + \tfrac{1}{9} x^9,$$

quae expressio posito $x = a$ abit in $\frac{4 \cdot 8}{5 \cdot 9} a^9$. Simili modo pro quarto termino habebimus

$$\int \partial x (a^4 - x^4)^3 = a^{13}\left(1 - \tfrac{3}{5} + \tfrac{3}{9} - \tfrac{1}{13}\right)^3 = \frac{4 \cdot 8 \cdot 12}{5 \cdot 9 \cdot 13} a^{13}.$$

Eodemque modo reperitur fore

$$\int \partial x\, (a^4 - x^4)^4 = \frac{4 \cdot 8 \cdot 12 \cdot 16}{5 \cdot 9 \cdot 13 \cdot 17} a^{17},$$

et ita porro. Hanc autem elegantem progressionis legem infra sum demonstraturus.

§. 4. His igitur valoribus substitutis, totus valor integralis quaesitus reperietur fore

$$a\sqrt{(\Delta + a^4)} \times \left[1 - \tfrac{1}{2} \cdot \tfrac{4}{5} \cdot \frac{a^4}{\Delta + a^4} - \frac{1.1}{2.4} \cdot \frac{4.8}{5.9}\left(\frac{a^4}{\Delta + a^4}\right)^2 - \frac{1.1.3}{2.4.6} \cdot \frac{4.8.12}{5.9.13}\left(\frac{a^4}{\Delta + a^4}\right)^3 - \text{etc.}\right].$$

Quoniam hic duplices coëfficientes occurrunt, si singulos factores priorum tam supra quam infra duplicemus, ista series contrahetur in sequentem

$$a\sqrt{(\Delta + a^4)} \times \left[1 - \tfrac{2}{5} \cdot \frac{a^4}{\Delta + a^4} - \frac{2.2}{5.9}\left(\frac{a^4}{\Delta + a^4}\right)^2 - \frac{2.2.6}{5.9.13}\left(\frac{a^4}{\Delta + a^4}\right)^3 - \text{etc.}\right],$$

quae series manifesto semper convergit, propterea quod non solum coëfficientes haud mediocriter decrescunt, sed etiam formula $\frac{a^4}{\Delta + a^4}$ unitate est minor.

§. 5. Jam nihil obstat quo minus loco a restituamus ipsam quantitatem variabilem x, sicque valor hujus formulae integralis $\int \partial x \sqrt{(\Delta + x^4)}$ exprimetur per sequentem seriem semper convergentem

$$x\sqrt{(\Delta + x^4)} \times \left[1 - \tfrac{2}{5} \cdot \frac{x^4}{\Delta + x^4} - \frac{2.2}{5.9}\left(\frac{x^4}{\Delta + x^4}\right)^2 - \frac{2.2.6}{5.9.13}\left(\frac{x^4}{\Delta + x^4}\right)^3 - \text{etc.}\right].$$

Hic casus quo ista series minime convergit, est ille ipse, quem initio commemoravimus, quo $\Delta = 0$, ipsumque integrale $= \frac{1}{3} x^3$. Posito igitur $\Delta = 0$ pervenimus ad sequentem seriem maxime notatu dignam

$$x^3 \left(1 - \tfrac{2}{5} - \frac{2.2}{5.9} - \frac{2.2.6}{5.9.13} - \frac{2.2.6.10}{5.9.13.17} - \text{etc.}\right),$$

cujus adeo summam novimus esse $= \frac{1}{3} x^3$, ita ut jam habeamus hanc summationem

$$\tfrac{1}{3} = 1 - \tfrac{2}{5} - \tfrac{2}{5} \cdot \tfrac{2}{9} - \tfrac{2}{5} \cdot \tfrac{2}{9} \cdot \tfrac{6}{13} - \tfrac{2}{5} \cdot \tfrac{2}{9} \cdot \tfrac{6}{13} \cdot \tfrac{10}{17} - \text{etc.}$$

cujus demonstratio altioris indaginis videtur. Interim tamen quoniam ejus summa est cognita, veritas sequenti modo ostendi potest. Hinc enim erit

$$\frac{2}{5} + \frac{2 \cdot 2}{3 \cdot 9} + \frac{2 \cdot 2 \cdot 6}{5 \cdot 9 \cdot 13} + \text{etc.} = \frac{2}{3}.$$

quae aequatio in $\frac{5}{2}$ ducta dat

$$1 + \frac{2}{9} + \frac{2 \cdot 6}{9 \cdot 13} + \frac{2 \cdot 6 \cdot 10}{9 \cdot 13 \cdot 17} + \text{etc.} = \frac{5}{3}.$$

Transponatur hic primus terminus in alteram partem, et multiplicando per $\frac{9}{2}$ prodibit

$$1 + \frac{6}{13} + \frac{6 \cdot 10}{13 \cdot 17} + \frac{6 \cdot 10 \cdot 14}{13 \cdot 17 \cdot 21} + \text{etc.} = \frac{9}{3}.$$

Translato iterum primo termino ad alteram partem factaque multiplicatione per $\frac{13}{6}$, colligitur

$$1 + \frac{10}{17} + \frac{10 \cdot 14}{17 \cdot 21} + \frac{10 \cdot 14 \cdot 18}{17 \cdot 21 \cdot 25} + \text{etc.} = \frac{13}{3}.$$

Simili modo progrediendo prodibit

$$1 + \frac{14}{21} + \frac{14 \cdot 18}{21 \cdot 25} + \frac{14 \cdot 18 \cdot 22}{21 \cdot 25 \cdot 29} + \text{etc.} = \frac{17}{3}.$$

$$1 + \frac{18}{25} + \frac{18 \cdot 22}{25 \cdot 29} + \frac{18 \cdot 22 \cdot 26}{25 \cdot 29 \cdot 33} + \text{etc.} = \frac{21}{3}.$$

Sicque innumerabiles nacti sumus series, quarum summa est cognita, et quoniam lege aequabili ulterius progrediuntur, signum hoc certum est summam primo datam esse justam. Hanc autem insignem veritatem infra, ubi rem in genere persequemur, accuratius demonstrabimus.

Problema generale.

Formulae integralis $\int x^{m-1} \partial x (\Delta + x^n)^\lambda$ *valorem a termino* $x = 0$ *usque ad* $x = a$ *extensum per seriem semper convergentem exprimere.*

Solutio.

§. 6. Formulam $\Delta + x^n$ sub hac forma repraesentemus $\Delta + a^n - (a^n - x^n)$, quae reducitur ad hanc

$$(\Delta + a^n)\left(1 - \frac{a^n - x^n}{\Delta + a^n}\right),$$

sicque formula integralis proposita erit

$$(\Delta + a^n)^\lambda \int x^{m-1}\partial x \left(1 - \frac{a^n - x^n}{\Delta + a^n}\right)^\lambda.$$

At facta evolutione est

$$\left(1 - \frac{a^n - x^n}{\Delta + a^n}\right)^\lambda = 1 - \frac{\lambda}{1}\left(\frac{a^n - x^n}{\Delta + a^n}\right) + \frac{\lambda(\lambda-1)}{2}\left(\frac{a^n - x^n}{\Delta + a^n}\right)^2 - \text{etc.}$$

quae ergo series ducta in $x^{m-1}\partial x$ ita integrari debet, ut integrale ab $x=0$ ad $x=a$ extendatur. Hinc patet totum negotium reduci ad hanc integrationem $\int x^{m-1}\partial x (a^n - x^n)^\theta$, cujus valor casu quo $\theta = 0$ manifesto est $\frac{x^m}{m} = \frac{a^m}{m}$. Casu vero quo $\theta = 1$ erit

$$\int x^{m-1}\partial x (a^n - x^n) = \frac{a^n x^m}{m} - \frac{x^{m+n}}{m+n},$$

qui valor, posito $x = a$, evadit $\frac{n}{m(m+n)} a^{m+n}$. Ac casu quo $\theta = 2$ erit

$$\int x^{m-1}\partial x (a^n - x^n)^2 = a^{2n}\frac{x^m}{m} - 2a^n\frac{x^{m+n}}{m+n} + \frac{x^{m+2n}}{m+2n},$$

quae expressio posito $x = a$ abit in hanc $\frac{n \cdot 2n}{m(m+n)(m+2n)} a^{m+2n}$. Simili modo calculo subducto reperietur

$$\int x^{m-1}\partial x (a^n - x^n)^3 = \frac{n \cdot 2n \cdot 3n}{m(m+n)(m+2n)(m+3n)} a^{m+3n}.$$

Ne autem hic inductioni nimium tribuamus, hanc progressionem sequenti modo accuratius demonstrabimus.

§. 7. Ponamus formulae $\int x^{m-1}\partial x (a^n - x^n)^\theta$ valorem jam esse inventum $= V$, hincque quaeramus valorem formulae se-

quentis $\int x^{m-1}\partial x(a^n-x^n)^{\theta+1}$. Hunc in finem ponamus

$$\int x^{m-1}\partial x(a^n-x^n)^{\theta+1}=A\int x^{m-1}\partial x(a^n-x^n)^{\theta}+Bx^m(a^n-x^n)^{\theta+1},$$

quae formula differentiata et per $x^{m-1}\partial x(a^n-x^n)^{\theta}$ divisa praebet

$$a^n-x^n=A+mB(a^n-x^n)-(\theta+1)nBx^n;$$

unde nascuntur hae duae determinationes

$$A+mBa^n=a^n \quad \text{et} \quad mB+(\theta+1)nB=1,$$

qui praebent

$$A=\frac{(\theta+1)na^n}{m+(\theta+1)n} \quad \text{et} \quad B=\frac{1}{m+(\theta+1)n}.$$

§. 8. Quoniam igitur post integrationem fieri debet $x=a$, membrum littera B affectum evanescit, eritque

$$\int x^{m-1}\partial x(a^n-x^n)^{\theta+1}=\frac{(\theta+1)na^n}{m+(\theta+1)n}.V.$$

Cum igitur casu $\theta=0$ sit $V=\frac{a^m}{m}$, erit

$$\int x^{m-1}\partial x(a^n-x^n)=\frac{n}{m(m+n)}a^{m+n},$$

$$\int x^{m-1}\partial x(a^n-x^n)^2=\frac{n.2n}{m(m+n)(m+2n)}a^{m+2n},$$

$$\int x^{m-1}\partial x(a^n-x^n)^3=\frac{n.2n.3n}{m(m+n)(m+2n)(m+3n)}a^{m+3n}.$$

Unde patet ordinem supra observatum in ipsa rei natura esse fundatum.

§. 9. Quia hic integralia ita capi debent, ut evanescant posito $x=0$, in reductione generali, qua sumus usi ubi postremum membrum erat $Bx^m(a^n-x^n)^{\theta+1}$, evidens est, hoc membrum non evanescere, nisi fuerit $m>0$; quamobrem, si forte ejus-

modi formulae occurrant, ubi exponens m fuerit vel 0 vel adeo negativus, reductiones hic inventae locum habere nequeunt.

§. 10. Singuli hi termini factorem involvunt comunem $\frac{a^m}{m}$, qui si cum multiplicatore generali conjungatur, series per integrationem orta erit

$$\frac{a^m}{m}(\Delta+a^n)^\lambda \left\{ \begin{array}{l} 1-\frac{\lambda}{1}\cdot\frac{n}{m+n}\left(\frac{a^n}{\Delta+a^n}\right) \\ +\frac{\lambda(\lambda-1)}{1.2}\cdot\frac{m.2n}{(m+n)(m+2n)}\left(\frac{a^n}{\Delta+a^n}\right)^2-\text{etc.} \end{array} \right\}$$

ubi coëfficientes sequenti modo contrahi poterunt

$$\frac{a^m}{m}(\Delta+a^n)^\lambda \left\{ \begin{array}{l} 1-\frac{\lambda n}{m+n}\left(\frac{a^n}{\Delta+a^n}\right)+\frac{\lambda n}{m+n}\cdot\frac{(\lambda-1)n}{m+2n}\left(\frac{a^n}{\Delta+a^n}\right)^2 \\ -\frac{\lambda n}{m+n}\cdot\frac{(\lambda-1)n}{m+2n}\cdot\frac{(\lambda-2)n}{m+3n}\left(\frac{a^n}{\Delta+a^n}\right)^3+\text{etc.} \end{array} \right\}.$$

Quod si jam hic loco a substituamus ipsam quantitatem variabilem x, haec series

$$\frac{x^m}{m}(\Delta+x^n)^\lambda \left\{ \begin{array}{l} 1-\frac{\lambda n}{m+n}\left(\frac{x^n}{\Delta+x^n}\right)+\frac{\lambda n}{m+n}\cdot\frac{(\lambda-1)n}{m+2n}\left(\frac{x^n}{\Delta+x^n}\right)^2 \\ -\frac{\lambda n}{m+n}\cdot\frac{(\lambda-1)n}{m+2n}\cdot\frac{(\lambda-2)n}{m+3n}\left(\frac{x^n}{\Delta+x^n}\right)^3+\text{etc.} \end{array} \right\}$$

exprimet valorem formulae integralis $\int x^{m-1}\partial x(\Delta+x^n)^\lambda$ a termino $x=0$ sumtum.

§. 11. Casibus quibus exponens λ est numerus integer positivus, veritas seriei inventae sponte elucescit; uti his casibus

1°) Si $\lambda=1$, erit

$$\int x^{m-1}\partial x(\Delta+x^n)=\frac{x^m}{m}(\Delta+x^n)\left(1-\frac{n}{m+n}\cdot\frac{x^n}{\Delta+x^n}\right),$$

quae expressio reducitur ad hanc $\frac{x^m}{m}(\Delta+x^n-\frac{n}{m+n}x^n)$: inte-

grale vero ordinario modo sumtum erit $\frac{\Delta x^m}{m} + \frac{x^{m+n}}{m+n}$, quod cum praecedente convenit.

2°) Si fuerit $\lambda = 2$, erit

$$\int x^{m-1}\partial x(\Delta+x^n)^2 = \frac{x^m}{m}(\Delta+x^n)^2\left[1 - \frac{2n}{m+n}\left(\frac{x^n}{\Delta+x^n}\right) + \frac{2n}{m+n}\cdot\frac{2}{m+2n}\left(\frac{x^n}{\Delta+x^n}\right)^2\right]$$

quae expressio reducitur ad hanc

$$\frac{x^m}{m}\left\{\begin{array}{lll} \Delta\Delta + 2\,\Delta x^n & & + x^{2n} \\ & -\frac{2n}{m+n}\Delta x^n & -\frac{2n}{m+n}x^{2n} \\ & & +\frac{n\,.\,2n}{(m+n)(m+2n)}x^{2n} \end{array}\right\},$$

sive ad hanc concinniorem

$$\frac{x^m}{m}\left(\Delta\Delta + \frac{2n}{m+n}\Delta x^n + \frac{m}{m+2n}x^{2n}\right)$$

quod egregie convenit cum integrali more solito sumto. Caeterum hic meminisse juvabit, haec integralia locum habere non posse, nisi m fuerit nihilo major, quia alioquin integrale non ita sumi posset, ut evanesceret casu $x = 0$.

§. 12. Sin autem exponens λ non fuerit numerus integer, series inventa in infinitum progreditur, ejusque veritas non amplius in oculos incurrit. His autem casibus forma nostri integralis simplicior et concinnior reddetur, si statuamus $\lambda = -\frac{\mu}{n}$; tum enim hujus formulae $\int x^{m-1}\partial x(\Delta + x^n)^{-\frac{\mu}{n}}$ integrale erit

$$\frac{x^m}{m(\Delta + x^n)^{\frac{\mu}{n}}}\left\{\begin{array}{l} 1 + \frac{\mu}{m+n}\left(\frac{x^n}{\Delta+x^n}\right) + \frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n}\left(\frac{x^n}{\Delta+x^n}\right)^2 \\ + \frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n}\cdot\frac{\mu+2n}{m+3n}\left(\frac{x^n}{\Delta+x^n}\right)^3 + \text{etc.} \end{array}\right\}.$$

§. 13. Hinc jam summam hujus seriei generalis assignare licebit

$$1+\frac{a}{b}\chi+\frac{a}{b}\cdot\frac{a+n}{b+n}\chi^2+\frac{a}{b}\cdot\frac{a+n}{b+n}\cdot\frac{a+2n}{b+2n}\chi^3+\text{etc.}$$

Si enim hanc seriem cum inventa comparemus, erit $\mu=a$, et $m+n=b$, ideoque $m=b-n$; tum vero erit $\chi=\frac{x^n}{\Delta+x^n}$, unde relatio inter χ et x innotescit. Tum igitur hujus seriei summa aequabitur huic formulae integrali $\int\frac{x^{b-n-1}\partial x}{(\Delta+x^n)^{\frac{a}{n}}}$ divisae per hanc quantitatem $\frac{x^{b-n}}{(b-n)(\Delta+x^n)^{\frac{a}{n}}}$; ideoque ista summa erit

$$\frac{(b-n)(\Delta+x^n)^{\frac{a}{n}}}{x^{b-n}}\cdot\int\frac{x^{b-n-1}\partial x}{(\Delta+x^n)^{\frac{a}{n}}},$$

quae autem summa subsistere nequit, nisi fuerit $b>n$. Caeterum evidens est, istam seriem semper esse convergentem, cum non solum fractio $\frac{x^n}{\Delta+x^n}$ sit unitate minor, sed etiam coëfficientes omnes sint uninate minores.

§. 14. Casus autem maxime memorabilis, qui hic occurrit, est quando $\Delta=0$; tum enim nostra formula integralis erit

$$\int x^{m-\mu-1}\partial x=\frac{x^{m-\mu}}{m-\mu},$$

huic ergo quantitati semper aequabitur sequens series

$$\frac{x^{m-\mu}}{m}\left(1+\frac{\mu}{m+n}+\frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n}+\frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n}\cdot\frac{\mu+2n}{m+3n}+\text{etc.}\right),$$

si modo fuerit m numerus positivus, uti jam aliquoties est animad-

versum. Consequenter hujus seriei

$$1 + \frac{\mu}{m+n} + \frac{\mu(\mu+n)}{(m+n)(m+2n)} + \frac{\mu(\mu+n)(\mu+2n)}{(m+n)(m+2n)(m+3n)} + \text{etc.}$$

summa est $\frac{m}{m-\mu}$, quae summatio est eo magis notatu digna, quod vix ulla via patet, ejus veritatem investigandi.

§. 15. Statim autem apparet, hanc summam subsistere non posse, nisi tam n quam $m-\mu$ fuerit numerus positivus. Cum enim formula nostra integralis casu $\Delta = 0$ sit $\int x^{m-1-\mu}\partial x$, quam ab $x=0$ inchoari oportet, evidens est hoc fieri non posse, nisi $m-\mu$ fuerit numerus positivus; praeterea etiam notandum est, exponentem n necessario positivum esse debere. Cum enim in Analysi supra exposita hoc integrale occurrat $\int x^{m-1}\partial x(a^n - x^n)^\theta$, manifestum est, si n esset numerus negativus, integrationem non ita institui posse, ut casu $x=0$ evanescat. His notatis istam seriem accuratius sum contemplaturus et quoniam ejus indoles non parum abscondita videtur, ejus veritatem duplici modo sum ostensurus. Primo scilicet ostendam, summam assignatam revera aequari summae totius progressionis; deinde analysin prorsus singularem apperiam, quae non solum directe ad ipsam hanc seriem perducet, sed etiam ejus summam indicabit.

Demonstratio hujus summationis:

$$\frac{m}{m-\mu} = 1 + \frac{\mu}{m+n} + \frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n} + \frac{\mu}{m+n}\cdot\frac{\mu+n}{m+2n}\cdot\frac{\mu+2n}{m+3n} + \text{etc.}$$

§. 16. Hic scilicet ostendam, si omnes hujus seriei termini a summa inventa $\frac{m}{m-\mu}$ successive subtrahantur, tandem revera nihil relictum iri. Subtracto enim primo termino 1 remanet $\frac{\mu}{m-\mu}$. Hinc terminus secundus ablatus relinquet $\frac{\mu(\mu+n)}{(m-\mu)(m+n)}$. Hinc

porro subtrahatur tertius terminus ac remanebit

$$\frac{\mu(\mu+n)(\mu+2n)}{(m-\mu)(m+n)(m+2n)}.$$

Hinc jam quartus terminus ablatus residuum praebet sequens

$$\frac{\mu(\mu+n)(\mu+2n)(\mu+3n)}{(m-\mu)(m+n)(m+2n)(m+3n)}.$$

Unde jam satis liquet, omnibus terminis ablatis tandem remansurum esse hoc productum in infinitum excurrens

$$\frac{\mu(\mu+n)(\mu+2n)(\mu+3n)(\mu+4n)\text{ (etc.}}{(m-\mu)(m+n)(m+2n)(m+3n)(m+4n)\text{ (etc.}}.$$

§. 17. Facile autem intelligitur valorem hujus producti revera nihilo esse aequalem. Omisso enim primo factore $\frac{\mu}{m-\mu}$, omnes reliqui factores sunt fractiones unitate minores, quia $\mu < m$, et quoniam tam numeratores quam denominatores in arithmetica progressione increscunt, jam satis constat, valorem talis producti revera evanescere. Hic autem probe tenendum est, ut productum infinitarum talium fractionum in nihilum abeat, non sufficere, ut singulae fractiones sint unitate minores, veluti evenit in hac forma

$$\tfrac{3}{4}\cdot\tfrac{8}{9}\cdot\tfrac{15}{16}\cdot\tfrac{24}{25}\cdot\tfrac{35}{36}\cdot\tfrac{48}{49}\cdot\text{ etc.}$$

cujus producti in infinitum protensi valor facile ostenditur esse $=\frac{1}{2}$.

§. 18. Quoniam in nostro producto singuli denominatores superant suos numeratores eadem quantitate $m=\mu$, istam formam generaliorem considerabo

$$\frac{a}{a+\Delta}\cdot\frac{b}{b+\Delta}\cdot\frac{c}{c+\Delta}\cdot\frac{d}{d+\Delta}\cdot\frac{e}{e+\Delta}\cdot\text{ etc.}$$

et perscrutabor, sub quibusnam conditionibus ejus valor in infinitum extensus, qui sit Π, revera in nihilum sit abiturus. Evidens

autem est, hoc evenire, si eadem forma inversa

$$\frac{1}{\Pi} = \frac{a+\Delta}{a} \cdot \frac{b+\Delta}{b} \cdot \frac{c+\Delta}{c} \cdot \frac{d+\Delta}{d} \cdot \text{etc.}$$

in infinitum excrescit. Sin autem ejus valor fuerit infinitus, etiam ejus logarithmus infinitus evadat necesse est. Cum igitur sit

$$l\frac{1}{\Pi} = l\frac{a+\Delta}{a} + l\frac{b+\Delta}{b} + l\frac{c+\Delta}{c} + l\frac{d+\Delta}{d} + \text{etc.}$$

facta evolutione reperietur

$$\begin{aligned} l\frac{1}{\Pi} = & \Delta\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e} + \text{etc.}\right) \\ & - \tfrac{1}{2}\Delta^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} + \frac{1}{d^2} + \frac{1}{e^2} + \text{etc.}\right) \\ & + \tfrac{1}{3}\Delta^3\left(\frac{1}{a^3} + \frac{1}{b^3} + \frac{1}{c^3} + \frac{1}{d^3} + \frac{1}{e^3} + \text{etc.}\right) \\ & - \text{etc.} \end{aligned}$$

quae expressio semper erit infinita, quoties summa primae seriei $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \text{etc.}$ fuerit infinita. Hanc autem summam semper esse infinitam, quoties numeri a, b, c, d, etc. in progressione arithmetica crescunt, jam notum est et per se perspicuum, quod cum in nostra serie contingat, certum est, illius producti infiniti valorem esse evanescentem.

§. 19. Circa nostram autem seriem id imprimis notatu dignum occurrit, quod ejus summa $\frac{m}{m-\mu}$ litteram n non involvat, ita ut ejus summa semper maneat eadem, quicunque valores litterae n tribuantur, quod quidem pro casu $n = 0$ per se statim fit manifestum, quandoquidem tum series nostra evadit

$$1 + \frac{\mu}{m} + \frac{\mu^2}{m^2} + \frac{\mu^3}{m^3} + \text{etc.}$$

quae cum sit progressio geometrica, ejus summa erit $\frac{m}{m-\mu}$. Quod vero summa perpetuo maneat eadem, quicunque valores ipsi n tri-

buantur, non tam facile perspicitur. etsi veritas a nobis jam sit demonstrata.

§. 20. Quin etiam demonstratio hic tradita multo latius patet, cum adeo in eadem forma valores ipsius n variare liceat. Ita si posito α loco n, pro ejus multiplis $2n$, $3n$, $4n$, $5n$, etc. scribamus novas litteras β, γ, δ, ε, etc. ut habeatur ista series

$$1 + \frac{\mu}{m+\alpha} + \frac{\mu}{m+\alpha} \cdot \frac{\mu+\alpha}{m+\beta} + \frac{\mu}{m+\alpha} \cdot \frac{\mu+\alpha}{m+\beta} \cdot \frac{\mu+\beta}{m+\gamma} + \text{etc.}$$

ejus summa etiam nunc erit $\frac{m}{m-\mu}$. Subtracto enim termino primo remanet $\frac{\mu}{m-\mu}$. Hinc terminus secundus subtractus relinquit

$$\frac{\mu(\mu+\alpha)}{(m-\mu)(m+\alpha)}.$$

Hinc tertius terminus subtractus

$$\frac{\mu(\mu+\alpha)(\mu+\beta)}{(m-\mu)(m+\alpha)(m+\beta)}.$$

Unde jam patet, in infinitum tandem prodiri productum

$$\frac{\mu(\mu+\alpha)(\mu+\beta)(m+\gamma)(m+\delta)\,(\text{etc.}}{(m-\mu)(m+\alpha)(m+\beta)(m+\gamma)(m+\delta)\,(\text{etc.}},$$

cujus valor semper erit evanescens, si modo haec series

$$\frac{1}{\mu+\alpha} + \frac{1}{\mu+\beta} + \frac{1}{\mu+\gamma} + \frac{1}{\mu+\delta} + \text{etc.}$$

habuerit summam infinite magnam, uti modo ante ostendimus.

Analysis singularis directe ad seriem supra inventam perducens.

§. 21. Ponamus

$$x^m (1-x^n)^\theta = A\int x^{m-1}\partial x\,(1-x^n)^\theta + B\int x^{m-1}\partial x\,(1-x^n)^{\theta-1},$$

et reperietur $A = m + \theta n$ et $B = -\theta n$; hinc ergo si ponamus

$$\int x^{m-1}\partial x\,(1-x^n)^\theta = P \text{ et } \int x^{m-1}\partial x\,(1-x^n)^{\theta-1} = Q,$$

erit

$$x^m (1 - x^n)^\theta = (m + \theta n) P - \theta n Q, \text{ ideoque}$$

$$Q = \frac{m+\theta n}{\theta n} . P - \frac{1}{\theta n} x^m (1 - x^n)^\theta.$$

Quodsi jam ambo integralia P et Q a termino $x = 0$ usque ad $x = 1$ extendamus, erit $Q = \frac{m+\theta n}{\theta n} \times P$; si modo fuerit tam $m > 0$ quam $\theta > 0$.

§. 22. Cum jam sit $\partial Q = \frac{\partial P}{1 - x^n}$, denominatore in seriem evoluto erit

$$\partial Q = \partial P (1 + x^n + x^{2n} + x^{3n} + x^{4n} + \text{etc.}):$$

consequenter habebitur

$$Q = P + \int x^n \partial P + \int x^{2n} \partial P + \int x^{3n} \partial P + \text{etc.}$$

quae singula integralia ita sunt comparata, ut quodlibet ad praecedens reduci possit, ope hujus reductionis

$$x^\alpha (1 - x^n)^{\theta + 1} = A \int x^{\alpha + n - 1} \partial x (1 - x^n)^\theta + B \int x^{\alpha - 1} \partial x (1 - x^n)^\theta,$$

pro qua reperitur $A = -\alpha - n(\theta + 1)$ et $B = \alpha$.

§. 23. Si etiam haec duo integralia a termino $x = 0$ usque ad $x = 1$ extendantur, fiet

$$0 = -[\alpha + n(\theta + 1)] \int x^{\alpha + n - 1} \partial x (1 - x^n)^\theta + \alpha \int x^{\alpha - 1} \partial x (1 - x^n)^\theta,$$

si modo fuerit $\alpha > 0$ et $\theta + 1 > 0$. Faciamus nunc $\alpha = m + \lambda n$, et quia ante posueramus $x^{m-1} \partial x (1 - x^n)^\theta = \partial P$, haec aequatio abibit in hanc formam

$$-[\alpha + n(\theta + 1)] \int x^{(\lambda + 1) n} \partial P + \alpha \int x^{\lambda n} \partial P,$$

quocirca habebimus hanc reductionem

$$\int x^{\lambda n + n} \partial P = \frac{\alpha}{\alpha + n(\theta + 1)} \int x^{\lambda n} \partial P = \frac{m + \alpha n}{m + n(\lambda + \theta + 1)} \int x^{\lambda n} \partial P.$$

§. 24. Haec formula generalis nobis jam suppeditat sequentes integrationes speciales

$$
\begin{array}{l|l}
1^\circ)\ \text{Si}\ \lambda=0 & \int x^n \partial P = \frac{m}{m+n(\theta+1)} P, \\
2^\circ)\ \text{Si}\ \lambda=1 & \int x^{2n} \partial P = \frac{m+n}{m+n(\theta+2)} \int x^n \partial P, \ \text{ideoque} \\
 & \int x^{2n} \partial P = \frac{m}{m+n(\theta+1)} \cdot \frac{m+n}{m+n(\theta+2)} P, \\
3^\circ)\ \text{Si}\ \lambda=3 & \int x^{3n} \partial P = \frac{m+2n}{m+n(\theta+3)} \int x^{2n} \partial P, \ \text{ideoque} \\
 & \int x^{3n} \partial P = \frac{m}{m+n(\theta+1)} \cdot \frac{m+n}{m+n(\theta+2)} \cdot \frac{m+2n}{m+n(\theta+3)} P, \\
4^\circ)\ \text{Si}\ \lambda=4 & \int x^{4n} \partial P = \frac{m}{m+n(\theta+1)} \cdot \frac{m+n}{m+n(\theta+2)} \cdot \frac{m+2n}{m+n(\theta+3)} \cdot \frac{m+3n}{m+n(\theta+4)} P. \\
\quad \text{etc.} & \quad \text{etc.}
\end{array}
$$

§. 25. Cum igitur ex superioribus fuisset

$$Q = \int \partial P (1 + x^n + x^{2n} + x^{3n} + x^{4n} + \text{etc.}),$$

si pro singulis terminis valores modo inventos substituamus, atque utrinque per P dividamus, nanciscemur hanc aequationem

$$\frac{Q}{P} = 1 + \frac{m}{m+n(\theta+1)} + \frac{m}{m+n(\theta+1)} \cdot \frac{m+n}{m+n(\theta+2)} + \frac{m}{m+n(\theta+1)} \cdot \frac{m+n}{m+n(\theta+2)} \cdot \frac{m+2n}{m+n(\theta+3)} + \text{etc.}$$

supra autem ostendimus esse $\frac{Q}{P} = \frac{m+\theta n}{\theta n}$, quae ergo fractio est summa istius seriei infinitae.

§. 26. Ut jam consensum hujus seriei cum supra inventa ostendamus, primo loco θ scribamus $\frac{\mu}{n}$, atque series nostra inventa hanc induet formam

$$\frac{m+\mu}{\mu} = \frac{m}{m+n+\mu} + \frac{m}{m+n+\mu} \cdot \frac{m+n}{m+2n+\mu} + \frac{m}{m+n+\mu} \cdot \frac{m+n}{m+2n+\mu} \cdot \frac{m+2n}{m+3n+\mu} + \text{etc.}$$

cujus veritas eodem modo quo ante fueram usus, demonstrari potest. Si enim a summa subtrahatur terminus primus relinquitur $\frac{m}{\mu}$. Subtracto hinc termino secundo remanet $\frac{m(m+n)}{\mu(m+n+\mu)}$. Hinc porro tertius terminus subtractus relinquit $\frac{m(m+n)(m+2n)}{\mu(m+n+\mu)(m+2n+\mu)}$ et ita porro, quae operatio si in infinitum continuetur, producti hujus resultantis valor est $= 0$. Tum vero evidens est, seriem

hanc inventam in eam ipsam quam supra dedimus transmutari, si hic loco m scribatur μ, at vero $m-\mu$ loco μ.

§. 27. Coronidis loco hic subjungam seriem multo generaliorem ejusdem generis, cujus summam pariter assignare licet, quam sequenti problemate sum complexurus.

Problema 1.

§. 28. *Proposita hac serie* $A+B\frac{\alpha}{a}+C\frac{\alpha\beta}{ab}+D\frac{\alpha\beta\gamma}{abc}+$ *etc. investigare conditiones, sub quibus ejus summam assignare liceat.*

Solutio.

Haec ergo series involvit ternas series: primam litterarum α, β, γ, δ, etc. quae numeratores seriei propositae constituunt; secundam litterarum a, b, c, d, etc. ex quibus denominatores formantur; tertiam litterarum A, B, C, D, etc. quae coëfficientes terminorum exhibent. Quemadmodum igitur ternae istae series comparatae esse debeant, ut seriei propositae summam per expressionem finitam atque adeo rationalem assignare liceat, hic investigabo.

§. 29. Statuamus hujus seriei summam esse $\frac{S}{t}$, atque eadem methodo utamur quam jam supra adhibuimus, scilicet ab hac summa primo subtrahamus primum terminum A et cum remaneat $\frac{S-At}{t}$, statuamus $S-At=\alpha$, ut habeamus $\frac{\alpha}{t}$; hinc subtrahamus secundum terminum $B\frac{\alpha}{a}$, et residuum erit $\frac{\alpha(a-Bt)}{t.a}$. Hic jam faciamus $a-Bt=\beta$, ut habeamus $\frac{\alpha\beta}{t.a}$; unde si subtrahatur tertius terminus $C\frac{\alpha\beta}{ab}$, residuum erit $\frac{\alpha\beta(b-Ct)}{t.ab}$. Fiat hic $b-Ct=\gamma$, ut habeamus $\frac{\alpha\beta\gamma}{tab}$, unde terminus quartus ablatus relinquit $\frac{\alpha\beta\gamma(c-Dt)}{t.abc}$. Fiat hic iterum $c-Dt=\delta$, ut habeamus $\frac{\alpha\beta\gamma\delta}{t.abc}$, unde quintum

terminum subtrahendo colligitur $\frac{\alpha\beta\gamma\delta(d - Et)}{t \cdot abcd}$. Haecque operationes in infinitum continuari intelligantur,

§. 30. Ex his igitur determinationibus tam littera S quam litterae *a*, *b*, *c*, *d*, etc. sequenti modo definientur

$$S = \alpha + At;\ a = \beta + Bt;\ b = \gamma + Ct;\ c = \delta + Dt;\ \text{etc.}$$

Atque his valoribus introductis residuum, postquam omnes seriei termini fuerint a formula $\frac{S}{t}$ ablati, remanebit hoc productum in infinitum excurrens $\frac{\alpha\beta\gamma\delta\varepsilon\zeta\ etc.}{t.\ abcdef\ etc.}$, quod ergo productum si in nihilum abeat, tum summa seriei propositae revera erit $= \frac{S}{t}$. Videamus igitur sub quibusnam conditionibus hoc productum evanescat.

§. 31. Designemus hoc productum littera Π, ut substitutis pro *a*, *b*, *c*, etc. valoribus inventis erit

$$\Pi = \frac{S}{t}\left(\frac{\alpha}{\alpha + At} \cdot \frac{\beta}{\beta + Bt} \cdot \frac{\gamma}{\gamma + Ct} \cdot \frac{\delta}{\delta + Dt} \cdot \text{etc.}\right)$$

ubi scilicet factorem $\frac{S}{t}$ praefiximus. Nunc igitur quaeritur sub quibusnam conditionibus istud productum in infinitum continuatum in nihilum sit abiturum. Evidens autem est hoc evenire, si productum istud invertatur, ejusque logarithmus eveniat infinite magnus. Hoc ergo locum inveniet, quando summa horum logarithmorum

$$l\left(1 + \frac{At}{\alpha}\right) + l\left(1 + \frac{Bt}{\beta}\right) + l\left(1 + \frac{Ct}{\gamma}\right) + l\left(1 + \frac{Dt}{\delta}\right) + \text{etc.} = \infty;$$

id quod semper continget, si sumtis tantum primis terminis, qui omnes factorem comunem habent t, series haec

$$\frac{A}{\alpha} + \frac{B}{\beta} + \frac{C}{\gamma} + \frac{D}{\delta} + \text{etc.}$$

habuerit summam infinite magnam, tum igitur nostrae seriei propositae summa semper erit $\frac{\alpha + At}{t}$.

§. 32. Neque vero absolute necesse est, ut productum Π penitus evanescat, sed quemcunque habuerit valorem scilicet Π, quoniam is oritur postquam tota summa seriei propositae, quam ponamus $= S$, ablata fuerit a formula $\frac{s}{t}$, ita ut sit $\Pi = \frac{s}{t} - S$, unde manifesto fit $S = \frac{s}{t} - \Pi$.

§. 33. Ut hoc exemplo illustramus, litteris $\alpha, \beta, \gamma, \delta$, etc. hos tribuemus valores $\alpha = 3$, $\beta = 15$, $\gamma = 35$, $\delta = 63$, etc. praeterea vero sit $t = 1$, atque insuper $A = B = C = D =$ etc. $= 1$: hinc ergo determinationes inventae praebebunt

$$s = 4, \; a = 16, \; b = 36, \; c = 64, \; d = 100, \text{ etc.}$$

Sicque series nostra jam erit

$$1 + \frac{3}{16} + \frac{3 \cdot 15}{16 \cdot 36} + \frac{3 \cdot 15 \cdot 35}{16 \cdot 36 \cdot 64} + \frac{3 \cdot 15 \cdot 35 \cdot 63}{16 \cdot 36 \cdot 64 \cdot 100} + \text{etc.}$$

pro cujus summa notetur esse $\Pi = 4 \cdot \frac{3 \cdot 15 \cdot 35 \cdot 63 \cdot 99}{4 \cdot 16 \cdot 36 \cdot 64 \cdot 100}$ etc. Constat autem ex quadratura circuli *Wallisiana* esse $\frac{3 \cdot 15 \cdot 35 \cdot 63 \cdot 99 \text{ etc}}{4 \cdot 16 \cdot 36 \cdot 64 \cdot 100 \text{ etc.}} = \frac{2}{\pi}$, existente π peripheria circuli cujus diameter est unitas. Hinc ergo erit $\Pi = \frac{8}{\pi}$, ideoque summa nostrae seriei $S = 4 - \frac{8}{\pi}$, ideoque proxime $\frac{16}{11}$.

§. 34. At vero series generalis, quam hoc modo sumus adepti, maxime est faecunda in formatione innumerabilium serierum specialium, cum non modo tam seriem litterarum $\alpha, \beta, \gamma, \delta$, etc. sed etiam litterarum A, B, C, D, etc. prorsus pro lubitu assumere liceat, quandoquidem inde litterae a, b, c, d, etc. sponte determinantur; tum autem talium serierum summam semper assignare licebit, si modo valor producti in infinitum excurrentis, quod littera Π indicavimus definiri poterit, ubi perinde est, utrum iste valor fuerit rationalis sive adeo transcendens quadraturam quamcunque involvens.

SUPPLEMENTUM III.

AD TOM. I. CAP. IV.

DE
INTEGRATIONE FORMULARUM LOGARITHMICARUM ET EXPONENTIALIUM.

1) Evolutio formulae integralis $\int x^{f-1}\partial x(lx)^{\frac{m}{n}}$, integratione a valore $x=0$ ad $x=1$ extensa. *Nov. Commentarii Acad. Imp. Sc. Petropolitanae. Tom. XVI. Pag.* 91 — 139.

Theorema 1.

§. 1. Si n denotat numerum integrum positivum quemcunque, et formulae $\int x^{f-1}\partial x(1-x^g)^n$ integratio a valore $x=0$ usque ad $x=1$ extendatur, erit ejus valor:

$$=\frac{g^n}{f}\cdot\frac{1.\quad 2.\quad 3.\quad\ldots\ldots\ldots\ldots n}{(f+g)(f+2g)(f+3g)\ldots\ldots(f+ng)}.$$

Demonstratio.

Notum est in genere, integrationem formulae

$$\int x^{f-1}\partial x(1-x^g)^m$$

reduci posse ad integrationem hujus $\int x^{f-1}\partial x(1-x^g)^{m-1}$, quoniam quantitates constantes A et B ita definire licet, ut fiat

$$\int x^{f-1}\partial x(1-x^g)^m = A\int x^{f-1}\partial x(1-x^g)^{m-1}+Bx^f(1-x^g)^m:$$

sumtis enim differentialibus prodit haec aequatio

$$x^{f-1}\partial x(1-x^g)^m = Ax^{f-1}\partial x(1-x^g)^{m-1} + Bfx^{f-1}\partial x(1-x^g)^m$$
$$- Bmgx^{f+g-1}\partial x(1-x^g)^{m-1},$$

quae per $x^{f-1}\partial x(1-x^g)^{m-1}$ divisa dat

$$1 - x^g = A + Bf(1-x^g) - Bmgx^g, \quad \text{seu}$$
$$1 - x^g = A - Bmg + B(f+mg)(1-x^g),$$

quae aequatio ut consistere possit, necesse est sit

$$1 = B(f+mg) \quad \text{et} \quad A = Bmg;$$

unde colligimus

$$B = \frac{1}{f+mg} \quad \text{et} \quad A = \frac{mg}{f+mg}.$$

Quocirca habebimus sequentem reductionem generalem

$$\int x^{f-1}\partial x(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}\partial x(1-x^g)^{m-1} + \frac{1}{f+mg}\cdot x^f(1-x^g)^m$$

quae cum evanescat posito $x = 0$, siquidem sit $f > 0$, constantis additione haud est opus. Quare extenso utroque integrali usque ad $x = 1$, pars integralis postrema sponte evanescit, eritque pro casu $x = 1$

$$\int x^{f-1}\partial x(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}\partial x(1-x^g)^{m-1}.$$

Cum igitur sumto $m = 1$ sit $\int x^{f-1}\partial x(1-x^g)^0 = \frac{1}{f}x^f = \frac{1}{f}$, posito $x = 1$, nanciscimur pro eodem casu $x = 1$ sequentes valores

$$\int x^{f-1}\partial x(1-x^g)^1 = \frac{g}{f}\cdot\frac{1}{f+g}$$
$$\int x^{f-1}\partial x(1-x^g)^2 = \frac{g^2}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g}$$
$$\int x^{f-1}\partial x(1-x^g)^3 = \frac{g^3}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g}\cdot\frac{3}{f+3g}$$

etc.

hincque pro numero quocunque integro positivo n concludimus fore

$$\int x^{f-1}\partial x(1-x^g)^n = \frac{g^n}{f}\cdot\frac{1}{f+g}\cdot\frac{2}{f+2g}\cdot\frac{3}{f+3g}\cdots\cdots\frac{n}{f+ng}$$

si modo numeri f et g sint positivi.

Corollarium I.

§. 2. Hinc ergo vicissim valor hujusmodi producti ex quotcunque factoribus formati, per formulam integralem exprimi potest, ita ut sit

$$\frac{1.}{(f+g)}\frac{2.}{(f+2g)}\frac{3.\ldots\ldots n}{(f+3g)\ldots(f+ng)} = \frac{f}{g^n}\int x^{f-1}\partial x\,(1-x^g)^n,$$

integrali hoc a valore $x=0$ usque ad $x=1$ extenso.

Corollarium 2.

§. 3. Quodsi ergo hujusmodi habeatur progressio

$$\frac{1}{f+g};\ \frac{1.}{(f+g)}\frac{2}{(f+2g)};\ \frac{1.}{(f+g)}\frac{2.}{(f+2g)}\frac{3}{(f+3g)};\ \frac{1.}{(f+g)}\frac{2.}{(f+2g)}\frac{3.}{(f+3g)}\frac{4.}{(f+4g)};\ \text{etc.}$$

ejus terminus generalis qui indici indefinito n convenit, commode hac forma integrali $\frac{f}{g^n}\int x^{f-1}\partial x\,(1-x^g)^n$ repraesentatur, cujus ope ea progressio interpolari, ejusque termini indicibus fractis respondentes exhiberi poterunt.

Corollarium 3.

§. 4. Si loco n scribamus $n-1$, habebimus

$$\frac{1.}{(f+g)}\frac{2.}{(f+2g)}\frac{3.\ldots\ldots(n-1)}{(f+3g)\ldots[f+(n-1)g]} = \frac{f}{g^{n-1}}\int x^{f-1}\partial x\,(1-x^g)^{n-1},$$

quae per $\frac{n}{f+ng}$ multiplicata praebet

$$\frac{1.}{(f+g)}\frac{2.}{(f+2g)}\frac{3.\ldots\ldots n}{(f+3g)\ldots(f+ng)} = \frac{f.ng}{g^n(f+ng)}\int x^{f-1}\partial x\,(1-x^g)^{n-1}.$$

Scholion 1.

§. 5. Hanc posteriorem formam immediate ex praecedente derivare licuisset, cum modo demonstraverimus esse

$$\int x^{f-1}\partial x\,(1-x^g)^n = \frac{ng}{f+ng}\int x^{f-1}\partial x\,(1-x^g)^{n-1};$$

siquidem utrumque integrale a valore $x=0$ usque ad $x=1$ extendatur; quam integralium determinationem in sequentibus ubique subintelligi oportet. Deinde etiam perpetuo est tenendum, quantitates f et g esse positivas, quippe quam conditionem demonstratio allata absolute postulat. Quod autem ad numerum n attinet, quatenus eo index cujusque termini progressionis (§. 3.) designatur, nihil impedit, quominus eo numeri quicunque sive positivi sive negativi denotentur, quandoquidem ejus progressionis omnes termini etiam indicibus negativis respondentes per formulam integralem datam exhiberi censentur. Interim tamen probe tenendum est, hanc reductionem

$$\int x^{f-1}\partial x\,(1-x^g)^m = \frac{mg}{f+mg}\int x^{f-1}\partial x\,(1-x^g)^{m-1}$$

non esse veritati consentaneam, nisi sit $m>0$; quia alioquin pars algebraica $\frac{1}{f+mg}x^f(1-x^g)^m$ non evanesceret posito $x=1$.

Scholion 2.

§. 6. Hujusmodi series, quas transcendentes appellare licet, quia termini indicibus fractis respondentes sunt quantitates transcendentes, jam olim in Comment. Petrop. Tomo V. (*) fusius sum prosecutus; unde hoc loco non tam istas progressiones, quam eximias formularum integralium comparationes, quae inde derivantur, diligentius sum scrutaturus. Cum scilicet ostendissem, hujus producti indefiniti 1. 2. 3. . . . n valorem hac formula integrali $\int\partial x\,(l\frac{1}{x})^n$ ab $x=0$ ad $x=1$ extensa exprimi, quae res quoties n est numerus integer positivus per ipsam integrationem est manifesta, eos casus examini subjeci, quibus pro n numeri fracti accipiuntur; ubi quidem ex ipsa formula integrali neutiquam patet, ad quodnam genus quantitatum transcendentium hi termini referri debeant. Singulari autem

(*) Institut. Calc. integralis Tom. I. Sect. I. Cap. IV.

artificio eosdem terminos ad quadraturas magis cognitas reduxi, quod propterea maxime dignum videtur, ut majori studio perpendatur.

Problema 1.

§. 7. *Cum demonstratum sit esse*

$$\frac{1. \quad 2. \quad 3 \ldots\ldots n}{(f+g)\,(f+2g)\,(f+3g)\ldots\,(f+ng)} = \frac{f}{g^n}\int x^{f-1}\partial x\,(1-x^g)^n$$

integrali ab $x=0$ ad $x=1$ extenso, ejusdem producti casu quo $g=0$ valorem per formulam integralem assignare.

Solutio.

Posito $g=0$ in formula integrali membrum $(1-x^g)^n$ evanescit, simul vero etiam denominator g^n, unde quaestio huc redit, ut fractionis $\frac{(1-x^g)^n}{g^n}$ valor definiatur casu $g=0$, quo tam numerator quam denominator evanescit. Hunc in finem spectetur g ut quantitas infinite parva, et cum sit $x^g = e^{g\,l\,x}$, fiet $x^g = 1+g\,l\,x$, ideoque

$$(1-x^g)^n = g^n(-l\,x)^n = g^n\,(l\tfrac{1}{x})^n;$$

ex quo pro hoc casu formula nostra integralis abit in

$$f\int x^{f-1}\partial x\,(l\tfrac{1}{x})^n;$$

ita ut jam hebeatur

$$\frac{1. \quad 2. \quad 3 \ldots\ldots n}{f^n} = f\int x^{f-1}\partial x\,(l\tfrac{1}{x})^n \text{ seu}$$

$$1. \quad 2. \quad 3 \ldots\ldots n = f^{n+1}\int x^{f-1}\partial x\,(l\tfrac{1}{x})^n.$$

Collarium 1.

§. 8. Quoties n est numerus integer positivus, integratio formulae $\int x^{f-1}\partial x\,(l\frac{1}{x})^n$ succedit, eaque ab $x=0$ ad $x=1$ extensa revera prodit

id productum, cui istam formulam aequalem invenimus. Sin autem pro n capiantur numeri fracti, eadem formula integralis inserviet huic progressioni hypergeometricae interpolandae

1; 1. 2; 1. 2. 3; 1. 2. 3. 4; 1. 2. 3. 4. 5; 1. 2. 3. 4. 5. 6; etc. seu
1; 2; 24; 120; 720; etc.

Corollarium 2.

§. 9. Si expressio modo inventa per principalem dividatur, orietur productum, cujus factores in progressione arithmetica quacunque progrediuntur.

$$(f+g)(f+2g)(f+3g)\ldots(f+ng)=f^n g^n \cdot \frac{\int x^{f-1}\partial x\,(l\frac{1}{x})^n}{\int x^{f-1}\partial x\,(1-x^g)^n},$$

cujus ergo etiam valores, si n sit numerus fractus, hinc assignare licebit.

Corollarium 3.

§. 10. Cum sit

$$\int x^{f-1}\partial x\,(1-x^g)^n=\frac{ng}{f+ng}\int x^{f-1}\partial x\,(1-x^g)^{n-1},$$

erit etiam simili modo pro casu $g=0$

$$\int x^{f-1}\partial x\,(l\tfrac{1}{x})^n=\frac{n}{f}\int x^{f-1}\partial x\,(l\tfrac{1}{x})^{n-1},$$

hincque per istas alteras formulas integrales

$$1.\ 2.\ 3\ldots\ldots n=nf^n\int x^{f-1}\partial x\,(l\tfrac{1}{x})^{n-1} \text{ et}$$

$$(f+g)(f+2g)\ldots(f+ng)=f^{n-1}g^{n-1}(f+ng)\cdot\frac{\int x^{f-1}\partial x\,(l\frac{1}{x})^{n-1}}{\int x^{f-1}\partial x\,(1-x^g)^{n-1}}.$$

Scholion.

§. 11. Cum invenerimus esse

$$1.\ 2.\ 3\ldots\ldots n=f^{n+1}\int x^{f-1}\partial x\,(l\tfrac{1}{x})^n,$$

patet hanc formulam integralem non a valore quantitatis f pendere, quod etiam facile perspicitur ponendo $x^f = y$, unde fit $f x^{f-1} \partial x = \partial y$, et $l\frac{1}{x} = -lx = -\frac{1}{f} l y = \frac{1}{f} l \frac{1}{y}$, ideoque $f^n (l\frac{1}{x})^n = (l\frac{1}{y})^n$, ita ut sit

$$1 \cdot 2 \cdot 3 \ldots\ldots n = \int \partial y \, (l\tfrac{1}{y})^n,$$

quae formula ex priori nascitur ponendo $f = 1$. Pro interpolatione ergo hujusmodi formarum totum negotium huc reducitur, ut istius formulae integralis $\int \partial x \, (l\frac{1}{x})^n$ valores difiniantur, quando exponens n est numerus fractus. Veluti si n sit $= \frac{1}{2}$, assignari oportet valorem hujus formulae $\int \partial x \, \sqrt{l\frac{1}{x}}$, quem olim jam ostendi esse $= \frac{1}{2}\sqrt{\pi}$, denotante π circuli peripheriam cujus diameter $= 1$: pro aliis autem numeris fractis cujus valorem ad quadraturas curvarum algebraicarum altioris ordinis revocare docui. Quae reductio cum minime sit obvia, atque tum solum locum habeat, quando formulae $\int \partial x \, (l\frac{1}{x})^n$ integratio a valore $x = 0$ ad $x = 1$ extenditur, singulari attentione digna videtur. Etsi autem jam olim hoc argumentum tractavi, tamen quia per plures ambages eo sum perductus, idem hic resumere et concinnius evolvere constitui.

Theorema 2.

§. 12. Si formulae integrales a valore $x = 0$ usque ad $x = 1$ extendantur, et n denotet numerum integrum positivum, erit

$$\frac{1 \cdot 2 \cdot 3 \ldots\ldots n}{(n+1)(n+2)(n+3)\ldots 2n} = \tfrac{1}{2} n g \int x^{f+ng-1} \partial x (1-x^g)^{n-1} \times \frac{\int x^{f-1} \partial x \, (1-x^g)^{n-1}}{\int x^{f-1} \partial x (1-x^g)^{2n-1}}$$

quicunque numeri positivi loco f et g accipiantur.

Demonstratio.

Cum supra (§. 4.) ostenderimus esse

$$\frac{1.\quad 2.\quad 3\ldots\ldots n}{(f+g)(f+2g)\ldots(f+ng)} = \frac{f.ng}{g^n(f+ng)}\int x^{f-1}\partial x(1-x^g)^{n-1},$$

habebimus, si loco n scribamus $2n$,

$$\frac{1.\quad 2.\quad 3\ldots\ldots 2n}{(f+g)(f+2g)\ldots(f+2ng)} = \frac{f.2ng}{g^{2n}(f+2ng)}\int x^{f-1}\partial x(1-x^g)^{2n-1}.$$

Dividatur nunc prima aequatio per secundam, ac prodibit ista tertia

$$\frac{[f+(n+1)g][f+(n+2)g]\ldots(f+2ng)}{(n+1)(n+2)\ldots\ldots 2n} = \frac{g^n(f+2ng)}{2(f+ng)}\cdot\frac{\int x^{f-1}\partial x(1-x^g)^{n-1}}{\int x^{f-1}\partial x(1-x^g)^{2n-1}}.$$

At si in prima aequatione loco f scribatur $f+ng$, orietur haec aequatio quarta

$$\frac{1.\quad 2.\quad 3\ldots\ldots\ldots\ldots n}{[f+(n+1)g][f+(n+2)g]\ldots(f+2ng)} = \frac{(f+ng)ng}{g^n(f+2ng)}\int x^{f+ng-1}\partial x(1-x^g)^{n-1}.$$

Multiplicetur haec quarta aequatio per illam tertiam, ac reperietur ipsa aequatio demonstranda:

$$\frac{1.\quad 2.\quad 3\ldots\ldots n}{(n+1)(n+2)(n+3)\ldots 2n} = \tfrac{1}{2}ng\int x^{f+ng-1}\partial x(1-x^g)^{n-1}\times\frac{\int x^{f-1}\partial x(1-x^g)^{n-1}}{\int x^{f-1}\partial x(1-x^g)^{2n-1}}.$$

Corollarium 1.

§. 13. Si in prima aequatione statuatur $f=n$ et $g=1$, orietur idem productum

$$\frac{1.\quad 2.\quad\ldots\ldots n}{(n+1)(n+2)\ldots\ldots 2n} = \tfrac{1}{2}n\int x^{n-1}\partial x(1-x)^{n-1},$$

qua aequatione cum illa collata adipiscimur

$$\frac{\int x^{n-1}\partial x(1-x)^{n-1}}{g\int x^{f+ng-1}\partial x(1-x^g)^{n-1}} = \frac{\int x^{f-1}\partial x(1-x^g)^{n-1}}{\int x^{f-1}\partial x(1-x^g)^{2n-1}}.$$

Corollarium 2.

§. 14. Si in illa aequatione loco x scribamus x^g, fiet

$$\frac{1.\ 2.\ 3 \ldots\ldots n}{(n+1)(n+2)\ldots 2n} = \tfrac{1}{2} n g \int x^{ng-1} \partial x (1-x^g)^{n-1};$$

ita ut jam consequamur istam comparationem inter sequentes formulas integrales

$$\int x^{ng-1} \partial x (1-x^g)^{n-1} = \int x^{f+ng-1} \partial x (1-x^g)^{n-1} \times \frac{\int x^{f-1} \partial x (1-x^g)^{n-1}}{\int x^{f-1} \partial x (1-x^g)^{2n-1}}.$$

Corollarium 3.

§. 15. Si in aequatione theorematis ponamus $g=0$, ob $(1-x^g)^m = g^m (l\frac{1}{x})^m$, potestates ipsius g se destruent, orieturque haec aequatio

$$\frac{1.\ 2.\ 3 \ldots\ldots n}{(n+1)(n+2)\ldots 2n} = \tfrac{1}{2} n \int x^{f-1} \partial x (l\tfrac{1}{x})^{n-1} \times \frac{\int x^{f-1} \partial x (l\frac{1}{x})^{n-1}}{\int x^{f-1} \partial x (l\frac{1}{x})^{2n-1}};$$

unde colligimus

$$\frac{[\int x^{f-1} \partial x (l\frac{1}{x})^{n-1}]^2}{\int x^{f-1} \partial x (l\frac{1}{x})^{2n-1}} = g \int x^{ng-1} \partial x (1-x^g)^{n-1},$$

seu ob

$$\int x^{f-1} \partial x (l\tfrac{1}{x})^{n-1} = \frac{f}{n} \int x^{f-1} \partial x (l\tfrac{1}{x})^n, \text{ hanc}$$

$$\frac{2f}{n} \cdot \frac{[\int x^{f-1} \partial x (l\frac{1}{x})^n]^2}{\int x^{f-1} \partial x (l\frac{1}{x})^{2n}} = g \int x^{ng-1} \partial x (1-x^g)^{n-1}.$$

Corollarium 4.

§. 16. Ponamus hic $f=1$, $g=2$ et $n=\frac{m}{2}$, ut m sit numerus integer positivus, et ob $\int \partial x (l\frac{1}{x})^m = 1.\ 2.\ 3 \ldots\ldots m$, erit

$$\frac{4}{m} \cdot \frac{[\int \partial x (l\frac{1}{x})^{\frac{m}{2}}]^2}{1.\ 2.\ 3 \ldots m} = 2 \int x^{m-1} \partial x (1-x^2)^{\frac{m}{2}-1},$$

hincque

$$\int \partial x (l\tfrac{1}{x})^{\frac{m}{2}} = \sqrt{1.\ 2.\ 3 \ldots m . \tfrac{m}{2} \int x^{m-1} \partial x (1-x^2)^{\frac{m}{2}-1}},$$

et sumendo $m = 1$, ob $\int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2}$ habebitur

$$\int \partial x \sqrt{l\tfrac{1}{x}} = \sqrt{\tfrac{1}{2} \int \frac{\partial x}{\sqrt{(1-xx)}}} = \tfrac{1}{2} \sqrt{\pi}.$$

Scholion.

§. 17. En ergo succinctam demonstrationem theorematis olim a me prolati, quod sit $\int \partial x \sqrt{l\frac{1}{x}} = \frac{1}{2}\sqrt{\pi}$, eamque ab interpolationis ratione, qua tum usus fueram, libera. Deducta scilicet hic ea ex hoc theoremate, quo inveni esse

$$\frac{[\int x^{f-1} \partial x (l\frac{1}{x})^{n-1}]^2}{\int x^{f-1} \partial x (l\frac{1}{x})^{2n-1}} = g \int x^{ng-1} \partial x (1-x^g)^{n-1}.$$

Principale autem theorema, unde hoc est deductum ita se habet

$$g \cdot \frac{\int x^{f-1} \partial x (1-x^g)^{n-1} \times \int x^{f+ng-1} \partial x (1-x^g)^{n-1}}{\int x^{f-1} \partial x (1-x^g)^{2n-1}} = \int x^{n-1} \partial x (1-x)^{n-1};$$

utrumque enim membrum per intergrationem ab $x = 0$ ad $x = 1$ extensam evolvitur in hoc productum numericum

$$\frac{1.\quad 2.\quad 3 \ldots\ldots (n-1)}{(n+1)(n+2)\ldots\ldots(2n-1)}.$$

Ac si alteri membro speciem latius patentem tribuere velimus, theorema ita proponi poterit ut sit

$$g \cdot \frac{\int x^{f-1} \partial x (1-x^g)^{n-1} \times \int x^{f+ng-1} \partial x (1-x^g)^{n-1}}{\int x^{f-1} \partial x (1-x^g)^{2n-1}} = k \int x^{nk-1} \partial x (1-x^k)^{n-1},$$

hicque si capiatur $g=0$, fit

$$\frac{[\int x^{f-1}\partial x\,(l\frac{1}{x})^{n-1}]^2}{\int x^{f-1}\partial x\,(l\frac{1}{x})^{2n-1}}=k\int x^{nk-1}\partial x\,(1-x^k)^{n-1}.$$

Imprimis igitur notandum est, quod illa aequalitas subsistat, quicunque numeri loco f et g accipiantur: casu quidem $f=g$, ea est manifesta, cum sit

$$\int x^{g-1}\partial x\,(1-x^g)^{n-1}=\frac{1-(1-x^g)^n}{ng}=\frac{1}{ng},$$

fiet enim

$$2g\int x^{ng+g-1}\partial x\,(1-x^g)^{n-1}=k\int x^{nk-1}\partial x\,(1-x^k)^{n-1},$$

et quia

$$\int x^{ng+g-1}\partial x\,(1-x^g)^{n-1}=\tfrac{1}{2}\int x^{ng-1}\partial x\,(1-x^g)^{n-1},$$

aequalitas est perspicua, quia k pro lubitu accipere licet. Eodem autem modo, quo ad hoc theorema perveni, ad alia similia pertingere licet.

Theorema 3.

§. 18. Si sequentes formulae integrales a valore $x=0$ ad $x=1$ extendantur, et n denotet numerum integrum positivum quemcunque, erit

$$\frac{1\,.\,2\,.\,3\,\ldots\ldots\,n}{(2n+1)\,(2n+2)\ldots3n}=\tfrac{3}{2}ng\int x^{f+2ng-1}\partial x\,(1-x^g)^{n-1}\times\frac{\int x^{f-1}\partial x\,(1-x^g)^{2n-1}}{\int x^{f-1}\partial x\,(1-x^g)^{3n-1}},$$

quicunque numeri positivi pro f et g accipiantur.

Demonstratio.

In praecedente theoremate jam vidimus esse

$$\frac{1. \quad 2. \quad 3 \ldots\ldots 2n}{(f+g)(f+2g)\ldots(f+2ng)g} = \frac{f.2ng}{g^{2n}(f+2ng)}\int x^{f-1}\partial x(1-x^g)^{2n-1}:$$

simili autem modo, si in forma principali loco n scribamus $3n$ habebimus

$$\frac{1. \quad 2. \quad 3 \ldots\ldots 3n}{(f+g)(f+2g)\ldots(f+3ng)} = \frac{f.3ng}{g^{3n}(f+3ng)}\int x^{f-1}\partial x(1-x^g)^{3n-1},$$

ex quo illa aequatio per hanc divisa producit

$$\frac{[f+(2n+1)g]\,[f+(2n+2)g]\ldots\ldots(f+3ng)}{(2n+1)(2n+2)\;\ldots\ldots\ldots\ldots\;3n} = \frac{2g^n(f+3ng)}{3(f+2ng)}\times$$
$$\frac{\int x^{f-1}\partial x(1-x^g)^{2n-1}}{\int x^{f-1}\partial x(1-x^g)^{3n-1}}.$$

Verum si in aequatione principali (§. 4.) loco f scribamus $f+2ng$, adipiscimur hanc aequationem

$$\frac{1. \quad 2. \quad 3. \ldots\ldots\ldots\ldots\ldots\ldots \quad n}{[f+(2n+1)g]\,[f+(2n+2)g]\ldots\ldots(f+3ng)} = \frac{(f+2ng).ng}{g^n(f+3ng)}\times$$
$$\int x^{f+2ng-1}\partial x(1-x^g)^{n-1}.$$

Multiplicetur nunc haec aequatio per praecedentem, et orietur ipsa aequatio, quam demonstrari oportet

$$\frac{1. \quad 2. \quad 3. \ldots\ldots n}{(2n+1)(2n+2)\ldots\ldots 3n} = \tfrac{2}{3}ng.\int x^{f+2ng-1}\partial x(1-x^g)^{n-1}\times$$
$$\frac{\int x^{f-1}\partial x(1-x^g)^{2n-1}}{\int x^{f-1}\partial x(1-x^g)^{3n-1}}.$$

Corollarium 1.

§. 19. Eundem valorem ex aequatione principali nanciscimur, ponendo $f=2n$ et $g=1$, ita ut sit

$$\frac{1. \quad 2 \quad 3. \ldots\ldots n}{(2n+1)(2n+2)\ldots 3n} = \tfrac{2}{3}n\int x^{2n-1}\partial x(1-x)^{n-1},$$

quae formula integralis, loco x scribendo x^k, transformatur in hanc

$$\tfrac{2}{3} n k \int x^{2nk-1} \partial x (1-x^k)^{n-1},$$

ita ut sit

$$g \int x^{f+2ng-1} \partial x (1-x^g)^{n-1} \times \frac{\int x^{f-1} \partial x (1-x^g)^{2n-1}}{\int x^{f-1} \partial x (1-x^g)^{3n-1}}$$
$$= k \int x^{2nk-1} \partial x (1-x^k)^{n-1}.$$

Corollarium 2.

§. 20. Si hic statuamus $g = 0$, ob $1 - x^g = g\, l \frac{1}{x}$ habebimus hanc aequationem

$$\int x^{f-1} \partial x (l \tfrac{1}{x})^{n-1} \times \frac{\int x^{f-1} \partial x (l \frac{1}{x})^{2n-1}}{\int x^{f-1} \partial x (l \frac{1}{x})^{3n-1}} = k \int x^{2nk-1} \partial x (1-x^k)^{n-1}:$$

cum igitur ante invenissemus

$$\frac{[\int x^{f-1} \partial x (l \frac{1}{x})^{n-1}]^2}{\int x^{f-1} \partial x (l \frac{1}{x})^{2n-1}} = k \int x^{nk-1} \partial x (1-x^k)^{n-1},$$

habebimus has aequationes in se multiplicando

$$\frac{[\int x^{f-1} \partial x (l \frac{1}{x})^{n-1}]^3}{\int x^{f-1} \partial x (l \frac{1}{x})^{3n-1}} = k^2 \int x^{nk-1} \partial x (1-x^k)^{n-1} \times \int x^{2nk-1} \partial x (1-x^k)^{n-1}.$$

Corollarium 3.

§. 21. Sine ulla restrictione hic ponere licet $f = 1$; tum ergo sumto $n = \frac{1}{3}$ et $k = 3$, erit

$$\frac{[\int \partial x (l \frac{1}{x})^{-\frac{2}{3}}]^3}{\int \partial x (l \frac{1}{x})^0} = 9 \int \partial x (1-x^3)^{-\frac{2}{3}} \times \int x \partial x (1-x^3)^{-\frac{2}{3}},$$

et ob

$$\int \partial x (l \tfrac{1}{x})^{-\frac{2}{3}} = 3 \int \partial x (l \tfrac{1}{x})^{\frac{1}{3}} \text{ et } \int \partial x (l \tfrac{1}{x})^0 = 1, \text{ obtinebimus}$$

$$[\int \partial x (l\tfrac{1}{x})^{\frac{1}{3}}]^3 = \int \partial x (1-x^3)^{-\frac{2}{3}} \times \int x \partial x (1-x^3)^{-\frac{2}{3}}$$

tum vero sumto $n = \frac{2}{3}$ et $k = 3$, erit

$$\frac{[\int \partial x (l\frac{1}{x})^{-\frac{1}{3}}]^3}{\int \partial x (l\frac{1}{x})} = 9 \int x \partial x (1-x^3)^{-\frac{1}{3}} \times \int x^3 \partial x (1-x^3)^{-\frac{1}{3}}$$

seu

$$[\int \partial x (l\tfrac{1}{x})^{\frac{2}{3}}]^3 = 4 \int x \partial x (1-x^3)^{-\frac{1}{3}} \times \int x^3 \partial x (1-x^3)^{-\frac{1}{3}}.$$

Theorema generale.

§. 22. Si sequentes formulae integrales a valore $x = 0$ usque ad $x = 1$ extendantur, et n denotet numerum integrum positivum quemcunque, erit

$$\frac{1. \quad 2. \quad 3. \ldots\ldots n}{(\lambda n+1)(\lambda n+2)\ldots(\lambda+1)n} = \frac{n}{\lambda+1} ng \int x^{f+\lambda ng-1} \partial x (1-x^g)^{n-1} \times$$
$$\frac{\int x^{f-1} \partial x (1-x^g)^{\lambda n-1}}{\int x^{f-1} \partial x (1-x^g)^{(\lambda+1)n-1}},$$

quicunque numeri positivi pro litteris f et g accipiantur.

Demonstratio.

Cum sit uti supra ostendimus

$$\frac{1. \quad 2. \ldots\ldots\ldots n}{(f+g)(f+2g)\ldots(f+ng)} = \frac{f.ng}{g^n(f+ng)} \int x^{f-1} \partial x (1-x^g)^{n-1},$$

si hic loco n scribamus primo λn, tum vero $(\lambda+1)n$, nanciscemur has duas aequationes

$$\frac{1. \quad 2. \ldots\ldots\ldots \lambda n}{(f+g)(f+2g)\ldots(f+\lambda ng)} = \frac{f \cdot \lambda ng}{g^{\lambda n}(f+\lambda ng)} \int x^{f-1} \partial x (1-x^g)^{\lambda n-1} \text{ et}$$

$$\frac{1.\quad 2.\ \ldots\ldots\ldots\ (\lambda+1)n}{(f+g)(f+2g)\ \ldots\ [f+(\lambda+1)ng]} = \frac{f\,.\,(\lambda+1)ng}{g^{(\lambda+1)n}[f+(\lambda+1)ng]} \times$$
$$\int x^{f-1}\partial x(1-x^g)^{(\lambda+1)n-1},$$

quarum illa per hanc divisa praebet

$$\frac{(f+\lambda ng+g)(f+\lambda ng+2g)\ \ldots\ldots\ (f+\lambda ng+ng)}{(\lambda n+1)(\lambda n+2)\ \ldots\ldots\ldots\ (\lambda n+n)}$$
$$= g^n \frac{\lambda(f+\lambda ng+ng)}{(\lambda+1)(f+\lambda ng)} \cdot \frac{\int x^{f-1}\partial x(1-x^g)^{\lambda n-1}}{\int x^{f-1}\partial x(1-x^g)^{(\lambda+1)n-1}}.$$

At si in aequatione prima loco f scribamus $f+\lambda ng$, obtinebus

$$\frac{1.\quad 2.\ \ldots\ldots\ldots\ h}{(f+\lambda ng+g)(f+\lambda ng+2g)\ \ldots\ (f+\lambda ng+ng)}$$
$$= \frac{(f+\lambda ng)ng}{g^n(f+\lambda ng+ng)} \cdot \int x^{f+\lambda ng-1}\partial x(1-x^g)^{n-1},$$

quae duae aequationes in se ductae producunt ipsam aequalitatem demonstrandam

$$\frac{1.\quad 2.\ \ldots\ldots\ n}{(\lambda n+1)(\lambda n+2)\ \ldots\ (\lambda n+n)} = \frac{\lambda ng}{\lambda+1}\int x^{f+\lambda ng-1}\partial x(1-x^g)^{n-1} \times$$
$$\frac{\int x^{f-1}\partial x(1-x^g)^{\lambda n-1}}{\int x^{f-1}\partial x(1-x^g)^{(\lambda+1)n-1}}.$$

Corollarium 1.

§. 23. Si in aequatione principali statuamus $f=\lambda n$ et $g=1$, reperiemus etiam

$$\frac{1.\quad 2.\ \ldots\ldots\ n}{(\lambda n+1)(\lambda n+2)\ \ldots\ (\lambda n+n)} = \frac{\lambda n}{\lambda+1}\int x^{\lambda n-1}\partial x(1-x)^{n-1},$$

quae forma loco x scribendo x^k abit in hanc

$$\frac{\lambda nk}{\lambda+1}\int x^{\lambda nk-1}\partial x(1-x^k)^{n-1};$$

ita ut habeamus hoc theorema latissime patens

$$g\int x^{f+\lambda ng-1}\partial x(1-x^g)^{n-1}\times\frac{\int x^{f-1}\partial x(1-x^g)^{\lambda n-1}}{\int x^{f-1}\partial x(1-x^g)^{\lambda n+n-1}}$$
$$=k\int x^{\lambda nk-1}\partial x(1-x^k)^{n-1}$$

Corollarium 2.

§. 24. Hoc jam theorema locum habet, etiamsi n non fit numerus integer; quin etiam cum numerum λ pro lubitu accipere liceat, loco λn scribamus m, et perveniemus ad hoc theorema

$$\frac{\int x^{f-1}\partial x(1-x^g)^{m-1}}{\int x^{f-1}\partial x(1-x^g)^{m+n-1}}=\frac{k\int x^{mk-1}\partial x(1-x^k)^{n-1}}{g\int x^{f+mg-1}\partial x(1-x^g)^{n-1}}.$$

Corollarium 3.

§. 25. Si ponamus $g=0$, ob $1-x^g=g\,l\frac{1}{x}$, hoc theorema istam induet formam

$$\frac{\int x^{f-1}\partial x(l\frac{1}{x})^{m-1}}{\int x^{f-1}\partial x(l\frac{1}{x})^{m+n-1}}=\frac{k\int x^{mk-1}\partial x(1-x^k)^{n-1}}{\int x^{f-1}\partial x(l\frac{1}{x})^{n-1}},$$

quae commodius ita repraesentatur

$$\frac{\int x^{f-1}\partial x(l\frac{1}{x})^{n-1}\times\int x^{f-1}\partial x(l\frac{1}{x})^{m-1}}{\int x^{f-1}\partial x(l\frac{1}{x})^{m+n-1}}=k\int x^{mk-1}\partial x(1-x^k)^{n-1};$$

ubi evidens est numeros m et n inter se permutari posse.

Scholion.

§. 26. Duplicem ergo deteximus fontem, unde innumerabiles formularum integralium comparationes haurire licet; alter fons §. 24. patefactus complectitur hujusmodi formulas integrales

$$\int x^{p-1}\partial x(1-x^g)^{q-1},$$

quas jam ante aliquod tempus pertractavi in observationibus circa integralia formularum (*)

$$\int x^{p-1}\partial x(1-x^n)^{\frac{q}{n}-1}$$

a valore $x=0$ usque ad $x=1$ extensa, ubi ostendi primo litteras p et q inter se permutari posse, ut sit

$$\int x^{p-1}\partial x\,(1-x^n)^{\frac{q}{n}-1}=\int x^{q-1}\partial x(1-x^n)^{\frac{p}{n}-1},$$

tum vero etiam esse

$$\int\frac{x^{p-1}\partial x}{(1-x^n)^{\frac{p}{m}}}=\frac{\pi}{n\sin.\frac{p\pi}{n}}:$$

imprimis autem demonstravi esse

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}\times\int\frac{x^{p+q-1}\partial x}{\sqrt[n]{(1-x^n)^{n-r}}}=\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-r}}}\times\int\frac{x^{p+r-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}},$$

in qua aequatione comparatio in §. 24. inventa jam continetur; ita ut hinc nihil novi, quod non jam evolvi, deduci queat. Alterum igitur fontem §. 25. indicatum hic potissimum investigandum suscipio, ubi cum sive ulla restrictione sumi queat $f=1$, aequasio nostra primaria erit

$$\frac{\int\partial x\,(l\frac{1}{x})^{n-1}\times\int\partial x\,(l\frac{1}{x})^{m-1}}{\int\partial x\,(l\frac{1}{x})^{m+n-1}}=k\int x^{mk-1}\partial x\,(1-x^k)^{n-1},$$

cujus beneficio valores formulae integralis $\int\partial x\,(l\frac{1}{x})^{\lambda}$, quando λ non est numerus integer, ad quadraturas curvarum algebraicarum revocare licebit; quandoquidem quoties λ est numerus integer, integratio habetur absoluta quoniam est

$$\int\partial x\,(l\tfrac{1}{x})^{\lambda}=1.\ 2.\ 3.\ \ldots\ldots\ \lambda.$$

Maximi autem momenti quaestio versatur circa eos casus, quibus λ est numerus fractus, quos ergo pro ratione denominationis hic successive sum definiturus.

(*) Miscellanea Taurinensia. Tom. III.

Problema 2.

§. 27. *Denotante i numerum integrum positivum, definire valorem formulae integralis $\int \partial x (l\frac{1}{x})^{\frac{i}{2}}$, integratione ab $x=0$ usque ad $x=1$ extensa.*

Solutio.

In aequatione nostra generali faciamus $m=n$, eritque

$$\frac{[\int \partial x (l\frac{1}{x})^{n-1}]^2}{\int \partial x (l\frac{1}{x})^{2n-1}} = k \int x^{nk-1} \partial x (1-x^k)^{n-1}.$$

Sit jam $n-1=\frac{i}{2}$, et ob $2n-1=i+1$, erit

$$\int \partial x (l\frac{1}{x})^{2n-1} = 1.\ 2.\ 3.\ \ldots\ldots\ (i+1):$$

sumatur porro $k=2$, ut sit $nk-1=i+1$, fietque

$$\frac{[\int \partial x \sqrt{(l\frac{1}{x})^i}]^2}{1.\ 2.\ 3.\ldots(i+1)} = 2 \int x^{i+1} \partial x (1-x^2)^{\frac{i}{2}},$$

ideoque

$$\frac{\int \partial x \sqrt{(l\frac{1}{x})^i}}{\sqrt{[1.2.3.\ldots(i+1)]}} = \sqrt{2 \int x^{i+1} \partial x \sqrt{(1-x^2)^i}},$$

ubi evidens est, pro i numeros tantum impares sumi convenire, quoniam pro paribus evolutio per se est manifesta.

Corollarium 1.

§. 28. Omnes autem casus facile reducuntur ad $i=1$, vel adeo ad $i=-1$; dummodo enim $i+1$ non sit numerus negativus, reductio inventa locum habet. Pro hoc ergo casu erit

$$\int \frac{\partial x}{\sqrt{l\frac{1}{x}}} = \sqrt{2 \int \frac{\partial x}{\sqrt{(1-xx)}}} = \sqrt{\pi}, \text{ ob } \int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2}.$$

Corollarium 2.

§. 29. Hoc autem casu principali expedito, ob

$$\int \partial x \left(l\tfrac{1}{x}\right)^n = n \int \partial x \left(l\tfrac{1}{x}\right)^{n-1}$$

habebimus

$$\int \partial x \sqrt{l\tfrac{1}{x}} = \tfrac{1}{2}\sqrt{\pi}; \quad \int \partial x \left(l\tfrac{1}{x}\right)^{\frac{3}{2}} = \frac{1.3}{2.2}\sqrt{\pi}$$

atque in genere

$$\int \partial x \left(l\tfrac{1}{x}\right)^{\frac{2n+1}{2}} = \tfrac{1}{2}\cdot\tfrac{3}{2}\cdot\tfrac{5}{2}\cdot\tfrac{7}{2}\cdot \ldots \ldots \tfrac{(2n+1)}{2}\sqrt{\pi}$$

Problema 3.

§. 30. *Denotante i numerum integrum positivum, definire valorem formulae integralis $\int \partial x \left(l\frac{1}{x}\right)^{\frac{i}{3}-1}$, integratione ab $x = 0$ ad $x = 1$ extensa.*

Solutio.

Inchoemus ab aequatione praecedentis problematis

$$\frac{\left[\int \partial x \left(l\frac{1}{x}\right)^{n-1}\right]^2}{\int \partial x \left(l\frac{1}{x}\right)^{2n-1}} = k \int x^{nk-1} \partial x \left(1 - x^k\right)^{n-1},$$

atque in forma generali statuamus $m = 2n$, ut habeatur

$$\frac{\int \partial x \left(l\frac{1}{x}\right)^{n-1} \times \int \partial x \left(l\frac{1}{x}\right)^{3n-1}}{\int \partial x \left(l\frac{1}{x}\right)^{3n-1}} = k \int x^{2nk-1} \partial x \left(1 - x^k\right)^{n-1},$$

ac multiplicando has duas aequalitates adipiscimur

$$\frac{\left[\int \partial x \left(l\frac{1}{x}\right)^{n-1}\right]^3}{\int \partial x \left(l\frac{1}{x}\right)^{3n-1}} = kk \int x^{nk-1} \partial x \left(1 - x^k\right)^{n-1} \times \int x^{2nk-1} \partial x \left(1 - x^k\right)^{n-1}.$$

Hic jam ponatur $n = \frac{i}{3}$ ut sit

$$\int \partial x \left(l\tfrac{1}{x}\right)^{i-1} = 1.\ 2.\ 3.\ \ldots\ldots\ (i-1),$$

sumaturque $k = 3$, ac prodibit

$$\frac{[\int \partial x \sqrt[3]{(l\frac{1}{x})^{i-3}}]^3}{1.\ 2\ 3\ldots(i-1)} = 9 \int x^{i-1} \partial x \sqrt[3]{(1-x^3)^{i-3}} \times \int x^{2i-1} \partial x \sqrt[3]{(1-x^3)^{i-3}};$$

unde concludimus

$$\frac{\int \partial x \sqrt[3]{(l\frac{1}{x})^{i-3}}}{\sqrt[3]{1.\ 2.\ 3\ldots(i-1)}} = \sqrt[3]{9} \int \frac{x^{i-1}\partial x}{\sqrt[3]{(1-x^3)^{3-i}}} \times \int \frac{x^{2i-1}\partial x}{\sqrt[3]{(1-x^3)^{3-i}}}.$$

Corollarium 1.

§ 31. Bini hic occurrunt casus principales, a quibus reliqui omnes pendent, ponendo scilicet vel $i = 1$ vel $i = 2$, qui sunt

$$\text{I.} \int \frac{\partial x}{\sqrt[3]{(l\frac{1}{x})^2}} = \sqrt[3]{9} \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} \times \int \frac{x \partial x}{\sqrt[3]{(1-x^3)^2}}$$

$$\text{II.} \int \frac{\partial x}{\sqrt[3]{l\frac{1}{x}}} = \sqrt[3]{9} \int \frac{x \partial x}{\sqrt[3]{(1-x^3)}} \times \int \frac{x \partial x}{\sqrt[3]{(1-x^3)}},$$

quae posterior forma ob

$$\int \frac{x^3 \partial x}{\sqrt[3]{(1-x^3)}} = \frac{1}{3} \int \frac{\partial x}{\sqrt[3]{(1-x^3)}}$$

abit in

$$\int \frac{\partial x}{\sqrt[3]{l\frac{1}{x}}} = \sqrt[3]{3} \int \frac{\partial x}{\sqrt[3]{(1-x^3)}} \times \int \frac{x \partial x}{\sqrt[3]{(1-x^3)}}.$$

Corollarium 2.

§. 32. Si uti in observationibus meis ante allegatis brevitatis gratia ponamus

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[3]{(1-x^3)^{3-q}}} = \left(\frac{p}{q}\right),$$

atque ut ibi pro hac classe

$$\left(\frac{2}{1}\right) = \frac{\pi}{3 \sin. \frac{\pi}{3}} = \alpha,$$

tum vero

$$\left(\frac{1}{1}\right) = \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = \mathrm{A}, \text{ erit}$$

$$\text{I. } \int \frac{\partial x}{\sqrt[3]{(l\frac{1}{x})^2}} = \sqrt[3]{9 \left(\frac{1}{1}\right)\left(\frac{2}{1}\right)} = \sqrt[3]{9\,\alpha\,\mathrm{A}},$$

$$\text{II. } \int \frac{\partial x}{\sqrt[3]{(l\frac{1}{x})^1}} = \sqrt[3]{3 \left(\frac{1}{2}\right)\left(\frac{2}{2}\right)} = \sqrt[3]{\frac{3\alpha\alpha}{\mathrm{A}}}.$$

Corollarium 3.

§. 33. Pro casu ergo priori habebimus

$$\int \partial x \sqrt[3]{(l\tfrac{1}{x})^{-2}} = \sqrt[3]{9\,\alpha\,\mathrm{A}}, \quad \int \partial x \sqrt[3]{l\tfrac{1}{x}} = \tfrac{1}{3}\sqrt[3]{9\,\alpha\,\mathrm{A}}, \text{ et}$$

$$\int \partial x \sqrt[3]{(l\tfrac{1}{x})^{3n+1}} = \tfrac{1}{3} \cdot \tfrac{4}{3} \cdot \tfrac{7}{3} \cdots\cdots \tfrac{3n+1}{3} \sqrt[3]{9\,\alpha\,\mathrm{A}}:$$

pro altero vero casu

$$\int \partial x \sqrt[3]{(l\tfrac{1}{x})^{-1}} = \sqrt[3]{\tfrac{3\alpha\alpha}{\mathrm{A}}}, \quad \int \partial x \sqrt[3]{(l\tfrac{1}{x})^2} = \tfrac{2}{3}\sqrt[3]{\tfrac{3\alpha\alpha}{\mathrm{A}}}, \text{ et}$$

$$\int \partial x \sqrt[3]{(l\tfrac{1}{x})^{3n-1}} = \tfrac{2}{3} \cdot \tfrac{5}{3} \cdot \tfrac{8}{3} \cdots\cdots \tfrac{3n-1}{3} \sqrt[3]{\tfrac{3\alpha\alpha}{\mathrm{A}}}.$$

Problema 4.

§ 34. *Denotante i numerum integrum positivum, definire valorem formulae integralis $\int \partial x\,(l\frac{1}{x})^{\frac{i}{4}-1}$, integratione ab $x=0$ ad $x=1$ extensa.*

Solutio.

In solutione problematis praecedentis perducti sumus ad hanc aequationem

$$\frac{[\int \partial x (l\frac{1}{x})^{n-1}]^3}{\int \partial x (l\frac{1}{x})^{3n-1}} = kk\int \frac{x^{nk-1}\partial x}{(1-x^k)^{1-n}} \times \int \frac{x^{2nk-1}\partial x}{(1-x^k)^{1-n}};$$

forma generalis autem sumendo $m = 3n$ praebet

$$\frac{\int \partial x (l\frac{1}{x})^{n-1} \times \int \partial x (l\frac{1}{x})^{3n-1}}{\int \partial x (l\frac{1}{x})^{4n-1}} = k\int \frac{x^{3nk-1}\partial x}{(1-x^k)^{1-n}},$$

quibus conjungendis adipiscimur

$$\frac{[\int \partial x (l\frac{1}{x})^{n-1}]^4}{\int \partial x (l\frac{1}{x})^{4n-1}} = k^3\int \frac{x^{nk-1}\partial x}{(1-x^k)^{1-n}} \times \int \frac{x^{2nk-1}\partial x}{(1-x^k)^{1-n}} \times \int \frac{x^{3nk-1}\partial x}{(1-x^k)^{1-n}}.$$

Sit nunc $n = \frac{i}{4}$, et sumatur $k = 4$, fietque

$$\frac{\int \partial x (l\frac{1}{x})^{\frac{i}{4}-1}}{\sqrt[4]{1.2.3....(i-1)}} = \sqrt[4]{4^3}\int \frac{x^{i-1}\partial x}{\sqrt[4]{(1-x^4)^{4-i}}} \times \int \frac{x^{2i-1}\partial x}{\sqrt[4]{(1-x^4)^{4-i}}} \times \int \frac{x^{3i-1}\partial x}{\sqrt[4]{(1-x^4)^{4-i}}}$$

Corollarium 1.

§. 35. Si igitur sit $i = 1$, habebimus

$$\int \partial x \sqrt[4]{(l\frac{1}{x})^{-3}} = \sqrt[4]{4^3}\int \frac{\partial x}{\sqrt[4]{(1-x^4)^3}} \times \int \frac{x\partial x}{\sqrt[4]{(1-x^4)^3}} \times \int \frac{xx\partial x}{\sqrt[4]{(1-x^4)^3}}$$

quae expressio si littera P designetur, erit in genere

$$\int \partial x \sqrt[4]{(l\frac{1}{x})^{4n-3}} = \frac{1}{4}\cdot\frac{5}{4}\cdot\frac{9}{4}\cdot\ldots\ldots\ldots\frac{4n-3}{4}\cdot \mathrm{P}.$$

Corollarium 2.

§. 36. Pro altero casu principali sumamus $i = 3$, eritque

$$\int \partial x \sqrt[4]{(l\tfrac{1}{x})^{-1}} = \sqrt[4]{2.4^3} \int \frac{x^2\,\partial x}{\sqrt[4]{(1-x^4)}} \times \int \frac{x^5\,\partial x}{\sqrt[4]{(1-x^4)}} \times \int \frac{x^8\,\partial x}{\sqrt[4]{(1-x^4)}},$$

seu facta reductione ad simpliciores formas

$$\int \partial x \sqrt[4]{(l\tfrac{1}{x})^{-1}} = \sqrt[4]{8} \int \frac{xx\,\partial x}{\sqrt[4]{(1-x^4)}} \times \int \frac{x\,\partial x}{\sqrt[4]{(1-x^4)}} \times \int \frac{\partial x}{\sqrt[4]{(1-x^4)}},$$

quae expressio si littera Q designetur, erit generatim

$$\int \partial x \sqrt[4]{(l\tfrac{1}{x})^{4n-1}} = \tfrac{3}{4} \cdot \tfrac{7}{4} \cdot \tfrac{11}{4} \ldots\ldots\ldots \tfrac{4n-1}{4} \cdot Q.$$

Scholion.

§. 37. Si formulam integralem

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[4]{(1-x^4)^{4-q}}}$$

hoc signo $(\frac{p}{q})$ indicemus, solutio problematis ita se habebit

$$\int \partial x \sqrt[4]{(l\tfrac{1}{x})^{i-4}} = \sqrt[4]{1.2.3 \ldots (i-1).4^3 (\tfrac{i}{i}) (\tfrac{2i}{i}) (\tfrac{3i}{i})},$$

et pro binis casibus evolutis fit

$$P = \sqrt[4]{4^3 (\tfrac{1}{1}) (\tfrac{2}{1}) (\tfrac{3}{1})} \text{ et } Q = \sqrt[4]{8 (\tfrac{3}{3}) (\tfrac{2}{3}) (\tfrac{1}{3})}.$$

Statuamus nunc pro iis formulis quae a circulo pendent

$$(\tfrac{3}{1}) = \frac{\pi}{4 \sin.\frac{\pi}{4}} = \alpha \text{ et } (\tfrac{2}{2}) = \frac{\pi}{4 \sin.\frac{2\pi}{4}} = \beta,$$

pro transcendentibus autem altioris ordinis

$$(\tfrac{2}{1}) = \int \frac{x\,\partial x}{\sqrt[4]{(1-x^4)^3}} = \int \frac{\partial x}{\sqrt[4]{(1-x^4)}} = A,$$

quippe a qua omnes reliquae pendent ac reperiemus

$$P = \sqrt[4]{4^5 \cdot \frac{\alpha\alpha}{\beta} \cdot AA} \text{ et } Q = \sqrt[4]{4 \cdot \alpha\alpha\beta \cdot \frac{1}{AA}};$$

unde patet esse

$$PQ = 4\alpha = \frac{\pi}{\sin.\frac{\pi}{4}}.$$

Cum autem sit

$$\alpha = \frac{\pi}{2\sqrt{2}} \text{ et } \beta = \frac{\pi}{4}, \text{ erit}$$

$$P = \sqrt[4]{32\pi AA}, \quad Q = \sqrt[4]{\frac{\pi^3}{8AA}} \text{ et } \frac{P}{Q} = \frac{4A}{\sqrt{\pi}}.$$

Problema 5.

§. 38. *Denotante i numerum integrum positivum, definire valorem formulae integralis* $\int \partial x \sqrt[5]{(l\frac{1}{x})^{i-5}}$, *integratione ab* $x=0$ *ad* $x=1$ *extensa.*

Solutio.

Ex praecedentibus solutionibus jam satis est perspicuum pro hoc casu perventum iri ad hanc formam

$$\frac{\int \partial x \sqrt[5]{(l\frac{1}{x})^{i-5}}}{\sqrt[5]{1.2.3\ldots(i-1)}} = \sqrt[5]{5^4} \int \frac{x^{i-1}\partial x}{\sqrt[5]{(1-x^5)^{5-i}}} \times \int \frac{x^{2i-1}\partial x}{\sqrt[5]{(1-x^5)^{5-i}}} \times \int \frac{x^{3i-1}\partial x}{\sqrt[5]{(1-x^5)^{5-i}}} \times \int \frac{x^{4i-1}\partial x}{\sqrt[5]{(1-x^5)^{5-i}}},$$

quae formulae integrales ad classem quintam dissertationis meae supra allegatae sunt referendae. Quare si modo ibi recepto signum $\left(\frac{p}{q}\right)$ denotet hanc formulam

$$\int \frac{x^{p-1}\partial x}{\sqrt[5]{(1-x^5)^{5-q}}},$$

valorem quaesitum ita commodius exprimere licebit, ut sit

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{i-5}} = \sqrt[5]{1.2.3\ldots(i-1)5^4 \left(\tfrac{i}{i}\right)\left(\tfrac{2i}{i}\right)\left(\tfrac{3i}{i}\right)\left(\tfrac{4i}{i}\right)},$$

ubi quidem sufficit ipsi i valores quinario minores tribuisse, quando autem numeratores quinarium superant, tenendum est esse

$$\left(\frac{5+m}{i}\right) = \frac{m}{m+i}\left(\frac{m}{i}\right);$$

tum vero porro

$$\left(\frac{10+m}{i}\right) = \frac{m}{m+i}\cdot\frac{m+5}{m+i+5}\left(\frac{m}{i}\right),$$

$$\left(\frac{15+m}{i}\right) = \frac{m}{m+i}\cdot\frac{m+5}{m+i+5}\cdot\frac{m+10}{m+i+10}\left(\frac{m}{i}\right).$$

Deinde vero pro hac classe binae formulae quadraturam circuli involvunt, quae sint

$$\left(\tfrac{4}{1}\right) = \frac{\pi}{5\,\sin.\frac{\pi}{5}} = \alpha \text{ et } \left(\tfrac{3}{2}\right) = \frac{\pi}{5\,\sin.\frac{2\pi}{5}} = \beta,$$

duae autem quadraturas altiores continent, quae ponantur

$$\left(\tfrac{3}{1}\right) = \int \frac{xx\,\partial x}{\sqrt[5]{(1-x^5)^4}} = \int \frac{\partial x}{\sqrt[5]{(1-x^5)^2}} = \text{A et}$$

$$\left(\tfrac{2}{2}\right) = \int \frac{x\,\partial x}{\sqrt[5]{(1-x^5)^3}} = \text{B};$$

atque ex his valores omnium reliquarum formularum hujus classis assignavi, scilicet

$$\left(\tfrac{5}{1}\right) = 1;\ \left(\tfrac{5}{2}\right) = \tfrac{1}{2};\ \left(\tfrac{5}{3}\right) = \tfrac{1}{3};\ \left(\tfrac{5}{4}\right) = \tfrac{1}{4};\ \left(\tfrac{5}{5}\right) = \tfrac{1}{5}$$

$$\left(\tfrac{4}{1}\right) = \alpha;\ \left(\tfrac{4}{2}\right) = \frac{\beta}{\text{A}};\ \left(\tfrac{4}{3}\right) = \frac{\beta}{2\text{B}};\ \left(\tfrac{4}{4}\right) = \frac{\alpha}{3\text{A}}$$

$$\left(\tfrac{3}{1}\right) = \text{A};\ \left(\tfrac{3}{2}\right) = \beta;\ \left(\tfrac{3}{3}\right) = \frac{\beta\beta}{\alpha\text{B}}$$

$$\left(\tfrac{2}{1}\right) = \frac{\alpha\text{B}}{\beta};\ \left(\tfrac{2}{2}\right) = \text{B}$$

$$\left(\tfrac{1}{1}\right) = \frac{\alpha\text{A}}{\beta}.$$

Corollarium 1.

§. 39. Sumto exponente $i = 1$, erit

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{-4}} = \sqrt[5]{5^4 \left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)} = \sqrt[5]{5^4 \cdot \tfrac{\alpha^3}{\beta^2} A^2 B};$$

unde in genere concludimus fore, denotante n numerum integrum quemcunque

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{5n-4}} = \tfrac{1}{5} \cdot \tfrac{6}{5} \cdot \tfrac{11}{5} \cdots \tfrac{5n-4}{5} \cdot \sqrt[5]{5^4 \cdot \tfrac{\alpha^3}{\beta^2} A^2 B}$$

Corollarium 2.

§. 40. Sit nunc $i = 2$, et cum prodeat

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{-3}} = \sqrt[5]{1.\, 5^4 \left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)\left(\tfrac{8}{2}\right)}, \text{ ob}$$
$$\left(\tfrac{6}{2}\right) = \tfrac{1}{3}\left(\tfrac{1}{2}\right) = \tfrac{1}{3}\left(\tfrac{2}{1}\right) \text{ et } \left(\tfrac{8}{2}\right) = \tfrac{3}{5}\left(\tfrac{3}{2}\right);$$

erit haec expressio

$$\sqrt[5]{5^3 \left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{2}\right)}\ \sqrt[5]{5^3 . \alpha\beta . \tfrac{BB}{A}}$$

et in genere

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{5n-3}} = \tfrac{2}{5} \cdot \tfrac{7}{5} \cdot \tfrac{12}{5} \cdots \tfrac{5n-3}{5} \sqrt[5]{5^3 . \alpha\beta . \tfrac{BB}{A}}.$$

Corollarium 3.

§. 41. Sit $i = 3$, et forma inventa

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{-2}} = \sqrt[5]{2.\, 5^4 \left(\tfrac{3}{3}\right)\left(\tfrac{6}{3}\right)\left(\tfrac{9}{3}\right)\left(\tfrac{12}{3}\right)}, \text{ ob}$$
$$\left(\tfrac{6}{3}\right) = \tfrac{1}{4}\left(\tfrac{1}{3}\right);\ \left(\tfrac{9}{3}\right) = \tfrac{4}{7}\left(\tfrac{4}{3}\right);\ \left(\tfrac{12}{3}\right) = \tfrac{2}{5} \cdot \tfrac{7}{10}\left(\tfrac{2}{3}\right), \text{ abit in}$$
$$\sqrt[5]{2.\, 5^2 \left(\tfrac{3}{3}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{3}{2}\right)} = \sqrt[5]{5^2 \cdot \tfrac{\beta^4}{\alpha} \cdot \tfrac{A}{BB}};$$

unde in genere colligitur

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{5n-2}} = \tfrac{3}{5} \cdot \tfrac{8}{5} \cdot \tfrac{13}{5} \cdots \tfrac{5n-2}{5} \sqrt[5]{5^2 \cdot \tfrac{\beta^4}{\alpha} \cdot \tfrac{A}{BB}}.$$

Corollarium 4.

§. 42. Posito denique $i=4$, forma nostra

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{-4}} = \sqrt[5]{6 \cdot 5^4 (\tfrac{4}{4})(\tfrac{8}{4})(\tfrac{12}{4})(\tfrac{16}{4})}, \text{ ob}$$

$$(\tfrac{8}{4}) = \tfrac{3}{7}(\tfrac{4}{3});\ (\tfrac{12}{4}) = \tfrac{2}{6}\cdot\tfrac{7}{11}(\tfrac{4}{2});\ (\tfrac{16}{4}) = \tfrac{1}{5}\cdot\tfrac{6}{10}\cdot\tfrac{11}{15}(\tfrac{4}{1}),$$

transformabitur in hanc

$$\sqrt[5]{6 \cdot 5 (\tfrac{4}{4})(\tfrac{4}{3})(\tfrac{4}{2})(\tfrac{4}{1})} = \sqrt[5]{5 \cdot \frac{\alpha\alpha\beta\beta}{AAB}};$$

ita ut sit in genere

$$\int \partial x \sqrt[5]{(l\tfrac{1}{x})^{5n-1}} = \tfrac{4}{5}\cdot\tfrac{9}{5}\cdot\tfrac{14}{5}\cdots\tfrac{5n-1}{5}\sqrt[5]{5\cdot\alpha\alpha\beta\beta\cdot\frac{1}{AAB}}.$$

Scholion.

§. 43. Si valorem formulae integralis $\int \partial x (l\tfrac{1}{x})^\lambda$ hoc signo $[\lambda]$ repraesentemus, casus hactenus evoluti praebent

$$[-\tfrac{4}{5}] = \sqrt[5]{5^4 \cdot \frac{\alpha^3}{\beta^2}\cdot A^2 B};\quad [+\tfrac{1}{5}] = \tfrac{1}{5}\sqrt[5]{5^4\cdot\frac{\alpha^3}{\beta^2}\cdot A^2 B}$$

$$[-\tfrac{3}{5}] = \sqrt[5]{5^3\cdot\alpha\beta\cdot\frac{BB}{A}};\quad [+\tfrac{2}{5}] = \tfrac{2}{5}\sqrt[5]{5^3\cdot\alpha\beta\cdot\frac{BB}{A}}$$

$$[-\tfrac{2}{5}] = \sqrt[5]{5^2\cdot\frac{\beta^4}{\alpha}\cdot\frac{A}{BB}};\quad [+\tfrac{3}{5}] = \tfrac{3}{5}\sqrt[5]{5^2\cdot\frac{\beta^4}{\alpha}\cdot\frac{A}{BB}}$$

$$[-\tfrac{1}{5}] = \sqrt[5]{5\cdot\alpha^2\beta^2\cdot\frac{1}{AAB}};\quad [+\tfrac{4}{5}] = \tfrac{4}{5}\sqrt[5]{5\cdot\alpha^2\beta^2\cdot\frac{1}{BBA}};$$

unde binis, quarum indices simul sumti fiunt $=0$, conjungendis colligimus

$$[+\tfrac{1}{5}]\cdot[-\tfrac{1}{5}] = \alpha = \frac{\pi}{5\sin.\frac{\pi}{5}}$$

$$[+\tfrac{2}{5}]\cdot[-\tfrac{2}{5}] = 2\beta = \frac{2\pi}{5\sin.\frac{2\pi}{5}}$$

$$[+\tfrac{3}{5}]\cdot[-\tfrac{3}{5}]=3\beta=\frac{3\pi}{5\sin.\frac{3\pi}{5}}$$

$$[+\tfrac{4}{5}]\cdot[-\tfrac{4}{5}]=4\alpha=\frac{2\pi}{5\sin.\frac{4\pi}{5}}$$

Ex antecedente autem problemate simili modo deducimus

$$[-\tfrac{3}{4}]=P=\sqrt[4]{4^3\cdot\tfrac{\alpha\alpha}{\beta}}\cdot AA;\ [+\tfrac{1}{4}]=\tfrac{1}{4}\sqrt[4]{4^3\cdot\tfrac{\alpha\alpha}{\beta}}\cdot AA$$

$$[-\tfrac{1}{4}]=Q=\sqrt[4]{4\cdot\alpha\alpha\beta}\cdot\tfrac{1}{AA};\ [+\tfrac{3}{4}]=\tfrac{3}{4}\sqrt[4]{4\cdot\alpha\alpha\beta}\cdot\tfrac{1}{AA}$$

hincque

$$[+\tfrac{1}{4}]\cdot[-\tfrac{1}{4}]=\alpha=\frac{\pi}{4\sin.\frac{\pi}{4}}$$

$$[+\tfrac{3}{4}]\cdot[-\tfrac{3}{4}]=3\alpha=\frac{3\pi}{4\sin.\frac{3\pi}{4}};$$

unde in genere hoc Theorema adipiscimur, quod sit

$$[\lambda]\cdot[-\lambda]=\frac{\lambda\pi}{\sin.\lambda\pi},$$

cujus ratio ex methodo interpolandi olim exposita ita reddi potest

$$\text{cum sit } [\lambda]=\frac{1^{1-\lambda}\cdot 2^{\lambda}}{1+\lambda}\cdot\frac{2^{1-\lambda}\cdot 3^{\lambda}}{2+\lambda}\cdot\frac{3^{1-\lambda}\cdot 4^{\lambda}}{3+\lambda}\ \text{etc.}$$

$$\text{erit } [-\lambda]=\frac{1^{1+\lambda}\cdot 2^{-\lambda}}{1-\lambda}\cdot\frac{2^{1+\lambda}\cdot 3^{-\lambda}}{2-\lambda}\cdot\frac{3^{1+\lambda}\cdot 4^{-\lambda}}{3-\lambda}\ \text{etc.}$$

hincque

$$[\lambda]\cdot[-\lambda]=\frac{1\cdot 1}{1-\lambda\lambda}\cdot\frac{2\cdot 2}{4-\lambda\lambda}\cdot\frac{3\cdot 3}{9-\lambda\lambda}\ \text{etc.}=\frac{\lambda\pi}{\sin.\lambda\pi};$$

uti alibi demonstravi.

Problema 6 generale.

§. 44. *Si litterae i et n denotent numeros integros positivos, definire valorem formulae integralis*

$$\int \partial x \left(l\tfrac{1}{x}\right)^{\frac{i-n}{n}}, \text{ seu } \int \partial x \sqrt[n]{\left(l\tfrac{1}{x}\right)^{i-n}},$$

integratione ab $x=0$ *ad* $x=1$ *extensa.*

Solutio.

Methodus hactenus usitata quaesitum valorem sequenti modo per quadraturas curvarum algebraicarum expressum exhibebit

$$\frac{\int \partial x \sqrt[n]{\left(l\frac{1}{x}\right)^{i-n}}}{\sqrt[n]{1.2.3..(i-1)}} = \sqrt[n]{n^{n-1}} \int \frac{x^{i-1}\partial x}{\sqrt[n]{(1-x^n)^{n-i}}} \times \int \frac{x^{2i-1}\partial x}{\sqrt[n]{(1-x^n)^{n-i}}} \times \ldots \int \frac{x^{(n-1)i-1}\partial x}{\sqrt[n]{(1-x^n)^{n-i}}}$$

Quod si jam brevitatis gratia formulam integralem

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \text{ hoc charactere } \left(\frac{p}{q}\right),$$

formulam vero $\int \partial x \sqrt[n]{\left(l\frac{1}{x}\right)^m}$ isthoc $\left[\frac{m}{n}\right]$ designemus, ita ut $\left[\frac{m}{n}\right]$ valorem hujus producti indefiniti 1. 2. 3 . . . . . z denotet, existente $z = \frac{m}{n}$, succinctius valor quaesitus hoc modo expressus prodibit

$$\left[\frac{i-n}{n}\right] = \sqrt[n]{1.2.3\ldots(i-1)\,n^{n-1}}.\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\cdots\left(\frac{ni-i}{i}\right);$$

unde etiam colligitur

$$\left[\frac{i}{n}\right] = \frac{i}{n}\sqrt[n]{1.2.3\ldots(i-1)\,n^{n-1}}.\left(\frac{i}{i}\right)\left(\frac{2i}{i}\right)\left(\frac{3i}{i}\right)\cdots\left(\frac{ni-i}{i}\right).$$

Hic semper numerum i ipso n minorem accepisse sufficiet, quoniam pro majoribus notum est esse

$$\left[\frac{i+n}{n}\right]=\frac{i}{n}\left[\frac{i}{n}\right],\text{ item }\left[\frac{i+2n}{n}\right]=\frac{i+n}{n}\cdot\frac{i}{n}\left[\frac{i}{n}\right]\text{ etc.}$$

hocque modo tota investigatio ad eos tantum casus reducitur, quibus fractionis $\frac{i}{n}$ numerator i denominatore n est minor. Praeterea vero de formulis integralibus

$$\int_0^1 \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}=\left(\frac{q}{p}\right),$$

sequentia notasse juvabit

I. Litteras p et q inter se esse permutabiles ut sit

$$\left(\frac{p}{q}\right)=\left(\frac{q}{p}\right).$$

II. Si alteruter numerorum p vel q ipsi exponenti n aequetur, valorem formulae integralis fore algebraicum, scilicet

$$\left(\frac{n}{p}\right)=\left(\frac{p}{n}\right)=\frac{1}{p},\text{ seu }\left(\frac{n}{q}\right)=\left(\frac{q}{n}\right)=\frac{1}{q}.$$

III. Si summa numerorum $p+q$ ipsi exponenti n aequatur, formulae integralis $\left(\frac{p}{q}\right)$ valorem per circulum exhiberi posse, cum sit

$$\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}};\text{ et }\left(\frac{q}{n-q}\right)=\left(\frac{n-q}{q}\right)=\frac{\pi}{n\sin.\frac{q\pi}{n}}.$$

IV. Si alteruter numerorum p vel q major sit exponente n, formulam integralem $\left(\frac{p}{q}\right)$ ad aliam revocari posse, cujus termini sint ipso n minores, quod fit ope hujus reductionis

$$\left(\frac{p+n}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right).$$

V. Inter plures hujusmodi formulas integrales talem relationem intercedere, ut sit

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right),$$

cujus ope omnes reductiones reperiuntur, quas in observationibus circa has formulas exposui.

Corollarium 1.

§. 45. Si hoc modo ope reductionis N°. IV. indicatae formam inventam ad singulos casus accommodemus, eos sequenti ratione simplicissime exhibere poterimus. Ac primo quidem pro casu $n=2$, quo nulla opus est reductione habebimus

$$\left[\frac{1}{2}\right]=\frac{1}{2}\sqrt[2]{2\left(\frac{1}{1}\right)}=\frac{1}{2}\sqrt[2]{\frac{\pi}{\sin.\frac{\pi}{2}}}=\frac{1}{2}\sqrt{\pi}.$$

Corollarium 2.

§. 46. Pro casu $n=3$ habebimus has reductiones

$$\left[\frac{1}{3}\right]=\frac{1}{3}\sqrt[3]{3^2.\left(\frac{1}{1}\right)\left(\frac{1}{2}\right)}$$

$$\left[\frac{2}{3}\right]=\frac{2}{3}\sqrt[3]{3.\,1.\left(\frac{2}{2}\right)\left(\frac{2}{1}\right)}.$$

Corollarium 3.

§. 47. Pro casu $n=4$ hae tres reductiones obtinentur

$$\left[\frac{1}{4}\right]=\frac{1}{4}\sqrt[4]{4^3\left(\frac{1}{1}\right)\left(\frac{1}{2}\right)\left(\frac{1}{3}\right)}$$

$$\left[\frac{2}{4}\right]=\frac{2}{4}\sqrt[4]{4^2.\,2.\left(\frac{2}{2}\right)^2\left(\frac{2}{4}\right)}=\frac{2}{4}\sqrt[2]{4\left(\frac{2}{2}\right)},\ \text{ob}\ \left(\frac{2}{4}\right)=\frac{1}{2}$$

$$\left[\frac{3}{4}\right]=\frac{3}{4}\sqrt[4]{4.\,1.\,2\left(\frac{3}{3}\right)\left(\frac{3}{2}\right)\left(\frac{3}{1}\right)};$$

cum in media sit $\left(\frac{2}{2}\right)=\left(\frac{2}{4-2}\right)=\frac{\pi}{4}$, erit utique ut ante

$$\left[\frac{2}{4}\right]=\left[\frac{1}{2}\right]=\frac{1}{2}\sqrt{\pi}.$$

Corollarium 4.

§. 48. Sit nunc $n = 5$, et prodeunt hae quatuor reductiones

$$\left[\tfrac{1}{5}\right] = \tfrac{1}{5}\sqrt[5]{5^4 \cdot \left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)}$$

$$\left[\tfrac{2}{5}\right] = \tfrac{2}{5}\sqrt[5]{5^3 \cdot 1 \left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)}$$

$$\left[\tfrac{3}{5}\right] = \tfrac{3}{5}\sqrt[5]{5^2 \cdot 1 . 2 \left(\tfrac{3}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{2}{3}\right)}$$

$$\left[\tfrac{4}{5}\right] = \tfrac{4}{5}\sqrt[5]{5 . 1 . 2 . 3 \left(\tfrac{4}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{1}{4}\right)}.$$

Corollarium 5.

§. 49. Sit $n = 6$, et habebimus has reductiones

$$\left[\tfrac{1}{6}\right] = \tfrac{1}{6}\sqrt[6]{6^5 \cdot \left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\tfrac{3}{1})\left(\tfrac{4}{1}\right)\left(\tfrac{5}{1}\right)}$$

$$\left[\tfrac{2}{6}\right] = \tfrac{2}{6}\sqrt[6]{6^4 \cdot 2 \left(\tfrac{2}{2}\right)^2\left(\tfrac{4}{2}\right)^2\left(\tfrac{6}{2}\right)} = \tfrac{1}{3}\sqrt[3]{6^2 \left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)}$$

$$\left[\tfrac{3}{6}\right] = \tfrac{3}{6}\sqrt[6]{6^3 \cdot 3 . 3 \left(\tfrac{3}{3}\right)^3\left(\tfrac{6}{3}\right)^2} = \tfrac{1}{2}\sqrt[2]{6 \left(\tfrac{3}{3}\right)}$$

$$\left[\tfrac{4}{6}\right] = \tfrac{4}{6}\sqrt[6]{6^2 \cdot 2 . 4 . 2 \left(\tfrac{4}{4}\right)^2\left(\tfrac{2}{4}\right)^2\left(\tfrac{6}{4}\right)} = \tfrac{2}{3}\sqrt[3]{6 . 2 \left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right)}$$

$$\left[\tfrac{5}{6}\right] = \tfrac{5}{6}\sqrt[6]{6 . 1 . 2 , 3 . 4 \left(\tfrac{5}{5}\right)\left(\tfrac{4}{5}\right)\left(\tfrac{3}{5}\right)\left(\tfrac{2}{5}\right)\left(\tfrac{1}{5}\right)}.$$

Corollarium 6.

§. 50. Posito $n = 7$, sequentes sex prodeunt aequationes

$$\left[\tfrac{1}{7}\right] = \tfrac{1}{7}\sqrt[7]{7^6 \left(\tfrac{1}{1}\right)\left(\tfrac{2}{1}\right)\left(\tfrac{3}{1}\right)\left(\tfrac{4}{1}\right)\left(\tfrac{5}{1}\right)\left(\tfrac{6}{1}\right)}$$

$$\left[\tfrac{2}{7}\right] = \tfrac{2}{7}\sqrt[7]{7^5 \cdot 1 \left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right)\left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)}$$

$$\left[\tfrac{3}{7}\right] = \tfrac{3}{7}\sqrt[7]{7^4 \cdot 1 . 2 \left(\tfrac{3}{3}\right)\left(\tfrac{6}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{5}{3}\right)\left(\tfrac{1}{3}\right)\left(\tfrac{4}{3}\right)}$$

$$\left[\tfrac{4}{7}\right] = \tfrac{4}{7}\sqrt[7]{7^3 \cdot 1 . 2 . 3 \left(\tfrac{4}{4}\right)\left(\tfrac{1}{4}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{6}{4}\right)\left(\tfrac{3}{4}\right)}$$

$$\left[\tfrac{5}{7}\right] = \tfrac{5}{7}\sqrt[7]{7^2.\,1.\,2.\,3.\,4\,(\tfrac{5}{5})(\tfrac{3}{5})(\tfrac{1}{5})(\tfrac{6}{5})(\tfrac{4}{5})(\tfrac{2}{5})}$$

$$\left[\tfrac{6}{7}\right] = \tfrac{6}{7}\sqrt[7]{7.\,1.\,2.\,3.\,4.\,5\,(\tfrac{6}{6})(\tfrac{5}{6})(\tfrac{4}{6})(\tfrac{3}{6})(\tfrac{2}{6})(\tfrac{1}{6})}.$$

Corollarium 7.

§. 51. Sit $n = 8$, et septem hae reductiones impetrabuntur

$$\left[\tfrac{1}{8}\right] = \tfrac{1}{8}\sqrt[8]{8^7\,(\tfrac{1}{1})(\tfrac{2}{1})(\tfrac{3}{1})(\tfrac{4}{1})(\tfrac{5}{1})(\tfrac{6}{1})(\tfrac{7}{1})}$$

$$\left[\tfrac{2}{8}\right] = \tfrac{2}{8}\sqrt[8]{8^6.\,2\,(\tfrac{2}{2})^2(\tfrac{4}{2})^2(\tfrac{6}{2})^2(\tfrac{8}{2})} = \tfrac{1}{4}\sqrt[4]{8^3\,(\tfrac{2}{1})(\tfrac{4}{1})(\tfrac{6}{1})}$$

$$\left[\tfrac{3}{8}\right] = \tfrac{3}{8}\sqrt[8]{8^5.\,1.\,2\,(\tfrac{3}{3})(\tfrac{6}{3})(\tfrac{1}{3})(\tfrac{4}{3})(\tfrac{7}{3})(\tfrac{2}{3})(\tfrac{5}{3})}$$

$$\left[\tfrac{4}{8}\right] = \tfrac{4}{8}\sqrt[8]{8^4.\,4.\,4.\,4\,(\tfrac{4}{4})^4(\tfrac{8}{4})^3} = \tfrac{1}{2}\sqrt[2]{8\,(\tfrac{4}{4})},$$

$$\left[\tfrac{5}{8}\right] = \tfrac{5}{8}\sqrt[8]{8^3.\,1.\,2.\,3.\,4\,(\tfrac{5}{5})(\tfrac{2}{5})(\tfrac{7}{5})(\tfrac{4}{5})(\tfrac{1}{5})(\tfrac{6}{5})(\tfrac{3}{5})}$$

$$\left[\tfrac{6}{8}\right] = \tfrac{6}{8}\sqrt[8]{8^2.\,4.\,2.\,6.\,4.\,2\,(\tfrac{6}{6})^2(\tfrac{4}{6})^2(\tfrac{2}{6})^2(\tfrac{8}{6})} = \tfrac{3}{4}\sqrt[4]{8.\,2.\,4\,(\tfrac{6}{6})(\tfrac{4}{6})(\tfrac{2}{6})}.$$

$$\left[\tfrac{7}{8}\right] = \tfrac{7}{8}\sqrt[8]{8.\,1.\,2.\,3.\,4.\,5.\,6\,(\tfrac{7}{7})(\tfrac{6}{7})(\tfrac{5}{7})(\tfrac{4}{7})(\tfrac{3}{7})(\tfrac{2}{7})(\tfrac{1}{7})}.$$

Scholion.

§. 52. Superfluum foret hos casus ulterius evolvere, cum ex allatis ordo istarum formularum satis perspiciatur. Si enim in formula proposita $\left[\frac{m}{n}\right]$ numeri m et n sint inter se primi lex est manifesta, cum fiat

$$\left[\tfrac{m}{n}\right] = \tfrac{m}{n}\sqrt[n]{n^{n-m}.\,1.\,2\ldots(m-1).\,(\tfrac{1}{m})(\tfrac{2}{m})(\tfrac{3}{m})\ldots(\tfrac{n-1}{m})},$$

sin autem hi numeri m et n communem habeant divisorem, expediet quidem fractionem $\frac{m}{n}$ ad minimam formam reduci, et ex casibus praecedentibus quaesitum valorem peti; interim tamen etiam operatio hoc modo

instituí poterit. Cum expressio quaesita certe hanc habeat formam

$$\left[\tfrac{m}{n}\right] = \tfrac{m}{n}\sqrt[n]{n^{n-m}\,P\,.\,Q},$$

ubi Q est productum ex $n-1$ formulis integralibus, P vero productum ex aliquot numeris absolutis, primum pro illo producto Q inveniendo, continuetur haec formularum series $\left(\frac{m}{m}\right)\left(\frac{2m}{m}\right)\left(\frac{3m}{m}\right)$, donec numerator superet exponentem n, ejusque loco excessus supra n scribatur, qui si ponatur $=\alpha$, ut jam formula nostra sit $\left(\frac{\alpha}{m}\right)$, hic ipse numerator α dabit factorem producti P, tum hinc fomularum series porro statuatur $\left(\frac{\alpha}{m}\right)\left(\frac{\alpha+m}{m}\right)\left(\frac{\alpha+2m}{m}\right)$ etc. donec iterum ad numeratorem exponente n majorem perveniatur, formulaque prodeat $\left(\frac{n+\beta}{m}\right)$, cujus loco scribi oportet $\left(\frac{\beta}{m}\right)$, simulque hinc factor β in productum P inferatur, sicque progredi conveniet, donec pro Q prodierint $n-1$ formulae. Quae operationes quo facilius intelligantur, casum formulae $\left[\frac{9}{12}\right] = \frac{9}{12}\sqrt[12]{12^3\,P\,.\,Q}$ hoc modo evolvamus, ubi investigatio litterarum Q et P ita instituetur,

$$\text{pro Q} \ldots\ldots \left(\tfrac{9}{9}\right)\left(\tfrac{6}{9}\right)\left(\tfrac{3}{9}\right)\left(\tfrac{12}{9}\right)\left(\tfrac{9}{9}\right)\left(\tfrac{6}{9}\right)\left(\tfrac{3}{9}\right)\left(\tfrac{12}{9}\right)\left(\tfrac{9}{9}\right)\left(\tfrac{6}{9}\right)\left(\tfrac{3}{9}\right),$$

$$\text{pro P} \ldots\ldots\ldots \quad 6.\ 3 \qquad 9.\ 6.\ 3 \qquad 9.\ 6.\ 3,$$

sicque reperitur

$$Q = \left(\tfrac{9}{9}\right)^3\left(\tfrac{6}{9}\right)^3\left(\tfrac{3}{9}\right)^3\left(\tfrac{12}{9}\right)^2 \text{ et}$$

$$P = 6^3.\ 3^3.\ 9^2.$$

Cum igitur sit $\left(\frac{12}{9}\right)=\frac{1}{9}$, fit $P\,Q = 6^3\cdot 3^3\left(\frac{9}{9}\right)^3\left(\frac{6}{9}\right)^3\left(\frac{3}{9}\right)^3$, ideoque

$$\left[\tfrac{9}{12}\right] = \tfrac{3}{4}\sqrt[4]{12.\ 6.\ 3.\left(\tfrac{9}{9}\right)\left(\tfrac{6}{9}\right)\left(\tfrac{3}{9}\right)}.$$

Theorema.

§. 33. Quicunque numeri integri positivi litteris m et n indicentur, erit semper signandi modo ante exposito

$$[\tfrac{m}{n}] = \tfrac{m}{n} \sqrt[n]{n^{n-m}} . 1 . 2 . 3 \ldots (m-1) (\tfrac{1}{m}) (\tfrac{2}{m}) (\tfrac{3}{m}) \cdots (\tfrac{n-1}{m}).$$

Demonstratio.

Pro casu, quo m et n sunt numeri inter se primi, veritas theorematis in antecedentibus est evicta; quod autem etiam locum habeat, si illi numeri m et n commune divisore gaudeant, inde quidem non liquet: verum ex hoc ipso, quod pro casibus, quibus m et n sunt numeri primi, veritas constat, tuto concludere licet, theorema in genere esse verum. Minime quidem diffiteor hoc concludendi genus prorsus esse singulare, ac plerisque suspectum videri debere. Quare quo nullum dubium relinquatur, quoniam pro casibus, quibus numeri m et n inter se sunt compositi, geminam expressionem sumus nacti, utriusque consensum pro casibus ante evolutis ostendisse juvabit. Insigne autem jam suppeditat firmamentum casus $m = n$, quo forma nostra manifesto unitatem producit.

Corollarium 1.

§. 54. Primus casus consensus demonstrationem postulans est quo $m = 2$ et $n = 4$, pro quo supra §. 47 invenimus

$$[\tfrac{2}{4}] = \tfrac{2}{4} \sqrt[4]{4^2} \cdot (\tfrac{2}{2})^2,$$

nunc autem vi theorematis est

$$[\tfrac{2}{4}] = \tfrac{2}{4} \sqrt[4]{4^2} . 1 (\tfrac{1}{2}) (\tfrac{2}{2}) (\tfrac{3}{2}),$$

unde comparatione instituta fit $(\tfrac{2}{2}) = (\tfrac{1}{2}) (\tfrac{3}{2})$, cujus veritas in observationibus supra allegatis est confirmata.

Corollarium 2.

§. 55. Si $m = 2$ et $n = 6$, ex superioribus §. 49 est

$$[\tfrac{2}{6}] = \tfrac{2}{6} \sqrt[3]{6^4} . (\tfrac{2}{2})^2 (\tfrac{4}{2})^2$$

nunc vero per theorema

$$\left[\tfrac{7}{6}\right] = \tfrac{2}{6}\sqrt[6]{6^4}\,.\,1\left(\tfrac{1}{2}\right)\left(\tfrac{2}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{5}{2}\right),$$

ideoque necesse est sit

$$\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right) = \left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right),$$

cujus veritas indidem patet.

Corollarium 3.

§. 56. Si $m = 3$ et $n = 6$, pervenitur ad hanc aequationem

$$\left(\tfrac{3}{3}\right)^2 = 1\,.\,2\,.\left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right)\left(\tfrac{4}{3}\right)\left(\tfrac{5}{3}\right),$$

at si $m = 4$ et $n = 6$, fit simili modo

$$2^2\left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right) = 1\,.\,2\,.\,3\,.\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right),\ \text{seu}$$

$$\left(\tfrac{4}{4}\right)\left(\tfrac{2}{4}\right) = \tfrac{3}{2}\left(\tfrac{1}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right),$$

quod etiam verum deprehenditur.

Corollarium 4.

§. 57. Casus $m = 2$ et $n = 8$ praebet hanc aequalitatem

$$\left(\tfrac{2}{2}\right)\left(\tfrac{4}{2}\right)\left(\tfrac{6}{2}\right) = \left(\tfrac{1}{2}\right)\left(\tfrac{3}{2}\right)\left(\tfrac{5}{2}\right)\left(\tfrac{7}{2}\right);$$

at casus $m = 4$ et $n = 8$ hanc

$$\left(\tfrac{4}{4}\right)^3 = 1\,.\,2\,.\,3\left(\tfrac{1}{4}\right)\left(\tfrac{2}{4}\right)\left(\tfrac{3}{4}\right)\left(\tfrac{5}{4}\right)\left(\tfrac{6}{4}\right)\left(\tfrac{7}{4}\right);$$

casus denique $m = 6$ et $n = 8$ istam

$$2\,.\,4\left(\tfrac{6}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{2}{6}\right) = 1\,.\,3\,.\,5\left(\tfrac{1}{6}\right)\left(\tfrac{3}{6}\right)\left(\tfrac{5}{6}\right)\left(\tfrac{7}{6}\right),$$

quae etiam veritati sunt consentaneae.

Scholion.

§. 58. In genere autem si numeri m et n communem habeant factorem 2, et formula proposita sit $\left[\frac{2m}{2n}\right] = \left[\frac{m}{n}\right]$ quia est

$$\left[\tfrac{m}{n}\right]=\tfrac{m}{n}\sqrt[n]{n^{n-m}.1.2.3\ldots.(m-1)\left(\tfrac{1}{m}\right)\left(\tfrac{2}{m}\right)\left(\tfrac{3}{m}\right)\cdots\left(\tfrac{n-1}{m}\right)},$$

erit eadem ad exponentem $2n$ reducta,

$$\frac{m}{n}\sqrt[2n]{2n^{2n-2m}.2^2.4^2.6^2\ldots(2m-2)^2\left(\frac{2}{2m}\right)^2\left(\frac{4}{2m}\right)^2\left(\frac{6}{2m}\right)^2\cdots\left(\frac{2n-2}{2m}\right)^2}.$$

Per theorema vero eadem expressio fit

$$\frac{m}{n}\sqrt[2n]{2n^{2n-2m}.1.2.3\ldots(2m-1)\left(\frac{1}{2m}\right)\left(\frac{2}{2m}\right)\left(\frac{3}{2m}\right)\cdots\left(\frac{2n-1}{2m}\right)},$$

unde pro exponente $2n$ erit

$$2.4.6\ldots.(2m-2)\left(\frac{2}{2m}\right)\left(\frac{4}{2m}\right)\left(\frac{6}{2m}\right)\cdots.\left(\frac{2n-2}{2m}\right)=$$
$$1.3.5\ldots.(2m-1)\left(\frac{1}{2m}\right)\left(\frac{3}{2m}\right)\left(\frac{5}{2m}\right)\cdots.\left(\frac{2n-1}{2m}\right).$$

Simili modo si communis divisor sit 3, pro exponente $3n$ reperietur

$$3^2.6^2.9^2\ldots\ldots(3m-3)^2\left(\frac{3}{3m}\right)^2\left(\frac{6}{3m}\right)^2\left(\frac{9}{3m}\right)^2\cdots\left(\frac{3n-3}{3m}\right)^2=$$
$$1.2.4.5..(3m-2)(3m-1)\left(\frac{1}{3m}\right)\left(\frac{2}{3m}\right)\left(\frac{4}{3m}\right)\left(\frac{5}{3m}\right)\cdots\left(\frac{3n-1}{3m}\right).$$

quae aequatio concinnius ita exhiberi potest

$$\frac{1.2.4.5.7.8.10\ldots\ldots(3m-2)(3m-1)}{3^2\,.\,6^2\,.\,9^2\ldots\ldots\ldots(3m-3)^2}=$$
$$\frac{\left(\frac{3}{3m}\right)^2.\left(\frac{6}{3m}\right)^2\ldots\ldots\ldots\left(\frac{3n-3}{3m}\right)^2}{\left(\frac{1}{3m}\right)\left(\frac{2}{3m}\right)\left(\frac{4}{3m}\right)\left(\frac{5}{3m}\right)\left(\frac{7}{3m}\right)\ldots\ldots\left(\frac{3n-2}{3m}\right)\left(\frac{3n-1}{3m}\right)}.$$

In genere autem si communis divisor sit d et exponens dn, habebitur

$$\left[d.2d.3d\ldots.(dm-d)\left(\frac{d}{dm}\right)\left(\frac{2d}{dm}\right)\left(\frac{3d}{dm}\right)\cdots\cdots\left(\frac{dn-d}{dm}\right)\right]^d=$$
$$1.2.3.4\ldots.(dm-1)\left(\frac{1}{dm}\right)\left(\frac{2}{dm}\right)\left(\frac{3}{dm}\right)\cdots\cdots\left(\frac{dn-1}{dm}\right),$$

quae aequatio facile ad quosvis casus accommodari potest, unde sequens Theorema notari meretur.

Theorema.

§. 59. Si α fuerit divisor communis numerorum m et n, haecque formula $\left(\frac{p}{q}\right)$ denotet valorem integralis

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$$

ab $x=0$ usque ad $x=1$ extensi, erit

$$\left[\alpha.2\alpha.3\alpha\ldots\ldots(m-\alpha)\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\cdots\cdots\left(\frac{n-\alpha}{m}\right)\right]^{\alpha}=$$
$$1.2.3\ldots\ldots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\cdots\left(\frac{n-1}{m}\right).$$

Demonstratio.

Ex praecedente scholio veritas hujus theorematis perspicitur, cum enim ibi divisor communis esset $= d$, binique numeri propositi dm et dn, horum loco hic scripsi m et n, loco divisoris eorum autem d litteram α, quam divisoris rationem aequalitas enunciata ita complectitur, ut in progressione arithmetica α, 2α, 3α, etc. continuata occurrere assumantur ipsi numeri m et n ideoque etiam $m-\alpha$ et $n-\alpha$. Caeterum fateri cogor, hanc demonstrationem utpote inductioni potissimum innixam, neutiquam pro rigorosa haberi posse: cum autem nihilominus de ejus veritate simus convicti, hoc theorema eo majori attentione dignum videtur, interim tamen nullum est dubium, quin uberior hujusmodi formularum integralium evolutio tandem perfectam demonstrationem sit largitura, quod autem jam ante hanc veritatem nobis perspicere licuerit, insigne hinc specimen analyticae investigationis elucet.

Corollarium 1.

§. 60. Si loco signorum adhibitorum ipsas formulas integrales substituamus, theorema nostrum ita se habebit ut sit,

$$\alpha.2\alpha.3\alpha\ldots(m-\alpha)\int\frac{x^{\alpha-1}\partial x}{\sqrt[n]{(1-x^n)^{n-m}}}\cdot\int\frac{x^{2\alpha-1}\partial x}{\sqrt[n]{(1-x^n)^{n-m}}}\ldots\int\frac{x^{n-\alpha-1}\partial x}{\sqrt[n]{(1-x^n)^{n-m}}}=$$

$$\sqrt{1.2.3\ldots(m-1)}\int\frac{\partial x}{\sqrt[n]{(1-x^n)^{n-m}}}\cdot\int\frac{x\,\partial x}{\sqrt[n]{(1-x^n)^{n-m}}}\ldots\int\frac{x^{n-2}\partial x}{\sqrt[n]{(1-x^n)^{n-m}}}.$$

Corollarium 2.

§. 60. Vel si ad abbreviandum statuamus

$\sqrt[n]{(1-x^n)^{n-m}}=X$, erit

$$\alpha.2\alpha.3\alpha\ldots(m-\alpha)\int\frac{x^{\alpha-1}dx}{X}\cdot\int\frac{x^{2\alpha-1}dx}{X}\ldots\ldots\int\frac{x^{n-\alpha-1}dx}{X}=$$

$$\sqrt[\alpha]{1.2.3\ldots(m-1)}\int\frac{dx}{X}\cdot\int\frac{x\,dx}{X}\cdot\int\frac{x^2dx}{X}\ldots\ldots\int\frac{x^{n-2}dx}{X}.$$

Theorema generale.

§. 62. Si binorum numerorum m et n divisores communes sint α, β, γ etc. formulaque $\left(\frac{p}{q}\right)$ denotet valorem integralis

$$\int\frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$$

ab $x=0$ ad $x=1$ extensi; sequentes expressiones ex hujusmodi formulis integralibus formatae inter se erunt aequales

$$\left[\alpha.2\alpha.3\alpha\ldots\ldots(m-\alpha)\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\ldots\ldots\left(\frac{n-\alpha}{m}\right)\right]^{\alpha}=$$

$$\left[\beta.2\beta.3\beta\ldots\ldots(m-\beta)\left(\frac{\beta}{m}\right)\left(\frac{2\beta}{m}\right)\left(\frac{3\beta}{m}\right)\ldots\ldots\left(\frac{n-\beta}{m}\right)\right]^{\beta}=$$

$$\left[2\gamma.\gamma.3\gamma\ldots\ldots(m-\gamma)\left(\frac{\gamma}{m}\right)\left(\frac{2\gamma}{m}\right)\left(\frac{3\gamma}{m}\right)\ldots\ldots\left(\frac{n-\gamma}{m}\right)\right]^{\gamma}=\text{etc.}$$

Demonstratio.

Ex precedente Theoremate hujus veritas manifesto sequitur, cum quaelibet harum expressionum seorsim aequetur huic

$$1.2.3\ldots\ldots(m-1)\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\ldots\left(\frac{n-1}{m}\right),$$

quae unitati utpote minimo communi divisori numerorum m et n convenit. Tot igitur hujusmodi expressiones inter se aequales exhiberi possunt, quot fuerint divisores communes binorum numerorum m et n.

Corollarium 1.

§. 63. Cum sit haec formula $\left(\frac{n}{m}\right)=\frac{1}{m}$, ideoque $m\left(\frac{n}{m}\right)=1$, expressiones nostrae aequales succinctius hoc modo repraesentari possunt

$$\left[\alpha.2\alpha.3\alpha\ldots\ldots m\left(\frac{\alpha}{m}\right)\left(\frac{2\alpha}{m}\right)\left(\frac{3\alpha}{m}\right)\ldots\ldots\left(\frac{n}{m}\right)\right]^{\alpha}=$$

$$\left[\beta.2\beta.3\beta\ldots\ldots m\left(\frac{\beta}{m}\right)\left(\frac{2\beta}{m}\right)\left(\frac{3\beta}{m}\right)\ldots\ldots\left(\frac{n}{m}\right)\right]^{\beta}=$$

$$\left[\gamma.2\gamma.3\gamma\ldots\ldots m\left(\frac{\gamma}{m}\right)\left(\frac{2\gamma}{m}\right)\left(\frac{3\gamma}{m}\right)\ldots\ldots\left(\frac{n}{m}\right)\right]^{\gamma}=\text{etc.}$$

Etsi enim hic factorum numerus est auctus, tamen ratio compositionis facilius in oculos incurrit.

Corollarium 2.

§. 64. Si ergo sit $m=6$ et $n=12$, ob horum numerorum divisores communes 6, 3, 2, 1, quatuor sequentes formae inter se aequales habebuntur

$$\left[6\left(\tfrac{6}{6}\right)\left(\tfrac{12}{6}\right)\right]^{6}=\left[3.6\left(\tfrac{3}{6}\right)\left(\tfrac{6}{6}\right)\left(\tfrac{9}{6}\right)\left(\tfrac{12}{6}\right)\right]^{3}=$$

$$\left[2.4.6\left(\tfrac{2}{6}\right)\left(\tfrac{4}{6}\right)\left(\tfrac{6}{6}\right)\left(\tfrac{8}{6}\right)\left(\tfrac{10}{6}\right)\left(\tfrac{12}{6}\right)\right]^{2}=$$

$$1.2.3.4.5.6\left(\tfrac{1}{6}\right)\left(\tfrac{2}{6}\right)\left(\tfrac{3}{6}\right)\ldots\ldots\left(\tfrac{12}{6}\right).$$

Corollarium 3.

§. 65. Si ultima cum penultima combinetur, nascetur haec aequatio

$$\frac{1.3.5}{2.4.6} = \frac{\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{6}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{3}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{9}{6}\right)\left(\frac{11}{6}\right)},$$

ultima autem cum antepenultima comparata praebet

$$\frac{1.2.4.5}{3.3.5.5} = \frac{\left(\frac{3}{6}\right)\left(\frac{3}{6}\right)\left(\frac{6}{6}\right)\left(\frac{6}{6}\right)\left(\frac{9}{6}\right)\left(\frac{9}{6}\right)\left(\frac{12}{6}\right)\left(\frac{12}{6}\right)}{\left(\frac{1}{6}\right)\left(\frac{2}{6}\right)\left(\frac{4}{6}\right)\left(\frac{5}{6}\right)\left(\frac{7}{6}\right)\left(\frac{8}{6}\right)\left(\frac{10}{6}\right)\left(\frac{11}{6}\right)}.$$

Scholion.

§. 66. Infinitae igitur hinc consequuntur relationes inter formulas integrales formae

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} = \left(\frac{p}{q}\right),$$

quae eo magis sunt notatu dignae, quod singulari prorsus methodo ad eas hic sumus perducti. Ac si quis de earum veritate adhuc dubitet, observationes meas circa has formulas integrales consulat, indeque pro quovis casu oblato de veritate facile convincetur. Etsi autem illa tractatio huic confirmandae inservit, tamen relationes hic erutae eo majoris sunt momenti, quod in iis certus ordo cernitur, eaeque per omnes classes, quantumvis exponentem n accipere lubeat, facili negotio continuentur; in priori vero tractatione calculus pro classibus altioribus continuo fiat operosior et intricatior.

Supplementum continens demonstrationem Theorematis §. 53. propositi.

§. 67. Demonstrationem hanc altius peti convenit; sumatur scilicet aequatio §. 23. data, quae posito $f = 1$ et mutatis litteris est

$$\frac{\int dx\,(l\frac{1}{x})^{\nu-1}\times\int dx\,(l\frac{1}{x})^{\mu-1}}{\int dx\,(l\frac{1}{x})^{\mu+\nu-1}}=\varkappa\int\frac{x^{\varkappa\mu-1}\,dx}{(1-x^{\varkappa})^{1-\nu}},$$

eaque per reductiones notas hac forma repraesentetur

$$\frac{\int dx\,(l\frac{1}{x})^{\nu}\times\int dx\,(l\frac{1}{x})^{\mu}}{\int dx\,(l\frac{1}{x})^{\mu+\nu}}=\frac{\varkappa\mu\nu}{\mu+\nu}\int\frac{x^{\varkappa\mu-1}\,dx}{(1-x^{\varkappa})^{1-\nu}}.$$

Statuatur nunc $\nu=\frac{m}{n}$ et $\mu=\frac{\lambda}{n}$, tum vero $\varkappa=n$, ut habeamus

$$\frac{\int dx\,(l\frac{1}{x})^{\frac{m}{n}}\times\int dx\,(l\frac{1}{x})^{\frac{\lambda}{n}}}{\int dx\,(l\frac{1}{x})^{\frac{\lambda+m}{n}}}=\frac{\lambda m}{\lambda+m}\int\frac{x^{\lambda-1}\,dx}{\sqrt[n]{(1-x^{n})^{n-m}}}$$

quae brevitatis gratia, more supra usitato, ita concinne referatur

$$\frac{\left[\frac{m}{n}\right]\cdot\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+m}{m}\right]}=\frac{\lambda m}{\lambda+m}\cdot\left(\frac{\lambda}{m}\right).$$

Jam loco λ successive scribantur numeri 1, 2, 3, 4 ..... n omnesque hae aequationes, quarum numerus est $=n$, in se invicem ducantur, et aequatio resultans erit

$$\left[\frac{m}{n}\right]^{n}\cdot\frac{\left[\frac{1}{n}\right]\left[\frac{2}{n}\right]\left[\frac{3}{n}\right]\cdots\cdots\left[\frac{n}{n}\right]}{\left[\frac{m+1}{n}\right]\left[\frac{m+2}{n}\right]\left[\frac{m+3}{n}\right]\cdots\left[\frac{m+n}{n}\right]}=$$

$$m^{n}\cdot\frac{1}{m+1}\cdot\frac{2}{m+2}\cdot\frac{3}{m+3}\cdots\cdots\frac{n}{m+n}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\left(\frac{n}{m}\right)=$$

$$m^{n}\cdot\frac{1\cdot2\cdot3\cdots\cdots m}{(n+1)(n+2)(n+3)\cdots\cdots(n+m)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\cdots\left(\frac{n}{m}\right).$$

Simili autem modo pars prior transformetur ut sit

$$\left[\frac{m}{n}\right]^{n}\cdot\frac{\left[\frac{1}{n}\right]\left[\frac{2}{n}\right]\left[\frac{3}{n}\right]\cdots\cdots\left[\frac{m}{n}\right]}{\left[\frac{n+1}{n}\right]\left[\frac{n+2}{n}\right]\left[\frac{n+3}{n}\right]\cdots\cdots\left[\frac{n+m}{n}\right]},$$

cujus convenientia cum forma praecedente multiplicando per crucem, ut ajunt, sponte se prodit. Cum vero ex natura harum formularum sit

$$\left[\frac{n+1}{n}\right]=\frac{n+1}{n}\left[\frac{1}{n}\right],\ \left[\frac{n+2}{n}\right]=\frac{n+2}{n}\left[\frac{2}{n}\right];\ \left[\frac{n+3}{n}\right]=\frac{n+3}{n}\left[\frac{3}{n}\right],\ \text{etc.}$$

ob harum formularum numerum $= m$, evadet haec prior pars

$$\left[\frac{m}{n}\right]^n \cdot \frac{n^m}{(n+1)(n+2)(n+3)\ldots\ldots(n+m)},$$

quae cum aequalis sit parti alteri ante exhibitae

$$m^n \cdot \frac{1.\quad 2.\quad 3.\ldots\ldots\ldots m}{(n+1)(n+2)(n+3)\ldots\ldots(n+m)}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\ldots\left(\frac{n}{m}\right),$$

adpiscimur hanc aequationem

$$\left[\frac{m}{n}\right]^n = \frac{m^n}{n^m}\, 1.2.3\ldots\ldots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\ldots\left(\frac{n}{m}\right),$$

ita ut sit

$$\left[\frac{m}{n}\right] = m\sqrt[n]{\frac{1.2.3\ldots\ldots m}{n^m}\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\ldots\ldots\left(\frac{n}{m}\right)},$$

quae cum proposita in (§. 53.) ob $\left(\frac{n}{m}\right)=\frac{1}{m}$ omnino congruit, ex quo ejus veritas nunc quidem ex principiis certissimis est evicta.

Demonstratio Theorematis §. 59. propositi.

§. 68. Etiam hoc Theorema firmiori demonstratione indiget, quam ex aequalitate ante stabilita

$$\frac{\left[\frac{m}{n}\right]\cdot\left[\frac{\lambda}{n}\right]}{\left[\frac{\lambda+n}{n}\right]} = \frac{\lambda m}{\lambda+m}\left(\frac{\lambda}{m}\right)$$

ita adorno. Existente α communi divisore numerorum m et n, loco λ successive scribantur numeri α, 2α, 3α, etc. usque ad n, quorum multitudo est $=\frac{n}{\alpha}$, atque omnes aequalitates hoc modo resultantes in se invicem ducantur, ut prodeat haec aequatio

$$\left[\frac{m}{n}\right]^{\frac{n}{a}} \cdot \frac{\left[\frac{a}{n}\right]\left[\frac{2a}{n}\right]\left[\frac{3a}{n}\right]\cdots\cdots\cdots\left[\frac{n}{n}\right]}{\left[\frac{m+a}{n}\right]\left[\frac{m+2a}{n}\right]\left[\frac{m+3a}{n}\right]\cdots\left[\frac{m+n}{n}\right]} =$$

$$m^{\frac{n}{a}} \cdot \frac{1a}{m+a} \cdot \frac{2a}{m+2a} \cdot \frac{3a}{m+3a} \cdots\cdots \frac{n}{m+n}\left(\frac{a}{m}\right)\left(\frac{2a}{m}\right)\left(\frac{3a}{m}\right)\cdots\left(\frac{n}{m}\right).$$

Jam prior pars in hanc formam ipsi aequalem transmutetur

$$\left[\frac{m}{n}\right]^{\frac{n}{a}} \cdot \frac{\left[\frac{a}{n}\right]\left[\frac{2a}{n}\right]\left[\frac{3a}{n}\right]\cdots\cdots\cdots\left[\frac{m}{n}\right]}{\left[\frac{n+a}{n}\right]\left[\frac{n+2a}{n}\right]\left[\frac{n+3a}{n}\right]\cdots\left[\frac{n+m}{n}\right]},$$

quae ob $\left[\frac{n+a}{n}\right] = \frac{n+a}{n}\left[\frac{a}{n}\right]$, sicque de caeteris, reducitur ad hanc

$$\left[\frac{m}{n}\right]^{\frac{n}{a}} \frac{n}{n+a} \cdot \frac{n}{n+2a} \cdot \frac{n}{n+3a} \cdots\cdots \frac{n}{n+m}.$$

Posterior vero aequationis pars simili modo transformatur in

$$m^{\frac{n}{a}} \frac{a}{n+a} \cdot \frac{2a}{n+2a} \cdot \frac{3a}{n+3a} \cdots\cdots \frac{m}{n+m}\left(\frac{a}{m}\right)\left(\frac{2a}{m}\right)\left(\frac{3a}{m}\right)\cdots\cdots\left(\frac{n}{m}\right);$$

unde enascitur haec aequatio

$$\left[\frac{m}{n}\right]^{\frac{n}{a}} n^{\frac{m}{a}} = m^{\frac{n}{a}}.\alpha.2\alpha.3\alpha\ldots\ldots m\left(\frac{a}{m}\right)\left(\frac{2a}{m}\right)\left(\frac{3a}{m}\right)\cdots\cdots\left(\frac{n}{m}\right),$$

hincque

$$\left[\tfrac{m}{n}\right] = m\sqrt[n]{\frac{1}{m^n}\left[\alpha.2\alpha\ 3\alpha\ldots m\left(\frac{a}{m}\right)\left(\frac{2a}{m}\right)\left(\frac{3a}{m}\right)\cdots\cdots\left(\frac{n}{m}\right)\right]^{a}},$$

quae expressio cum praecedente comparata praebet hanc aequationem

$$\left[\alpha.2\alpha.3\alpha\ldots\ldots m\left(\frac{a}{m}\right)\left(\frac{2a}{m}\right)\left(\frac{3a}{m}\right)\cdots\cdots\left(\frac{n}{m}\right)\right]^{a} =$$

$$1.2.3\ldots\ldots m\left(\frac{1}{m}\right)\left(\frac{2}{m}\right)\left(\frac{3}{m}\right)\cdots\cdots\left(\frac{n}{m}\right),$$

quod de omnibus divisoribus communibus binorum numerorum m et n est intelligendum.

2) De valore formulae integralis $\int \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}} \cdot \frac{dz}{z} (lz)^{\mu}$, casu quo post integrationem ponitur $z = 1$. *Nov. Commentarii Acad. Imp. Sc. Petropolitanae. Tom XIX. Pag.* 30 — 64.

§. 69. Ex consideratione innumerabilium arcuum circularium, qui communem habent vel sinum vel tangentem, jam olim summationem duarum serierum infinitarum deduxi, quae ob summam generalitatem maxime memoratu dignae videbantur. Si enim litterae m et n numeros quoscunque denotant, posita diametri ratione ad peripheriam ut 1 ad π, illae duae summationes hoc modo se habebant

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.} = \frac{\pi}{n \sin. \frac{m\pi}{n}} \text{ et}$$

$$\frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.} = \frac{\pi}{n \text{ tang.} \frac{m\pi}{n}},$$

atque ex his duabus seriebus jam tum temporis elicueram summationes omnium serierum illarum, quarum denominatores secundum potestates numerorum naturalium progrediuntur, quemadmodum in introductione in analysin infinitorum et alibi fusius exposui. Nunc autem eaedem series me perduxerunt ad integrationem formulae in titulo expressae, quae eo magis attentione digna videtur, quod hujusmodi integrationes aliis methodis neutiquam exsequi liceat.

§. 70 Statim autem patet, has duas series infinitas oriri ex evolutione quarundam formularum integralium, si post integrationem quantitati variabili certus valor, veluti unitas tribuatur; ita prior series deducitur ex evolutione hujus formulae integralis

$$\int \frac{z^{m-1} + z^{n-m-1}}{1 + z^n} dz,$$

posterior vero ex evolutione istius

$$\int \frac{z^{m-1} - z^{n-m-1}}{1 - z^n} dz,$$

si quidem post integrationem statuatur $z = 1$. Deinceps autem ex ipsis principiis calculi integralis demonstravi, valorem integralis prioris harum duarum formularum, si quidem ponatur $z = 1$, reduci ad hanc formulam simplicem

$$\frac{\pi}{n \sin. \frac{m\pi}{n}},$$

integrale autem posterius, eodem casu $z = 1$, ad istam

$$\frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}},$$

ita, ut ex ipsis calculi integralis principiis certum sit esse

$$\int \frac{z^{m-1} + z^{n-m-1}}{1 + z^n} dz = \frac{\pi}{n \sin. \frac{m\pi}{n}} \text{ et}$$

$$\int \frac{z^{m-1} - z^{n-m-1}}{1 - z^n} dz = \frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}},$$

si quidem post integrationem ita institutam, ut integrale evanescat posito $z = 0$, statuatur $z = 1$.

§. 71. Quo jam hanc duplicem integrationem ad formam propositam reducamus, faciamus $n = 2\lambda$ et $m = \lambda - \omega$, unde binae illae series infinitae hanc induent formam:

$$\frac{1}{\lambda - \omega} + \frac{1}{\lambda + \omega} - \frac{1}{3\lambda - \omega} - \frac{1}{3\lambda + \omega} + \frac{1}{5\lambda - \omega} + \frac{1}{5\lambda + \omega} - \text{etc. et}$$

$$\frac{1}{\lambda - \omega} - \frac{1}{\lambda + \omega} + \frac{1}{3\lambda - \omega} - \frac{1}{3\lambda + \omega} + \frac{1}{5\lambda - \omega} - \frac{1}{5\lambda + \omega} + \text{etc.}$$

harum igitur serierum prioris summa erit

$$\frac{\pi}{2\lambda \sin. \frac{\pi(\lambda-\omega)}{2\lambda}} = \frac{\pi}{2\lambda \cos. \frac{\pi\omega}{2\lambda}},$$

posterioris vero summa erit

$$\frac{\pi}{2\lambda \operatorname{tang}. \frac{\pi(\lambda-\omega)}{2\lambda}} = \frac{\pi}{2\lambda \operatorname{cotang}. \frac{\pi\omega}{2\lambda}} = \frac{\pi \operatorname{tang}. \frac{\pi\omega}{2\lambda}}{2\lambda}.$$

Quod si ergo brevitatis gratia ponamus

$$\frac{\pi}{2\lambda \cos. \frac{\pi\omega}{2\lambda}} = S, \text{ et } \frac{\pi}{2\lambda} \operatorname{tang}. \frac{\pi\omega}{2\lambda} = T,$$

habebimus sequentes duas integrationes

$$\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} = S, \text{ et}$$

$$\int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} = T.$$

§. 72. Circa has binas integrationes ante omnia observo, eas perinde locum habere, sive pro litteris λ et ω accipiantur numeri integri, sive fracti. Sint enim λ et ω numeri fracti quicunque, qui evadant integri, si multiplicentur per α, quo posito fiat $z = x^\alpha$, eritque $\frac{dz}{z} = \frac{\alpha dx}{x}$, et potestas quaecunque $z^\delta = x^{\alpha\delta}$; prior igitur formula erit

$$\int \frac{x^{\alpha(\lambda-\omega)} + x^{\alpha(\lambda+\omega)}}{1+x^{2\alpha\lambda}} \cdot \frac{\alpha dx}{x},$$

ubi, cum jam omnes exponentes sint numeri integri, valor hujus formulae posito post integrationem $x = 1$, quandoquidem tunc etiam fit $z = 1$, a praecedente eo tantum differt, quod hic habemus $\alpha\lambda$ et $\alpha\omega$ loco λ et ω, ac praeterea hic adsit factor α, quocirca valor istius formulae erit

$$\alpha.\frac{\pi}{2\alpha\lambda\cos.\frac{\pi\omega}{2\lambda}}=\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}},$$

qui ergo valor est $= S$ prorsus ut ante; quae identitas etiam manifesto est in altera formula, unde patet, etiamsi pro λ et ω fractiones quaecunque accipiantur, integrationem hic exhibitam nihilo minus locum esse habituram; quae circumstantia probe notari meretur, quoniam in sequentibus litteram ω tanquam variabilem sumus tractaturi.

§. 73. Postquam igitur binae istae formulae integrales litteris S et T indicatae fuerint integratae, ita ut evanescant posito $z=0$, integralia spectari poterunt non solum ut functiones quantitatis z, sed etiam ut functiones binarum variabilium z et ω, quandoquidem numerum ω tanquam quantitatem variabilem tractare licet, quin etiam exponentem λ pro quantitate variabili habere liceret; sed quia hinc formulae integrales alius generis essent proditurae, atque hic contemplari constitui, solam quantitatem ω, praeter ipsam variabilem z, hic ut quantitatem variabilem sum tractaturus.

§. 74. Cum igitur sit

$$S=\int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}$$

in qua integratione sola z ut variabilis spectatur, erit utique secundum signandi morem jam satis usu receptum

$$\left(\frac{dS}{dz}\right)=\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{1}{z};$$

haec jam formula denuo differentietur, posita sola littera ω variabili, eritque

$$\left(\frac{ddS}{dzd\omega}\right)=\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{1}{z}lz,$$

quae formula ducta in dz, ac denuo integrata sola z habita pro variabili, dabit

$$\int dz\left(\frac{ddS}{dzd\omega}\right)=\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}lz,$$

ubi notetur esse

$$S=\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}};$$

ita ut hinc deducamus

$$\left(\frac{dS}{d\omega}\right)=\frac{\pi\pi\sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2},$$

hoc igitur valore substituto, nanciscimur hanc integrationem

$$\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}lz=\frac{\pi\pi\sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2}.$$

§. 75. Quod si jam altera formula simili modo tractetur, cum sit $T=\frac{\pi}{2\lambda}\text{tang.}\frac{\pi\omega}{2\lambda}$, erit

$$\left(\frac{dT}{d\omega}\right)=\frac{\pi\pi}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2};$$

ex formula autem integrali erit

$$\left(\frac{dT}{d\omega}\right)=\int\frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}lz,$$

unde colligimus sequentem integrationem

$$\int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}lz=\frac{-\pi\pi}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2}.$$

§. 76. Quoniam litteras S et T etiam per series expressas dedimus, erit etiam per similes series

$$\left(\frac{dS}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} - \frac{1}{(\lambda+\omega)^2} - \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} - \text{etc.}$$

$$= \frac{\pi\pi \sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2}.$$

Similique modo etiam pro altera serie

$$\left(\frac{dT}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} + \frac{1}{(\lambda+\omega)^2} + \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} + \text{etc.}$$

$$= \frac{\pi\pi}{4\lambda\lambda\left(\cos.\frac{\pi\omega}{2\lambda}\right)^2};$$

sicque summas harum serierum quoque duplici modo repraesentavimus, scilicet per formulam evolutam quantitatem π involventem, tum vero etiam per formulam integralem, quae ita est comparata, ut ejus integrale nulla methodo adhuc consueta assignari possit.

§. 77. Applicemus has integrationes ad aliquot casus particulares: ac primo quidem sumamus $\omega = 0$, quo quidem casu prior integratio sponte in oculos incurrit, at posterior praebet

$$\int \frac{2z^\lambda}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz = -\frac{\pi\pi}{4\lambda\lambda}, \text{ sive}$$

$$\int \frac{z^{\lambda-1}dz\,lz}{1-z^{2\lambda}} = -\frac{\pi\pi}{8\lambda\lambda};$$

hincque simul istam summationem adipiscimur

$$\frac{1}{\lambda\lambda} + \frac{1}{\lambda\lambda} + \frac{1}{9\lambda\lambda} + \frac{1}{9\lambda\lambda} + \frac{1}{25\lambda\lambda} + \frac{1}{25\lambda\lambda} + \text{etc.} = \frac{\pi\pi}{4\lambda\lambda}, \text{ sive}$$

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{etc.} = \frac{\pi\pi}{8},$$

id quod jam dudum a me est demonstratum.

§. 78. Hic statim patet, perinde esse, quinam numerus pro λ accipiatur; sit igitur $\lambda = 1$, et habebitur ista integratio

$$\int \frac{dz\, lz}{1 - z^2} = -\frac{\pi\pi}{8};$$

ex qua sequentia integralia simpliciora

$$\int \frac{dz\, lz}{1 - z} \text{ et } \int \frac{dz\, lz}{1 + z}$$

derivare licet ope hujus ratiocinii; statuatur

$$\int \frac{z\, dz\, lz}{1 - zz} = P,$$

et posito $zz = v$, ut sit $zdz = \frac{dv}{2}$ et $lz = \frac{1}{2} lv$, prodibit

$$\frac{1}{4} \int \frac{dv\, lv}{1 - v} = P,$$

si scilicet post integrationem fiat $v = 1$, quippe quo casu etiam sit $z = 1$; sic igitur erit

$$\int \frac{dv\, lv}{1 - v} = 4P:$$

nunc prior illa formula addatur ad inventam, eritque

$$\int \frac{dz\, lz + z\, dz\, lz}{1 - zz} = P - \frac{\pi\pi}{8},$$

haec autem formula sponte reducitur ad hanc

$$\int \frac{dz\, lz}{1 - z} = P - \frac{\pi\pi}{8},$$

modo autem vidimus esse

$$\int \frac{dv\, lv}{1 - v} \text{ sive } \int \frac{dz\, lz}{1 - z} = 4P, \text{ ita ut sit } 4P = P - \frac{\pi\pi}{8},$$

unde manifesto sit $P = -\frac{\pi\pi}{24}$, ex quo sequitur fore

$$\int \frac{dz\, lz}{1 - z} = -\frac{\pi\pi}{6};$$

simili modo erit

$$\int \frac{dz\, lz - z\, dz\, lz}{1 - zz} = -P - \frac{\pi\pi}{8} = -\frac{\pi\pi}{12},$$

quae, supra et infra per $1 - z$ dividendo, praebet

$$\int \frac{dz\, lz}{1 + z} = -\frac{\pi\pi}{12},$$

quare jam adepti sumus tres integrationes memoratu maxime dignas

$$\text{I.} \quad \int \frac{dz\,lz}{1+z} = -\frac{\pi\pi}{12},$$

$$\text{II.} \quad \int \frac{dz\,lz}{1-z} = -\frac{\pi\pi}{6},$$

$$\text{III.} \quad \int \frac{dz\,lz}{1-zz} = -\frac{\pi\pi}{8},$$

quibus adiungi potest

$$\text{IV.} \quad \int \frac{z\,dz\,lz}{1-zz} = -\frac{\pi\pi}{24}.$$

§. 79. Quemadmodum igitur hae formulae ex ipsis calculi integralis principiis sunt deductae, ita etiam earum veritas per resolutionem in series facile comprobatur; cum enim sit

$$\frac{1}{1+z} = 1 - z + zz - z^3 + z^4 - z^5 + \text{etc.},$$

et in genere

$$\int z^n\,dz\,lz = \frac{z^{n+1}}{n+1}\,lz - \frac{z^{n+1}}{(n+1)^2},$$

qui valor posito $z = 1$ reducitur ad $\frac{-1}{(n+1)^2}$, patet fore

$$\int \frac{dz\,lz}{1+z} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \text{etc.} = -\frac{\pi\pi}{12}, \text{ sive}$$

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \text{etc.} = \frac{\pi\pi}{12},$$

simili modo ob

$$\frac{1}{1-z} = 1 + z + zz + z^3 + z^4 + \text{etc. erit}$$

$$\int \frac{dz\,lz}{1-z} = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \frac{1}{25} - \text{etc.} = -\frac{\pi\pi}{6}, \text{ seu}$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.} = \frac{\pi\pi}{6},$$

tum vero ob

$$\frac{1}{1-zz} = 1 + zz + z^4 + z^6 + z^8 + \text{etc. erit}$$

$$\int \frac{dz\,lz}{1-zz} = -1 - \frac{1}{9} - \frac{1}{25} - \frac{1}{49} - \frac{1}{81} - \text{etc.} = -\frac{\pi\pi}{8}, \text{ sive}$$

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{etc.} = \frac{\pi\pi}{8}.$$

Eodem modo etiam

$$\int\frac{z\,dz\,lz}{1-zz}=-\frac{1}{4}-\frac{1}{16}-\frac{1}{36}-\frac{1}{64}-\text{etc.}=-\frac{\pi\pi}{24},\ \text{sive}$$

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\frac{1}{64}+\text{etc.}=\frac{\pi\pi}{24},$$

quae quidem summationes jam sunt notissimae. Neque tamen quisquam adhuc methodo directa ostendit esse

$$\int\frac{dz\,lz}{1+z}=-\frac{\pi\pi}{12}.$$

§. 80. Ponamus nunc $\omega=1$, et nostrae integrationes has induent formas

$$1^0.\ \int\frac{-z^{\lambda-2}(1-zz)\,dz\,lz}{1+z^{2\lambda}}=\frac{\pi\pi\sin.\frac{\pi}{2\lambda}}{4\lambda\lambda(\cos.\frac{\pi}{2\lambda})^2}\ \text{et}$$

$$2^0.\ \int\frac{-z^{\lambda-2}(1+zz)\,dz\,lz}{1-z^{2\lambda}}=+\frac{\pi\pi}{4\lambda\lambda(\cos.\frac{\pi}{2\lambda})^2},$$

unde pro diversis valoribus ipsius λ, quos quidem binario non minores accipere licet, sequentes obtinentur integrationes

I⁰. si $\lambda=2$, erit

$$1^0.\ \int\frac{-(1-zz)\,dz\,lz}{1+z^4}=\frac{\pi\pi}{8\sqrt{2}},$$

$$2^0.\ \int\frac{-(1+zz)\,dz\,lz}{1-z^4}=+\frac{\pi\pi}{8},\ \text{sive}\int\frac{-dz\,lz}{1-zz}=+\frac{\pi\pi}{8}.$$

II⁰. si $\lambda=3$, habebimus

$$1^0.\ \int\frac{-z(1-zz)\,dz\,lz}{1+z^6}=\frac{\pi\pi}{54},\ \text{et}$$

$$2^0.\ \int\frac{-z(1+zz)\,dz\,lz}{1-z^6}=\int\frac{-z\,dz\,lz}{1-zz+z^4}=\frac{\pi\pi}{27}.$$

Hae autem duae formulae ponendo $zz=v$, abibunt in sequentes

$$1^0.\ \int\frac{-dv(1-v)\,lv}{1+v^3}=\frac{2\pi\pi}{27},\ \text{et}$$

$$2^0.\ \int\frac{dv\,lv}{1-v+vv}=\frac{4\pi\pi}{27}.$$

III°. Sit $\lambda = 4$ et consequemur

$$1^0. \int \frac{-zz(1-zz)\,dz\,lz}{1+z} = \frac{\pi\pi\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}}{16(2+\sqrt{2})} = \frac{\pi\pi\sqrt{(2-\sqrt{2})}}{32(2+\sqrt{2})} \text{ et}$$

$$2^0. \int \frac{-zz(1+zz)\,dz\,lz}{1-z^8} = \int \frac{-zz\,dz\,lz}{(1-zz)(1+z^4)} = \frac{\pi\pi}{16(2+\sqrt{2})},$$

quae postrema forma reducitur ad hanc

$$\int -\frac{dz\,lz}{1-zz} + \int \frac{(1-zz)\,dz\,lz}{1+z^4} = \frac{\pi\pi}{8(2+\sqrt{2})},$$

est vero $\int \frac{-dz\,lz}{1-zz} = \frac{\pi\pi}{8}$, unde reperitur

$$\int \frac{(1-zz)\,dz\,lz}{1+z^4} = -\frac{\pi\pi(1+\sqrt{2})}{8(2+\sqrt{2})} = -\frac{\pi\pi}{8\sqrt{2}},$$

qui valor jam in superiori casu $\lambda = 2$ est inventus.

§. 81. Nihil autem impedit, quo minus etiam faciamus $\lambda = 1$, dummodo integralia ita capiantur ut evanescant, posito $z = 0$, tum autem reperiemus

$$1^0. \int \frac{-(1-zz)\,dz\,lz}{z(1+zz)} = \infty \text{ et}$$

$$2^0. \int \frac{-(1+zz)\,dz\,lz}{z(1-zz)} = \infty,$$

unde hinc nihil concludere licet. Caeterum etiam nostrae series supra inventae manifesto declarant, earum summas esse infinitas, quandoquidem primus terminus utriusque $\frac{1}{(\lambda-\omega)^2}$ fit infinitus, sumto uti fecimus $\lambda = 1$ et $\omega = 1$.

§. 82. His casibus evolutis, ulterius progrediamur ac ponamus formulas integrales inventas

$$\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}\,lz = S' \text{ et}$$

$$\int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z}\,lz = T'.$$

ita ut sit

$$S' = \frac{\pi\pi \sin. \frac{\pi\omega}{2\lambda}}{4\lambda\lambda \left(\cos. \frac{\pi\omega}{2\lambda}\right)^2}, \text{ et } T' = \frac{\pi\pi}{4\lambda\lambda \left(\cos. \frac{\pi\omega}{2\lambda}\right)^2},$$

atque ut ante jam differentiemus solo numero ω pro variabili habito: quo facto sequentes nanciscimur integrationes

$$\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \left(\frac{dS'}{d\omega}\right), \text{ et}$$

$$\int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \left(\frac{dT'}{d\omega}\right).$$

Hunc in finem ponamus brevitatis ergo angulum $\frac{\pi\omega}{2\lambda} = \varphi$, ut sit

$$S' = \frac{\pi\pi \sin. \varphi}{4\lambda\lambda \cos.^2 \varphi} = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{\sin. \varphi}{\cos.^2 \varphi}, \text{ et}$$

$$T' = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{1}{\cos.^2 \varphi},$$

ac reperiemus

$$d \cdot \frac{\sin. \varphi}{\cos.^2 \varphi} = \frac{\cos.^2 \varphi + 2 \sin.^2 \varphi}{\cos.^3 \varphi} d\varphi = \frac{1 + \sin.^2 \varphi}{\cos.^3 \varphi} d\varphi,$$

ubi est $d\varphi = \frac{\pi d\omega}{2\lambda}$; unde colligimus

$$\left(\frac{dS'}{d\omega}\right) = \frac{\pi^3}{8\lambda^3} \left(\frac{1 + \left(\sin. \frac{\pi\omega}{2\lambda}\right)^2}{\left(\cos. \frac{\pi\omega}{2\lambda}\right)^3}\right) = \frac{\pi^3}{8\lambda^3} \left(\frac{2}{\left(\cos. \frac{\pi\omega}{2\lambda}\right)^3} - \frac{1}{\cos. \frac{\pi\omega}{2\lambda}}\right);$$

simili modo ob $T' = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{1}{\cos.^2 \varphi}$, erit

$$d \cdot \frac{1}{\cos.^2 \varphi} = \frac{2 d\varphi \sin. \varphi}{\cos.^3 \varphi},$$

hincque

$$\left(\frac{dT'}{d\omega}\right) = \frac{\pi^3}{8\lambda^3} \cdot \frac{2 \sin. \frac{\pi\omega}{2\lambda}}{\left(\cos. \frac{\pi\omega}{2\lambda}\right)^3}.$$

Consequenter integrationes hinc natae erunt

$$\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}(lz)^2=\frac{\pi^3}{8\lambda^3}\left(\frac{2}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}-\frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right),$$

$$\int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}(lz)^2=\frac{\pi^3}{8\lambda^3}\,\frac{2\sin.\frac{\pi\omega}{2\lambda}}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}.$$

§. 83. Si jam eodem modo series §. 76. inventas denuo differentiemus, sumta sola ω variabili, perveniamus ad sequentes summationes

$$\frac{\pi^3}{8\lambda^3}\left\{\frac{2}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}-\frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right\}=+\frac{2}{(\lambda-\omega)^3}+\frac{2}{(\lambda+\omega)^3}-\frac{2}{(3\lambda-\omega)^3}-\frac{2}{(3\lambda+\omega)^3}$$
$$+\frac{2}{(5\lambda-\omega)^3}+\frac{2}{(5\lambda+\omega^3)}-\text{etc.}$$

$$\frac{\pi^3}{8\lambda^3}\cdot\frac{2\sin.\frac{\pi\omega}{2\lambda}}{\left(\cos.\frac{\pi\omega}{2\lambda}\right)^3}=\frac{2}{(\lambda-\omega)^3}-\frac{2}{(\lambda+\omega)^3}+\frac{2}{(3\lambda-\omega)^3}-\frac{2}{(3\lambda+\omega)^3}+\frac{2}{(5\lambda-\omega)^3}-\text{etc.}$$

§. 84. Si jam hic sumamus $\omega=0$ et $\lambda=1$, prior integratio hanc induit formam

$$\int\frac{2dz(lz)^2}{1+zz}=\frac{\pi^3}{8}=\frac{2}{1^3}+\frac{2}{1^3}-\frac{2}{3^3}-\frac{2}{3^3}+\frac{2}{5^3}+\frac{2}{5^3}-\frac{2}{7^3}-\frac{2}{7^3}+\text{etc.}$$

ita ut sit

$$\frac{1}{1^3}-\frac{1}{3^3}+\frac{1}{5^3}-\frac{1}{7^3}+\frac{1}{9^3}-\frac{1}{11^3}+\text{etc.}=\frac{\pi^3}{32},$$

quemadmodum jam dudum demonstravi. Altera autem integratio hoc casu in nihilum abit. Ex priori vero integrali

$$\int\frac{dz\,lz^2}{1+zz}=\frac{\pi^3}{16},$$

alia derivare non licet, uti supra fecimus ex formula

$$\int\frac{dz\,lz}{1-zz}=-\frac{\pi\pi}{8},$$

propterea quod hic denominator $1+zz$ non habet factores reales.

§. 85. Sumamus igitur $\lambda = 2$ et $\omega = 1$, ac prior integratio dabit

$$\int \frac{(1+zz)\,dz\,(lz)^2}{1+z^4} = \frac{3\pi^3}{32\sqrt{2}};$$

series autem hinc nata erit

$$\frac{2}{1^3} + \frac{2}{3^3} - \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} + \frac{2}{11^3} - \text{etc.},$$

ita ut sit

$$\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - \text{etc.} = \frac{3\pi^3}{64\sqrt{2}},$$

quae superiori addita praebet

$$\frac{1}{1^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{23^3} + \text{etc.} = \frac{\pi^3(3+\sqrt{2})}{128\sqrt{2}}.$$

Altera vero integratio hoc casu dat

$$\int \frac{dz\,(lz)^2}{1+zz} = \frac{\pi^3}{16},$$

quae cum paragrapho praecedenti perfecte congruit, quemadmodum etiam series hinc nata est

$$\frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} - \text{etc.}$$

§. 86. Quo autem facilius sequentes integrationes per continuam differentiationem elicere valeamus, eas in genere repraesentemus; et cum pro priore sit

$$S = \frac{\pi}{2\lambda \cos.\frac{\pi\omega}{2\lambda}},$$

integrationes hinc ortae ita ordine procedent

$$\text{I.}\quad \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} = S,$$

$$\text{II.}\quad \int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}\, lz = \left(\frac{dS}{d\omega}\right),$$

$$\text{III.}\quad \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}\,(lz)^2 = \left(\frac{ddS}{d\omega^2}\right),$$

$$\text{IV.} \quad \int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^3 = \left(\frac{d^3 S}{d\omega^3}\right),$$

$$\text{V.} \quad \int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^4 = \left(\frac{d^4 S}{d\omega^4}\right),$$

$$\text{VI.} \quad \int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^5 = \left(\frac{d^5 S}{d\omega^5}\right),$$

$$\text{VII.} \quad \int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^6 = \left(\frac{d^6 S}{d\omega^6}\right).$$

etc. etc. etc.

§. 87. Pro his differentiationibus continuis facilius absoluendis, ponamus brevitatis ergo $\frac{\pi}{2\lambda} = \alpha$, ut sit

$$S = \frac{\alpha}{\cos. \alpha\omega};$$

tum vero sit

$$\sin \alpha\omega = p \text{ et } \cos. \alpha\omega = q,$$

eritque

$$dp = \alpha q\, d\omega \text{ et } dq = -\alpha p\, d\omega.$$

Praeterea vero notetur esse

$$d \cdot \frac{p^n}{q^{n+1}} = \alpha\, d\omega \left\{ \frac{n p^{n-1}}{q^n} + \frac{(n+1) p^{n+1}}{q^{n+2}} \right\}.$$

His praemissis ob $S = \alpha \cdot \frac{1}{q}$ erit

$$\left(\frac{dS}{d\omega}\right) = \alpha^2 \cdot \frac{p}{qq}, \text{ deinde}$$

$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3 \left(\frac{1}{q} + \frac{2pp}{q^3}\right), \text{ porro}$$

$$\left(\frac{d^3 S}{d\omega^3}\right) = \alpha^4 \left(\frac{5p}{qq} + \frac{6p^3}{q^4}\right),$$

$$\left(\frac{d^4 S}{d\omega^4}\right) = \alpha^5 \left(\frac{5}{q} + \frac{28pp}{q^3} + \frac{24p^4}{q^5}\right),$$

$$\left(\frac{d^5 S}{d\omega^5}\right) = \alpha^6 \left(\frac{61p}{qq} + \frac{180p^3}{q^4} + \frac{120p^5}{q^6}\right),$$

$$\left(\frac{d^6 S}{d\omega^6}\right) = \alpha^7 \left(\frac{61}{q} + \frac{662\,pp}{q^3} + \frac{1520\,p^4}{q^5} + \frac{720\,p^6}{q^7}\right),$$
$$\left(\frac{d^7 S}{d\omega^7}\right) = \alpha^8 \left(\frac{1385\,p}{qq} + \frac{7266\,p^3}{q^4} + \frac{10920\,p^5}{q^6} + \frac{5040\,p^7}{q^8}\right), \text{ etc.}$$

hi autem valores ob $pp = 1 - qq$ ad sequentes reducuntur

$$S = \alpha \cdot \frac{1}{q},$$
$$\left(\frac{dS}{d\omega}\right) = \alpha^2 p \cdot \frac{1}{qq},$$
$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3 \left(\frac{1.2}{q^3} - \frac{1}{q}\right),$$
$$\left(\frac{d^3 S}{d\omega^3}\right) = \alpha^4 p \left(\frac{1.2.3}{q^4} - \frac{1}{qq}\right),$$
$$\left(\frac{d^4 S}{d\omega^4}\right) = \alpha^5 \left(\frac{1.2.3.4}{q^5} - \frac{20}{q^3} + \frac{1}{q}\right),$$
$$\left(\frac{d^5 S}{d\omega^5}\right) = \alpha^6 p \left(\frac{1.2.3.4.5}{q^6} - \frac{60}{q^4} + \frac{1}{qq}\right),$$
$$\left(\frac{d^6 S}{d\omega^6}\right) = \alpha^7 \left(\frac{1 \ldots\ldots 6}{q^7} - \frac{840}{q^5} + \frac{182}{q^3} - \frac{1}{q}\right), \text{ etc.}$$

§. 88. Has posteriores formas reperire licet ope horum duorum lemmatum

$$\text{I.} \quad \partial \cdot \frac{1}{q^{n+1}} = \alpha\, d\omega \frac{(n+1)\,p}{q^{n+2}}, \text{ et}$$
$$\text{II.} \quad \partial \cdot \frac{p}{q^{n+1}} = \alpha\, d\omega \left\{\frac{n+1}{q^{n+2}} - \frac{n}{q^n}\right\}.$$

hinc enim reperiemus

$$S = \alpha \frac{1}{q},$$
$$\left(\frac{dS}{d\omega}\right) = \alpha^2 \cdot \frac{p}{qq},$$
$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3 \left(\frac{2}{q^3} - \frac{1}{q}\right),$$
$$\left(\frac{d^3 S}{d^3\omega}\right) = \alpha^4 \left(\frac{2.3\,p}{q^4} - \frac{p}{qq}\right),$$
$$\left(\frac{d^4 S}{d\omega^4}\right) = \alpha^5 \left(\frac{2.3.4}{q^5} - \frac{20}{q^3} + \frac{1}{q}\right),$$
$$\left(\frac{d^5 S}{d\omega^5}\right) = \alpha^6 \left(\frac{2.3.4.5\,p}{q^6} - \frac{3.20\,p}{q^4} + \frac{p}{qq}\right),$$

$$\left(\frac{d^6 S}{d\omega^6}\right) = \alpha^7 \left(\frac{2.3.4.5.6}{q^7} - \frac{840}{q^5} + \frac{182}{q^3} - \frac{1}{q}\right),$$

$$\left(\frac{d^7 S}{d\omega^7}\right) = \alpha^8 \left(\frac{2 \ldots . 7\, p}{q^8} - \frac{5.840\, p}{q^6} + \frac{3.182\, p}{q^4} - \frac{p}{qq}\right) \text{ etc.}$$

§. 89. Ipsae autem series his formulis respondentes erunt

$$S = \frac{1}{\lambda-\omega} + \frac{1}{\lambda+\omega} - \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} + \frac{1}{5\lambda+\omega} - \text{etc,}$$

$$\left(\frac{dS}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} - \frac{1}{(\lambda+\omega)^2} - \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} - \frac{1}{(5\lambda+\omega)^2} - \text{etc.}$$

$$\left(\frac{ddS}{d\omega^2}\right) = \frac{1.2}{(\lambda-\omega)^3} + \frac{1.2}{(\lambda+\omega)^3} - \frac{1.2}{(3\lambda-\omega)^3} - \frac{1.2}{(3\lambda+\omega)^3} + \frac{1.2}{(5\lambda-\omega)^3} + \text{etc.}$$

$$\left(\frac{d^3 S}{d\omega^3}\right) = \frac{1.2.3}{(\lambda-\omega)^4} - \frac{1.2.3}{(\lambda+\omega)^4} - \frac{1.2.3}{(3\lambda-\omega)^4} + \frac{1.2.3}{(3\lambda+\omega)^4} + \frac{1.2.3}{(5\lambda-\omega)^4} - \text{etc.}$$

$$\left(\frac{d^4 S}{d\omega^4}\right) = \frac{1.2.3.4}{(\lambda-\omega)^5} + \frac{1.2.3.4}{(\lambda+\omega)^5} - \frac{1.2.3.4}{(3\lambda-\omega)^5} - \frac{1.2.3.4}{(3\lambda+\omega)^5} + \frac{1.2.3.4}{(5\lambda-\omega)^5} + \text{etc.}$$

$$\left(\frac{d^5 S}{d\omega^5}\right) = \frac{1.2.3.4.5}{(\lambda-\omega)^6} - \frac{1.2.3.4.5}{(\lambda+\omega)^6} - \frac{1.2.3.4.5}{(3\lambda-\omega)^6} + \frac{1.2.3.4.5}{(3\lambda+\omega)^6} + \frac{1.2.3.4.5}{(5\lambda-\omega)^6} - \text{etc.}$$

$$\left(\frac{d^6 S}{d\omega^6}\right) = \frac{1 \ldots\ldots 6}{(\lambda-\omega)^7} + \frac{1 \ldots\ldots 6}{(\lambda+\omega)^7} - \frac{1 \ldots\ldots 6}{(3\lambda-\omega)^7} - \frac{1 \ldots\ldots 6}{(3\lambda+\omega)^7} + \frac{1 \ldots\ldots 6}{(5\lambda-\omega)^7} + \text{etc.}$$

$$\left(\frac{d^7 S}{d\omega^7}\right) = \frac{1 \ldots\ldots 7}{(\lambda-\omega)^8} - \frac{1 \ldots\ldots 7}{(\lambda+\omega)^8} - \frac{1 \ldots\ldots 7}{(3\lambda-\omega)^8} + \frac{1 \ldots\ldots 7}{(3\lambda+\omega)^8} + \frac{1 \ldots\ldots 7}{(5\lambda-\omega)^8} - \text{etc.}$$

etc. etc. etc.

Circa hos autem valores probe meminisse opportet, esse

$$\alpha = \frac{\pi}{2\lambda},\ p = \sin. \alpha\omega = \sin. \frac{\pi\omega}{2\lambda}, \text{ et } q = \cos. \alpha\omega = \cos. \frac{\pi\omega}{2\lambda}.$$

§. 90. Eodem modo expediamus valores seu formulas integrales alterius generis, pro quibus est

$$T = \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda},$$

unde continuo differentiando oriuntur sequentes integrationes

$$\text{I.} \quad \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} = T,$$

$$\text{II.} \quad \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z}\, lz = \left(\frac{dT}{d\omega}\right),$$

$$\text{III.} \quad \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \left(\frac{ddT}{d\omega^2}\right),$$

$$\text{IV.} \quad \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^3 = \left(\frac{d^3T}{d\omega^3}\right),$$

$$\text{V.} \quad \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^4 = \left(\frac{d^4T}{d\omega^4}\right),$$

$$\text{VI.} \quad \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^5 = \left(\frac{d^5T}{d\omega^5}\right),$$

$$\text{VII.} \quad \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^6 = \left(\frac{d^6T}{d\omega^6}\right).$$

etc.

§. 91. Ponatur iterum $\frac{\pi}{2\lambda} = \alpha$, sin. $\alpha\omega = p$, et cos. $\alpha\omega = q$, ut sit

$$T = \frac{\alpha p}{q},$$

quae formula secundum lemmata §. 88. continuo differentiata dabit

$$T = \alpha \cdot \frac{p}{q},$$

$$\left(\frac{dT}{d\omega}\right) = \alpha^2 \cdot \frac{1}{qq},$$

$$\left(\frac{ddT}{d\omega^2}\right) = \alpha^3 \frac{2p}{q^3},$$

$$\left(\frac{d^3T}{d\omega^3}\right) = \alpha^4 \left(\frac{6}{q^4} - \frac{4}{qq}\right),$$

$$\left(\frac{d^4T}{d\omega^4}\right) = \alpha^5 \left(\frac{24p}{q^5} - \frac{8p}{q^3}\right),$$

$$\left(\frac{d^5T}{d\omega^5}\right) = \alpha^6 \left(\frac{120}{q^6} - \frac{120}{q^4} + \frac{16}{qq}\right),$$

$$\left(\frac{d^6T}{d\omega^6}\right) = \alpha^7 \left(\frac{720p}{q^7} - \frac{480p}{q^5} + \frac{32p}{q^3}\right),$$

$$\left(\frac{d^7T}{d\omega^7}\right) = \alpha^8 \left(\frac{5040}{q^8} - \frac{6720}{q^6} + \frac{2016}{q^4} - \frac{64}{qq}\right).$$

etc.

§. 92. Series autem infinitae, quae hinc nascuntur, erunt

$$T = \frac{1}{\lambda-\omega} - \frac{1}{\lambda+\omega} + \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} - \frac{1}{5\lambda+\omega} + \text{etc.}$$

$$\left(\frac{dT}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} + \frac{1}{(\lambda+\omega)^2} + \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} + \text{etc.}$$

$$\left(\frac{ddT}{d\omega^2}\right)=\frac{1.2}{(\lambda-\omega)^3}-\frac{1.2}{(\lambda+\omega)^3}+\frac{1.2}{(3\lambda-\omega)^3}-\frac{1.2}{(3\lambda+\omega)^3}+\frac{1.2}{(5\lambda-\omega)^3}-\text{etc.}$$

$$\left(\frac{d^3T}{d\omega^3}\right)=\frac{1.2.3}{(\lambda-\omega)^4}+\frac{1.2.3}{(\lambda+\omega)^4}+\frac{1.2.3}{(3\lambda-\omega)^4}+\frac{1.2.3}{(3\lambda+\omega)^4}+\frac{1.2.3}{(5\lambda-\omega)^4}+\text{etc.}$$

$$\left(\frac{d^4T}{d\omega^4}\right)=\frac{1.2.3.4}{(\lambda-\omega)^5}-\frac{1.2.3.4}{(\lambda+\omega)^5}+\frac{1.2.3.4}{(3\lambda-\omega)^5}-\frac{1.2.3.4}{(3\lambda+\omega)^5}+\frac{1.2.3.4}{(5\lambda-\omega)^5}-\text{etc.}$$

$$\left(\frac{d^5T}{d\omega^5}\right)=\frac{1.2.3.4.5}{(\lambda-\omega)^6}+\frac{1.2.3.4.5}{(\lambda+\omega)^6}+\frac{1.2.3.4.5}{(3\lambda-\omega)^6}+\frac{1.2.3.4.5}{(3\lambda+\omega)^6}+\frac{1.2.3.4.5}{(5\lambda-\omega)^6}+\text{etc.}$$

$$\left(\frac{d^6T}{d\omega^6}\right)=\frac{1\ldots\ldots6}{(\lambda-\omega)^7}-\frac{1\ldots\ldots6}{(\lambda+\omega)^7}+\frac{1\ldots\ldots6}{(3\lambda-\omega)^7}-\frac{1\ldots\ldots6}{(3\lambda+\omega)^7}+\frac{1\ldots\ldots6}{(5\lambda-\omega)^7}-\text{etc.}$$

etc. etc. etc.

§. 93. Operae pretium erit, hinc casus simplicissimos evolvere, qui oriuntur ponendo $\lambda=1$ et $\omega=0$, ita ut sit $\alpha=\frac{\pi}{2}$, $p=0$ et $q=1$, unde habebimus

Pro ordine priore	Pro ordine posteriore
$S=\frac{\pi}{2}$	$T=0$
$\left(\frac{dS}{d\omega}\right)=0$	$\left(\frac{dT}{d\omega}\right)=\frac{\pi\pi}{4}$
$\left(\frac{ddS}{d\omega^2}\right)=\frac{\pi^3}{8}$	$\left(\frac{ddT}{d\omega^2}\right)=0$
$\left(\frac{d^3S}{d\omega^3}\right)=0$	$\left(\frac{d^3T}{d\omega^3}\right)=\frac{\pi^4}{2}$
$\left(\frac{d^4S}{d\omega^4}\right)=\frac{5\pi^5}{32}$	$\left(\frac{d^4T}{d\omega^4}\right)=0$
$\left(\frac{d^5S}{d\omega^5}\right)=0$	$\left(\frac{d^5T}{d\omega^5}\right)=\frac{\pi^6}{4}$
$\left(\frac{d^6S}{d\omega^6}\right)=\frac{61\pi^7}{128}$	$\left(\frac{d^6T}{d\omega^6}\right)=0$
$\left(\frac{d^7S}{d\omega^7}\right)=0$	$\left(\frac{d^7T}{d\omega^7}\right)=\frac{79\pi^8}{32}$
etc.	etc.

§. 94. Hinc ergo, omissis valoribus evanescentibus, ex priore ordine habebimus sequentes formulas integrales cum seriebus inde natis

$$\int\frac{dz}{1+zz}=\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\text{etc.}$$

$$\int\frac{dz(lz)^2}{1+zz}=\frac{\pi^3}{16}=\frac{2}{1^3}-\frac{2}{3^3}+\frac{2}{5^3}-\frac{2}{7^3}+\frac{2}{9^3}-\frac{2}{11^3}+\text{etc.}$$

$$\int\frac{dz\,(lz)^4}{1+zz}=\frac{5\pi^5}{64}=\frac{24}{1^5}-\frac{24}{3^5}+\frac{24}{5^5}-\frac{24}{7^5}+\frac{24}{9^5}-\frac{24}{11^5}+\text{etc.}$$

$$\int\frac{dz\,(lz)^6}{1+zz}=\frac{61\,\pi^7}{256}=\frac{720}{1^7}-\frac{720}{3^7}+\frac{720}{5^7}-\frac{720}{7^7}+\frac{720}{9^7}-\text{etc.}$$

etc. etc. etc.

§. 95 Ex altero autem ordine pro eodem casu oriuntur

$$\int\frac{-dz\,lz}{1-zz}=\frac{\pi\pi}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\frac{1}{13^2}\text{ etc.}$$

$$\int\frac{-dz\,(lz)^3}{1-zz}=\frac{\pi^4}{16}=\frac{6}{1^4}+\frac{6}{3^4}+\frac{6}{5^4}+\frac{6}{7^4}+\frac{6}{9^4}+\frac{6}{11^4}+\frac{6}{13^4}+\text{etc.}$$

$$\int\frac{-dz\,(lz)^5}{1-zz}=\frac{\pi^6}{8}=\frac{120}{1^6}+\frac{120}{3^6}+\frac{120}{5^6}+\frac{120}{7^6}+\frac{120}{9^6}+\frac{120}{11^6}+\frac{120}{13^6}+\text{etc.}$$

etc. etc. etc.

§. 96. Quemadmodum ex primo integrali ordinis posterioris deduximus has formulas

$$\int\frac{dz\,lz}{1-z}=-\frac{\pi\pi}{6},\ \text{et}\ \int\frac{dz\,lz}{1+z}=-\frac{\pi\pi}{12},$$

similes quoque formulae integrales ex sequentibus deduci possunt; cum enim sit

$$\int\frac{dz\,(lz)^3}{1-zz}=-\frac{\pi^4}{16},$$

ponamus esse

$$\int\frac{z\,dz\,(lz)^3}{1-zz}=\text{P},\ \text{eritque}$$

$$\int\frac{dz\,(lz)^3}{1-z}=\text{P}-\frac{\pi^4}{16},\ \text{et}$$

$$\int\frac{dz\,(lz)^3}{1+z}=-\text{P}-\frac{\pi^4}{16},$$

nunc vero statuatur $zz=v$, ut sit $z\,dz=\frac{1}{2}dv$, et $lz=\frac{1}{2}lv$, ideoque $(lz)^3=\frac{1}{8}(lv)^3$, quibus substitutis erit

$$\text{P}=\frac{1}{16}\int\frac{dv\,(lv)^3}{1-v}=\frac{1}{16}\left(\text{P}-\frac{\pi^4}{16}\right),$$

unde fit

$$16\,\text{P}=\text{P}-\frac{\pi^4}{16},\ \text{ideoque}\ \text{P}=-\frac{\pi^4}{240},$$

sicque has duas habebimus integrationes novas

$$\int \frac{dz\,(lz)^3}{1-z} = -\frac{\pi^4}{15}, \text{ et}$$

$$\int \frac{dz\,(lz)^3}{1+z} = -\frac{7\pi^4}{120}:$$

hinc autem per series erit

$$\int \frac{-dz\,(lz)^3}{1-z} = +\frac{\pi^4}{15} = 6\left(1+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\frac{1}{7^4}+\frac{1}{8^4}+\text{etc.}\right) \text{ et}$$

$$\int \frac{-dz\,(lz)^3}{1+z} = +\frac{7\pi^4}{120} = 6\left(1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\frac{1}{6^4}+\frac{1}{7^4}-\frac{1}{8^4}+\text{etc.}\right)$$

§. 97. Porro cum $\int \frac{dz\,(lz)^5}{1-zz} = -\frac{\pi^6}{8}$, ponamus esse $\int \frac{z\,dz\,(lz)^5}{1-zz} = \mathrm{P}$, ut hinc obtineamus

$$\int \frac{dz\,(lz)^5}{1-z} = \mathrm{P} - \frac{\pi^6}{8}, \quad \text{et} \quad \int \frac{dz\,(lz)^5}{1+z} = -\mathrm{P} - \frac{\pi^6}{8},$$

nunc igitur statuamus $zz = v$, eritque

$$\mathrm{P} = \frac{1}{64}\int \frac{dv\,(lv)^5}{1-v} = \frac{1}{64}\left(\mathrm{P} - \frac{\pi^6}{8}\right),$$

unde sit

$$\mathrm{P} = -\frac{\pi^6}{504},$$

novaeque integrationes hinc deductae sunt

$$\int \frac{dz\,(lz)^5}{1-z} = -\frac{8\pi^6}{63}, \text{ et}$$

$$\int \frac{dz\,(lz)^5}{1+z} = -\frac{31\pi^6}{252}:$$

et vero per series reperitur

$$\int \frac{dz\,(lz)^5}{1-z} = -\frac{8\pi^6}{63} = -120\left(1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\frac{1}{7^6}+\text{etc.}\right), \text{ et}$$

$$\int \frac{dz\,(lz)^5}{1+z} = -\frac{31\pi^6}{252} = -120\left(1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\frac{1}{6^6}+\frac{1}{7^6}-\text{etc.}\right)$$

ita ut sit

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\frac{1}{7^6}+\text{etc.} = \frac{\pi^6}{945}, \text{ et}$$

$$1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\frac{1}{6^6}+\frac{1}{7^6}-\frac{1}{8^6}+\text{etc.} = \frac{31\pi^6}{30240} = \frac{31\pi^6}{32.945}.$$

§. 98. Consideremus etiam casus, quibus $\lambda = 2$ et $\omega = 1$, ita ut sit $\alpha = \frac{\pi}{4}$, et $\alpha\omega = \frac{\pi}{4}$, hinc $p = q = \frac{1}{\sqrt{2}}$, unde pro utroque ordine sequentes habebimus valores

Pro ordine priore	Pro ordine posteriore
$S = \frac{\pi}{2\sqrt{2}}$	$T = \frac{\pi}{4}$
$\left(\frac{dS}{d\omega}\right) = \frac{\pi\pi}{8\sqrt{2}}$	$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{8}$
$\left(\frac{ddS}{d\omega^2}\right) = \frac{5\pi^3}{32\sqrt{2}}$	$\left(\frac{ddT}{d\omega^2}\right) = \frac{\pi^3}{16}$
$\left(\frac{d^3S}{d\omega^3}\right) = \frac{11\pi^4}{128\sqrt{2}}$	$\left(\frac{d^3T}{d\omega^3}\right) = \frac{\pi^4}{16}$
$\left(\frac{d^4S}{d\omega^4}\right) = \frac{57\pi^5}{512\sqrt{2}}$	$\left(\frac{d^4T}{d\omega^4}\right) = \frac{5\pi^5}{64}$
$\left(\frac{d^5S}{d\omega^5}\right) = \frac{361\pi^6}{2048\sqrt{2}}$	$\left(\frac{d^5T}{d\omega^5}\right) = \frac{\pi^6}{8}$
$\left(\frac{d^6S}{d\omega^6}\right) = \frac{2763\pi^7}{8192\sqrt{2}}$	$\left(\frac{d^6T}{d\omega^6}\right) = \frac{61\pi^7}{256}$
$\left(\frac{d^7S}{d\omega^7}\right) = \frac{24611\pi^8}{32768\sqrt{2}}$	$\left(\frac{d^7T}{d\omega^7}\right) = \frac{79\pi^8}{32}$
etc.	etc.

§. 99. Hinc igitur sequentes integrationes, cum seriebus respondentibus resultant; ac primo quidem ex ordine primo

$$\int\frac{(1+zz)\,dz}{1+z^4} = \frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \text{etc.}$$

$$\int\frac{-(1-zz)\,dz\,lz}{1+z^4} = \frac{\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \text{etc}$$

$$\int\frac{(1+zz)\,dz\,(lz)^2}{1+z^4} = \frac{5\pi^3}{32\sqrt{2}} = \frac{2}{1^3} + \frac{2}{3^3} - \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} + \frac{2}{11^3} - \frac{2}{13^3} - \text{etc.}$$

$$\int\frac{-(1-zz)\,dz\,(lz)^3}{1+z^4} = \frac{11\pi^4}{128\sqrt{2}} = \frac{6}{1^4} - \frac{6}{3^4} - \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} - \frac{6}{11^4} - \frac{6}{13^4} + \text{etc.}$$

$$\int\frac{(1+zz)\,dz\,(lz)^4}{1+z^4} = \frac{57\pi^5}{512\sqrt{2}} = \frac{24}{1^5} + \frac{24}{3^5} - \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} + \frac{24}{11^5} - \frac{24}{13^5} - \text{etc}$$

$$\int\frac{-(1-zz)\,dz\,(lz)^5}{1+z^4} = \frac{361\pi^6}{2048\sqrt{2}} = \frac{120}{1^6} - \frac{120}{3^6} - \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} - \frac{120}{11^6} - \frac{120}{13^6} + \text{etc.}$$

$$\int\frac{(1+zz)\,dz\,(lz)^6}{1+z^4} = \frac{2763\pi^7}{8192\sqrt{2}} = \frac{720}{1^7} + \frac{720}{3^7} - \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} + \frac{720}{11^7} - \frac{720}{13^7} - \text{etc.}$$

$$\int\frac{-(1-zz)\,dz\,(lz)^7}{1+z^4} = \frac{24611\pi^8}{32768\sqrt{2}} = \frac{5040}{1^8} - \frac{5040}{3^8} - \frac{5040}{5^8} + \frac{5040}{7^8} + \frac{5040}{9^8} - \frac{5040}{11^8} - \frac{5040}{13^8} + \text{etc.}$$

etc. etc. etc.

§. 100. Eodem modo integrationes alterius ordinis cum seriebus erunt

$$\int\frac{dz}{1+zz}=\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\text{etc.}$$

$$\int\frac{-dz\,lz}{1-zz}=\frac{\pi\pi}{8}=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\frac{1}{13^2}+\text{etc.}$$

$$\int\frac{dz\,(lz)^2}{1+zz}=\frac{\pi^3}{16}=\frac{2}{1^3}-\frac{2}{3^3}+\frac{2}{5^3}-\frac{2}{7^3}+\frac{2}{9^3}-\frac{2}{11^3}+\frac{2}{13^3}-\text{etc.}$$

$$\int\frac{-dz\,(lz)^3}{1-zz}=\frac{\pi^4}{16}=\frac{6}{1^4}+\frac{6}{3^4}+\frac{6}{5^4}+\frac{6}{7^4}+\frac{6}{9^4}+\frac{6}{11^4}+\frac{6}{13^4}+\text{etc.}$$

$$\int\frac{dz\,(lz)^4}{1+zz}=\frac{5\pi^5}{64}=\frac{24}{1^5}-\frac{24}{3^5}+\frac{24}{5^5}-\frac{24}{7^5}+\frac{24}{9^5}-\frac{24}{11^5}+\frac{24}{13^5}-\text{etc.}$$

$$\int\frac{-dz\,(lz)^5}{1-zz}=\frac{\pi^6}{8}=\frac{120}{1^6}+\frac{120}{3^6}+\frac{120}{5^6}+\frac{120}{7^6}+\frac{120}{9^6}+\frac{120}{11^6}+\text{etc.}$$

$$\int\frac{dz\,(lz)^6}{1+zz}=\frac{61\,\pi^7}{256}=\frac{720}{1^7}-\frac{720}{3^7}+\frac{720}{5^7}-\frac{720}{7^7}+\frac{720}{9^7}-\frac{720}{11^7}+\text{etc.}$$

$$\int\frac{-dz\,(lz)^7}{1-zz}=\frac{17\,\pi^8}{32}=\frac{5040}{1^8}+\frac{5040}{3^8}+\frac{5040}{5^8}+\frac{5040}{7^8}+\frac{5040}{9^8}+\frac{5040}{11^8}+\text{etc.}$$

etc. etc. etc.

Hae autem series sunt eae ipsae, quas jam supra §§. 94. et 95. sumus consecuti.

§. 101. Praeterea autem ii casus imprimis notari merentur, quibus formulae integrales in formas simpliciores resolvi possunt. Haec autem resolutio tantum spectat ad fractionem

$$\pm\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1\pm z^{2\lambda}},$$

omisso factore $\frac{dz}{z}(lz)^\mu$; ad quod ostendendum sumamus primo $\lambda=3$ et $\omega=1$, unde fit $\alpha=\frac{\pi}{6}$, $p=\sin.\frac{\pi}{6}$, et $q=\cos.\frac{\pi}{6}$, tum autem, in priori ordine occurrunt alternatim sequentes fractiones

$$\text{I. } \frac{zz(1+zz)}{1+z^6}=\frac{zz}{1-zz+z^4},$$

quae posito $zz=v$ abit in $\frac{v}{1-v+vv}$: ergo cum sit

$$\frac{dz}{z}=\frac{1}{2}\frac{dv}{v}, \text{ et } lz=\frac{1}{2}lv,$$

hinc talis forma

$$\frac{1}{2^{2i+1}}\int\frac{dv\,(l\,v)^{2i}}{1-v+vv}$$

integrari poterit, casu scilicet $v=1$.

$$\text{II.}\quad -\frac{zz(1-zz)}{1+z^6}=+\frac{2}{6(1+zz)}-\frac{(2-zz)}{6(1-zz+z^4)},$$

quae posito $zz=v$, abit in $\frac{2}{6(1+v)}+\frac{2-v}{6(1-v+vv)}$, quae ergo forma ducta in $\frac{dz}{z}(l\,z)^{2i+1}$ vel in

$$\frac{1}{2^{2i+2}}\cdot\frac{dv}{v}(l\,v)^{2i+1},$$

semper integrari potest posito $v=1$.

§. 102. Eodem casu ordo posterior sequentes suppeditat resolutiones

$$\text{I.}\quad \frac{zz(1-zz)}{1-z^6}=\frac{zz}{1+zz+z^4}=\frac{v}{1+v+vv},$$

quae in $\frac{dz}{z}(l\,z)^{2i}$, vel in $\frac{1}{2^{2i+1}}\cdot\frac{dv}{v}(l\,z)^{2i}$ ducta semper est integrabilis

$$\text{II.}\quad \frac{-zz(1+zz)}{1-z^6}=\frac{-2}{6(1-zz)}+\frac{2+zz}{6(1+zz+z^4)},$$

quae facto $zz=v$ fit

$$\frac{-2}{6(1-v)}+\frac{2+v}{6(1+v+vv)},$$

quae ergo formulae in $\frac{dv}{v}(l\,v)^{2i+1}$ ductae fiunt integrabiles; quia autem in hac resolutione numeratores per z vel v dividere non licet, alia resolutione est opus, quae reperitur

$$\frac{-zz(1+zz)}{1-z^6}=\frac{-2zz}{6(1-zz)}-\frac{zz(1+2zz)}{6(1+zz+z^4)},\ \text{sive}$$

$$\frac{-2v}{6(1-v)}-\frac{v(1+2v)}{6(1+v+vv)},$$

quae formulae ductae in $\frac{dz}{z}(lz)^{2i+1}$, vel in $\frac{1}{2^{2i+2}}\cdot\frac{dv}{v}(lv)^{2i+1}$, integrationem quoque admittunt.

§. 103. Porro manente $\lambda = 3$ sumatur $\omega = 2$, ut sit $\alpha = \frac{\pi}{6}$, $p = \sin\frac{\pi}{3}$, et $q = \cos.\frac{\pi}{3}$, et ex ordine priore oriuntur sequentes reductiones.

$$\text{I.}\quad \frac{z(1+z^4)}{1+z^6} = \frac{2z}{3(1+zz)} + \frac{z(1+zz)}{3(1-zz+z^4)},$$

unde multiplicando per $\frac{dz}{z}lz^{2i}$ oriuntur formulae integrationem admittentes cusu $z = 1$.

$$\text{II.}\quad \frac{-z(1-z^4)}{1+z^6} = -\frac{z(1-zz)}{1-zz+z^4},$$

quae per $\frac{dz}{z}(lz)^{2i+1}$ multiplicata integrari poterit casu $z = 1$. Ex ordine vero posteriori sequentes prodibunt reductiones.

$$\text{I.}\quad \frac{z(1-z^4)}{1-z^6} = \frac{z(1+zz)}{1+zz+z^4},$$

quae ducta in $\frac{dz}{z}(lz)^{2i}$ fit integrabilis.

$$\text{II.}\quad \frac{-z(1+z^4)}{1-z^6} = \frac{-2z}{3(1-zz)} - \frac{z(1-zz)}{3(1+zz+z^4)},$$

quae formulae in $\frac{dz}{z}(lz)^{2i+1}$ ductae fiunt integrabiles.

§. 104. Operae jam erit pretium haec integralia actu evolvere, quare ex §. 101. ubi $\omega = 1$, ejusque numero I nanciscimur sequentes integrationes

$$1^0.\quad \frac{1}{2}\int\frac{dv}{1-v+vv} = \alpha\frac{1}{q} = \frac{\pi}{3\sqrt{3}}$$

$$2^0.\quad \frac{1}{8}\int\frac{dv(lv)^2}{1-v+vv} = \alpha^3\left(\frac{2}{q^3} - \frac{l}{q}\right) = \frac{5\pi^3}{324\sqrt{3}},$$

deinde vero ex ejusdem §. numero II. ubi etiam haec reductio locum habet

$$-\frac{zz(1-zz)}{1+z^6} = -\frac{2zz}{3(1+zz)} - \frac{zz(1-2zz)}{3(1-zz+z^4)} = -\frac{2v}{3(1+v)} - \frac{v(1-2v)}{3(1-v+vv)},$$

quae ducta in $\frac{1}{4}\cdot\frac{dv}{v}lv$ dabit

$$-\frac{1}{6}\int\frac{dv\,lv}{1+v}-\frac{1}{12}\int\frac{dv(1-2v)lv}{1-v+vv}=\alpha\alpha\frac{p}{qq}=\frac{\pi\pi}{54},$$

quarum formularum prior integrationem admittit, est enim

$$\int\frac{dv\,lv}{1+v}=-\frac{\pi\pi}{12},$$

unde invenitur posterior

$$\int\frac{dv(1-2v)lv}{1-v+vv}=-\frac{\pi\pi}{18}.$$

§. 105. Ex §. 102. ejusque numero I sequitur

$$1^\circ.\quad \frac{1}{2}\int\frac{dv}{1+v+vv}=\frac{ap}{q}=\frac{\pi}{6\sqrt{3}}$$

$$2^\circ.\quad \frac{1}{8}\int\frac{dv(lv)^2}{1+v+vv}=\alpha^3\frac{2p}{q^3}=\frac{\pi^3}{81\sqrt{3}};$$

deinde vero ex numero II fit

$$-\frac{1}{6}\int\frac{dv\,lv}{1-v}-\frac{1}{12}\int\frac{dv(1+2v)lv}{1+v+vv}=\alpha\alpha\cdot\frac{1}{qq}=\frac{\pi\pi}{27};$$

supra autem invenimus esse

$$\int\frac{dv\,lv}{1-v}=-\frac{\pi\pi}{6},$$

quo valore substituto fit

$$\int\frac{dv(1+2v)lv}{1+v+vv}=-\frac{\pi\pi}{9}:$$

maxime igitur operae pretium est visum, has postremas integrationes evolvisse.

§. 106. Quod si ambae formulae integrales

$$\int\frac{dv(1-2v)lv}{1-v+vv}\ \text{et}\ \int\frac{dv(1+2v)lv}{1+v+vv}$$

in series convertantur, reperitur

$$\int\frac{dv(1-2v)lv}{1-v+vv}=-1+\frac{1}{4}+\frac{2}{9}+\frac{1}{16}-\frac{1}{25}-\frac{2}{36}-\frac{1}{49}+\text{etc. et}$$

$$\int\frac{dv(1+2v)lv}{1+v+vv}=-1-\frac{1}{4}+\frac{2}{9}-\frac{1}{16}-\frac{1}{25}+\frac{2}{36}-\frac{1}{49}-\text{etc.}$$

unde has duas summationes attentione nostra non indignas assequimur

$$\text{I.}\quad 1-\frac{1}{4}-\frac{2}{9}-\frac{1}{16}+\frac{1}{25}+\frac{2}{36}+\frac{1}{49}-\frac{1}{64}-\frac{2}{81}-\frac{1}{100}+\text{etc.}=\frac{\pi\pi}{18},$$

$$\text{II.}\quad 1+\frac{1}{4}-\frac{2}{9}+\frac{1}{16}+\frac{1}{25}-\frac{2}{36}+\frac{1}{49}+\frac{1}{64}-\frac{2}{81}+\frac{1}{100}+\text{etc.}=\frac{\pi\pi}{9},$$

quarum prior a posteriore ablata praebet

$$\frac{2}{4}+\frac{2}{16}-\frac{4}{36}+\frac{2}{64}+\frac{2}{100}\text{ etc.}=\frac{\pi\pi}{18},$$

cujus duplum perducit ad hanc

$$1+\frac{1}{4}-\frac{2}{9}+\frac{1}{16}+\frac{1}{25}-\frac{2}{36}+\text{etc.}=\frac{\pi\pi}{9},$$

quae quoniam cum secunda congruit, veritas utriusque summationis satis confirmatur. Quod si vero secunda a duplo primae subtrahatur, remanebit ista series memorabilis

$$1-\frac{3}{4}-\frac{2}{9}-\frac{3}{16}+\frac{1}{25}+\frac{6}{36}+\frac{1}{49}-\frac{3}{64}-\frac{2}{81}-\frac{3}{100}+\text{etc.}=0$$

quae in periodos sex terminos complectentes distributa, manifestum ordinem in numerationibus declarat, quippe qui sunt

$$1-3-2-3+1+6.$$

Additamentum.

§. 107. Quemadmodum superiores integrationes per continuam differentiationem formularum S et T deduximus, ita etiam per integrationem alias et prorsus singulares integrationes impetrabimus; si enim ut supra fuerit $S=\int\frac{T\,dz}{z}$, existente T formula illa

$$+\frac{z^{\lambda-\omega}\pm z^{\lambda+\omega}}{1\pm z^{2\lambda}},$$

quae praeter z etiam exponentem variabilem ω involvere concipitur, erit per naturam integralium duas variabiles involventium

$$\int S\,d\omega=\int\frac{dz}{z}\int T\,d\omega,$$

ubi in priore formula integrali $\int S\,d\omega$, ubi z pro constanti habetur, statim scribi potest $z=1$; hoc igitur lemmate praemisso, quia est

$$\int T\,d\omega = \frac{-z^{\lambda-\omega} \pm z^{\lambda+\omega}}{(1 \pm z^{2\lambda})\, l z},$$

ambas formulas supra tractatas nempe S et T hoc modo evolvamus, et quia utramque triplici modo expressam dedimus; primo scilicet per seriem infinitam, secundo, per formulam finitam, ac tertio per formulam integralem, etiam quantitates, quae pro integralibus $\int S\,d\omega$ et $\int T\,d\omega$ resultabunt, erunt inter se aequales.

§. 108 Incipiamus a formula S, et cum per seriem fuerit

$$S = \frac{1}{\lambda-\omega} + \frac{1}{\lambda+\omega} - \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} + \frac{1}{5\lambda+\omega} - \text{etc.}$$

erit

$$\int S\,d\omega = -l(\lambda-\omega) + l(\lambda+\omega) + l(3\lambda-\omega) - l(3\lambda+\omega) - \text{etc.} + C,$$

quam constantem ita definire decet, ut integrale evanescat posito $\omega = 0$, quo facto erit

$$\int S\,d\omega = l\frac{\lambda+\omega}{\lambda-\omega} + l\frac{3\lambda-\omega}{3\lambda+\omega} + l\frac{5\lambda+\omega}{5\lambda-\omega} + l\frac{7\lambda-\omega}{7\lambda+\omega} + \text{etc.}$$

quae expressio reducitur ad sequentem

$$\int S\,d\omega = l\frac{(\lambda+\omega)(3\lambda-\omega)(5\lambda+\omega)(7\lambda-\omega)(9\lambda+\omega)\ \text{etc.}}{(\lambda-\omega)(3\lambda+\omega)(5\lambda-\omega)(7\lambda+\omega)(9\lambda-\omega)\ \text{etc.}}.$$

Deinde quia per formulam finitam erat

$$S = \frac{\pi}{2\lambda \cos.\frac{\pi\omega}{2\lambda}}, \text{ erit } \int S\,d\omega = \int \frac{\pi\,d\omega}{2\lambda \cos.\frac{\pi\omega}{2\lambda}},$$

ubi si brevitatis gratia ponatur $\frac{\pi\omega}{2\lambda} = \varphi$, ut sit

$$d\omega = \frac{2\lambda\,d\varphi}{\pi}, \text{ erit } \int S\,d\omega = \int \frac{d\varphi}{\cos.\varphi};$$

quia igitur novimus esse

$$\int \frac{d\theta}{\sin.\theta} = l \text{ tang. } \tfrac{1}{2}\theta,$$

sumamus $\sin.\theta = \cos.\varphi$, sive $\theta = 90^\circ - \varphi = \frac{\pi}{2} - \varphi$, eritque $d\theta = -d\varphi$, unde fit

$$\int \frac{-d\varphi}{\cos.\varphi} = l \text{ tang. } \left(\frac{\pi}{4} - \frac{1}{2}\varphi\right);$$

quoniam autem est

$$\varphi = \frac{\pi\omega}{2\lambda}, \text{ erit } \frac{\pi}{4} - \frac{1}{2}\varphi = \frac{\pi(\lambda - \omega)}{4\lambda},$$

unde nostrum integrale erit

$$\int S\, d\omega = -l \text{ tang. } \frac{\pi(\lambda - \omega)}{4\lambda} = +l \text{ tang. } \frac{\pi(\lambda + \omega)}{4\lambda}.$$

Ex tertia autem formula integrali

$$S = \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z} \text{ colligitur fore}$$

$$\int S\, d\omega = \int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z\, l z},$$

quod integrale a termino $z = 0$ usque ad terminum $z = 1$ extendi assumitur; sicque tres isti valores inventi inter se erunt aequales. Ac ne ob constantes forte addendas ullum dubium supersit, singulae istae expressiones sponte evanescunt casu $\omega = 0$.

§. 109. Consideremus hinc primo aequalitatem inter formulam primam et secundam: et quia utraque est logarithmus, erit

$$\text{tang. } \frac{\pi(\lambda + \omega)}{4\lambda} = \frac{(\lambda + \omega)(3\lambda - \omega)(5\lambda + \omega)(7\lambda - \omega) \text{ etc.}}{(\lambda - \omega)(3\lambda + \omega)(5\lambda - \omega)(7\lambda + \omega) \text{ etc.}}$$

cum igitur hujus fractionis numerator evanescat casibus, vel $\omega = -\lambda$, vel $\omega = +3\lambda$, vel $\omega = -5\lambda$, vel $\omega = +7\lambda$ etc. evidens est iisdem casibus quoque tangentem fieri $= 0$; denominator vero evanescit casibus vel $\omega = \lambda$, vel $\omega = -3\lambda$, vel $\omega = +5\lambda$, vel $\omega = -7\lambda$ etc. quibus ergo casibus tangens in infinitum excrescere debet, id quod etiam pulcherrime evenit. Caeterum haec expressio congruit cum ea, quam jam dudum inveni et in introductione exposui.

§. 110. Productum autem istud infinitum per principia alibi stabilita ad formulas integrales reduci potest ope hujus lemmatis latissime patentis

$$\frac{a(c+b)(a+k)(c+b+k)(a+2k)(c+b+2k)\text{ etc.}}{b(c+a)(b+k)(c+a+k)(b+2k)(c+a+2k)\text{ etc.}} = \frac{\int z^{c-1}dz(1-z^k)^{\frac{b-k}{k}}}{\int z^{c-1}dz(1-z^k)^{\frac{a-k}{k}}},$$

si quidem post utramque integrationem fiat $z=1$. Nostro igitur casu erit $a=\lambda+\omega$, $b=\lambda-\omega$, $c=2\lambda$, et $k=4\lambda$; unde valor nostri producti erit

$$\frac{\int z^{2\lambda-1}dz(1-z^{4\lambda})^{\frac{-3\lambda-\omega}{4\lambda}}}{\int z^{2\lambda-1}dz(1-z^{4\lambda})^{\frac{-3\lambda+\omega}{4\lambda}}} = \text{tang.}\,\frac{\pi(\lambda+\omega)}{4\lambda}:$$

formulae autem istae integrales concinniores evadunt, statuendo $z^{2\lambda}=y$, tum enim erit

$$\text{tang.}\,\frac{\pi(\lambda+\omega)}{4\lambda} = \frac{\int dy(1-yy)^{\frac{-3\lambda-\omega}{4\lambda}}}{\int dy(1-yy)^{\frac{-3\lambda+\omega}{4\lambda}}},$$

quae expressio utique omni attentione digna videtur. Denique ex formula integrali inventa erit quoque

$$\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z\,l\,z} = l\,\text{tang.}\,\frac{\pi(\lambda+\omega)}{4\lambda}.$$

§. 111. Operae erit pretium, etiam aliquot casus particulares evolvere: sit igitur primo $\lambda=2$ et $\omega=1$, ac per expressionem infinitam erit

$$\int S\,d\omega = l\frac{3.5}{1.7}\cdot\frac{11.13}{9.15}\cdot\frac{19.21}{17.23}\cdot\frac{27.29}{25.31}\cdot\frac{35.37}{33.39}\cdot\text{etc.}$$

deinde per expressionem finitam habebimus

$$\int S\, d\omega = l \operatorname{tang.} \frac{5\pi}{8},$$

at per formulam integralem

$$\int S\, d\omega = \int \frac{-(1-zz)}{1+z^4} \cdot \frac{dz}{lz}.$$

Tum vero ex aequalitate duarum priorum expressionum

$$\operatorname{tang.} \frac{5\pi}{8} = \frac{5.5}{1.7} \cdot \frac{11.13}{9.15} \cdot \frac{19.21}{17.23} \text{ etc.}$$

hincque per binas formulas integrales

$$\operatorname{tang.} \frac{5\pi}{8} = \frac{\int dy\,(1-yy)^{-\frac{7}{8}}}{\int dy\,(1-yy)^{-\frac{5}{8}}}.$$

§. 112. Ponamus nunc esse $\lambda = 3$ et $\omega = 1$, ac per expressionem infinitam erit

$$\int S\, d\omega = l \frac{2}{1} \cdot \frac{4}{5} \cdot \frac{8}{7} \cdot \frac{10}{11} \cdot \frac{14}{13} \cdot \frac{16}{17} \cdot \frac{20}{19} \cdot \frac{22}{23} \text{ etc.}$$

secundo, per expressionem finitam

$$\int S\, d\omega = l \operatorname{tang.} \frac{\pi}{3} = l\sqrt{3} = \tfrac{1}{2} l\, 3,$$

ita, ut futurum sit

$$\sqrt{3} = \frac{2.4}{1.5} \cdot \frac{8.10}{7.11} \cdot \frac{14.16}{13.17} \text{ etc.}$$

hujusque producti valor per formulas integrales erit

$$\frac{\int dy\,(1-yy)^{-\frac{5}{6}}}{\int dy\,(1-yy)^{-\frac{2}{6}}}.$$

Denique formula integralis praebebit

$$\int S\, d\omega = \int \frac{-z(1-zz)}{1+z^6} \cdot \frac{dz}{lz}.$$

§. 113. Eodem modo etiam evolvamus alteram formulam T, cujus valor per seriem erat

$$T = \frac{1}{\lambda-\omega} - \frac{1}{\lambda+\omega} + \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} - \frac{1}{5\lambda+\omega} + \text{etc.}$$

unde sit

$$\int T\,d\omega = -l(\lambda-\omega)-l(\lambda+\omega)-l(3\lambda-\omega)-l(3\lambda+\omega)-\text{etc.}$$

quae expressio, ut evanescat posito $\omega=0$, erit

$$\int T\,d\omega = l\frac{\lambda\lambda}{\lambda\lambda-\omega\omega}\cdot\frac{9\lambda\lambda}{9\lambda\lambda-\omega\omega}\cdot\frac{25\lambda\lambda}{25\lambda\lambda-\omega\omega}\ \text{etc.}$$

deinde vero cum per formulam finitam fuerit

$$T=\frac{\pi}{2\lambda}\ \text{tang.}\ \frac{\pi\omega}{2\lambda},\ \text{erit}$$

$$\int T\,d\omega = \int\frac{\pi\,d\omega}{2\lambda}\ \text{tang.}\ \frac{\pi\omega}{2\lambda},\ \text{ubi posito}\ \frac{\pi\omega}{2\lambda}=\varphi,\ \text{erit}$$

$$\int T\,d\omega = \int d\varphi\ \text{tang.}\ \varphi = -l\cos.\varphi,\ \text{ita ut sit}$$

$$\int T\,d\omega = -l\cos.\frac{\pi\omega}{2\lambda};$$

cujus valor casu $\omega=0$ fit sponte $=0$; denique per formulam integralem habebimus

$$\int T\,d\omega = -\frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z\,lz},$$

ubi integrale itidem a termino $z=0$ usque ad terminum $z=1$ extendi debet.

§. 114. Jam comparatio duorum priorum valorum hanc praebet aequationem.

$$\frac{1}{\cos.\frac{\pi\omega}{2\lambda}}=\frac{\lambda\lambda}{\lambda\lambda-\omega\omega}\cdot\frac{9\lambda\lambda}{9\lambda\lambda-\omega\omega}\cdot\frac{25\lambda\lambda}{25\lambda\lambda-\omega\omega}\cdot\frac{49\lambda\lambda}{49\lambda\lambda-\omega\omega}\ \text{etc. vel}$$

$$\cos.\frac{\pi\omega}{2\lambda}=\left(1-\frac{\omega\omega}{\lambda\lambda}\right)\left(1-\frac{\omega\omega}{9\lambda\lambda}\right)\left(1-\frac{\omega\omega}{25\lambda\lambda}\right)\left(1-\frac{\omega\omega}{49\lambda\lambda}\right)\ \text{etc.}$$

vel si factores singuli iterum in simplices evolvantur,

$$\cos.\frac{\pi\omega}{2\lambda}=\frac{\lambda+\omega}{\lambda}\cdot\frac{\lambda-\omega}{\lambda}\cdot\frac{3\lambda+\omega}{3\lambda}\cdot\frac{3\lambda-\omega}{3\lambda}\cdot\frac{5\lambda+\omega}{5\lambda}\cdot\frac{5\lambda-\omega}{5\lambda}\ \text{etc.}$$

quae formula cum reductione generali supra allata comparata dat, $a=\lambda+\omega$, $b=\lambda$, $c=-\omega$, et $k=2\lambda$, unde colligimus

$$\cos.\frac{\pi\omega}{2\lambda}=\frac{\int z^{-\omega-1}dz(1-z^{2\lambda})^{-\frac{1}{2}}}{\int z^{-\omega-1}dz(1-z^{2\lambda})^{\frac{\omega-\lambda}{2\lambda}}}.$$

Ut autem exponentes negativos $z^{-\omega-1}$ evitemus, superius productum ita repraesentemus

$$\cos.\frac{\pi\omega}{2\lambda}=\frac{\lambda-\omega}{\lambda}\cdot\frac{\lambda+\omega}{\lambda}\cdot\frac{3\lambda-\omega}{3\lambda}\cdot\frac{3\lambda+\omega}{3\lambda}\text{ etc.}$$

eritque facta comparatione $a=\lambda-\omega$, $b=\lambda$, $c=+\omega$, et $k=2\lambda$, sicque per formulas integrales erit

$$\cos.\frac{\pi\omega}{2\lambda}=\frac{\int z^{\omega-1}dz(1-z^{2\lambda})^{-\frac{1}{2}}}{\int z^{\omega-1}dz(1-z^{2\lambda})^{\frac{-\lambda-\omega}{2\lambda}}},$$

quae expressio ad simpliciorem formam reduci nequit.

§. 115. Sit nunc etiam $\lambda=2$, et $\omega=1$, eruntque ternae nostrae expressiones

$$\text{I. } \int T\,d\omega=l\frac{4}{3}\cdot\frac{36}{35}\cdot\frac{100}{99}\cdot\frac{196}{195}\text{ etc. sive}$$

$$\int T\,d\omega=l\frac{2.2}{1.3}\cdot\frac{6.6}{5.7}\cdot\frac{10.10}{9.11}\cdot\frac{14.14}{13.15}\text{ etc.}$$

$$\text{II. } \int T\,d\omega=-l\cos.\frac{\pi}{4}=+\frac{1}{2}l2,\text{ ita ut sit}$$

$$\sqrt{2}=\frac{2.2}{1.3}\cdot\frac{6.6}{5.7}\cdot\frac{10.10}{9.11}\cdot\frac{14.14}{13.15}\text{ etc.}$$

quod productum per formulas integrales ita exprimitur

$$\frac{\int dz(1-z^4)^{-\frac{1}{2}}}{\int dz(1-z^4)^{-\frac{3}{4}}}=\tfrac{1}{2}\sqrt{2}:$$

$$\text{III. } \int T\,d\omega=\int\frac{-(1+zz)}{1-z^4}\cdot\frac{dz}{lz}=\int\frac{-dz}{(1-zz)lz};$$

quod ergo integrale a termino $z=0$ usque ad $z=1$ extensum praebet eundem valorem $+\frac{1}{2}l2$, cujus aequalitatis ratio utique difficillime patet.

§. 116. Sit denique ut supra $\lambda=3$ et $\omega=1$, ac ternae formulae ita se habebunt

$$\text{I. } \int T\,d\omega=l\frac{9}{8}\cdot\frac{81}{80}\cdot\frac{225}{224}\text{ etc.}=l\frac{3.3}{2.4}\cdot\frac{9.9}{8.10}\cdot\frac{15.15}{14.16}\cdot\frac{21.21}{20.22}\text{ etc.}$$

II. $\int T.d\omega = -l\cos.\frac{\pi}{6} = -l\frac{\sqrt{3}}{2} = +l\frac{2}{\sqrt{3}}$, ita ut sit

$$\frac{2}{\sqrt{3}} = \frac{3.3}{2.4}\cdot\frac{9.9}{8.10}\cdot\frac{15.15}{14.16}\cdot\frac{21.21}{20.22};$$

ideoque per binas formulas integrales

$$\frac{3}{4}\,\frac{2}{\sqrt{3}} = \frac{\int dz(1-z^6)^{-\frac{1}{2}}}{\int dz(1-z^6)^{-\frac{1}{3}}}.$$

III. $\int T\,d\omega = \int\frac{-(1+zz)}{1-z^6}\cdot\frac{dz}{lz}$,

quae posito $zz = v$ abit in hanc

$$\int T\,d\omega = \int\frac{-dv(1+v)}{(1-v^3)\,lv}.$$

Hinc igitur patet, hac methodo plane nova perveniri ad formulas integrales, quas per methodos adhuc cognitas nullo modo evolvere, vel saltem inter se comparare, licuit.

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3). De integratione formulae $\int\frac{dx\,lx}{\sqrt{(1-xx)}}$, ab $x = 0$ ad $x = 1$ extensa. *Acta Acad. Imp. Sc. Tom. I. P. II. Pag.* 3—28.

§. 117. Methodus maxime naturalis hujusmodi formulas $\int p\,dx\,lx$ tractandi in hoc consistit, ut eae ad alias hujusmodi formas $\int q\,dx$ reducantur, in quibus littera q sit functio algebraica ipsius x; quandoquidem regulae integrandi potissimum ad tales formulas sunt accommodatae. Hujusmodi autem reductio nulla prorsus laborat difficultate, quando functio p ita est comparata, ut integrale $\int p\,dx$ algebraice exhiberi queat. Si enim fuerit $\int p\,dx = P$, ita ut formula proposita sit $\int dP\,lx$, ea sponte reducitur ad hanc expressionem $P\,lx - \int\frac{P\,dx}{x}$, sicque jam totum negotium ad

integrationem hujus formulae $\int \frac{P\,dx}{x}$ est perductum. Quando vero formula $\int p\,dx$ integrationem algebraicam non admittit, quemadmodum evenit in nostra formula proposita $\int \frac{dx\,lx}{\sqrt{(1-xx)}}$, talis reductio successu penitus caret. Cum enim sit $\int \frac{dx}{\sqrt{(1-xx)}} = \text{A}.\sin.x$, ista reductio daret

$$\int \frac{dx\,lx}{\sqrt{(1-xx)}} = \text{A}.\sin.x \times lx - \int \frac{dx}{x} \cdot \text{A}\sin.x,$$

sicque post signum integrationis nova quantitas transcendens A sin. x occurreret, cujus integratio aeque est abscondita ac ipsius propositae. Quare cum nuper singulari methodo invenissem esse

$$\int \frac{dx\,lx}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix} = -\frac{1}{2}\pi\,l\,2,$$

expressio integralis eo majori attentione digna est censenda, quod ejus investigatio neutiquam est obvia; unde operae pretium esse duxi ejus veritatem etiam ex aliis fontibus ostendisse, ante quam ipsam methodum, quae me eo perduxit, exponerem.

Prima demonstratio integrationis propositae.

§. 118. Quoniam hic potissimum ad series infinitas est recurrendum, formula autem lx talem resolutionem simplicem respuit, adhibeamus substitutionem $\sqrt{(1-xx)} = y$, unde fit $x = \sqrt{(1-yy)}$, hincque porro

$$lx = -\frac{yy}{2} - \frac{y^4}{4} - \frac{y^6}{6} - \frac{y^8}{8} - \text{etc.}$$

hoc igitur modo formula integralis proposita $\int \frac{dx\,lx}{\sqrt{(1-xx)}}$ transformatur in sequentem formam

$$\int \frac{dy}{\sqrt{(1-yy)}} \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \frac{y^8}{8} + \text{etc.}\right)$$

ubi, cum sit $y = \sqrt{(1-xx)}$, notetur integrationem extendi debere ab $y=1$ usque ad $y=0$; quare si hos terminos integrationis permutare velimus, signum totius formae mutari oportet.

§. 119. Quo autem minus tali signorum mutatione confundamur, designemus valorem quesitum littera S, ut sit

$$S = \int \frac{dx\, lx}{\sqrt{(1-xx)}} \begin{bmatrix} a\, b\, x = 0 \\ a\, d\, x = 1 \end{bmatrix}$$

atque facta substitutione $y = \sqrt{(1-xx)}$, habebimus, uti modo monuimus

$$S = -\int \frac{dy}{\sqrt{(1-yy)}} \left(\frac{yy}{2} + \frac{y^4}{4} + \frac{y^6}{6} + \text{etc.} \right) \begin{bmatrix} a\, b\, y = 0 \\ a\, d\, y = 1 \end{bmatrix}.$$

Sub his autem integrationis terminis, scilicet ab $y = 0$ ad $y = 1$, jam satis notum est, singulas partes, quae hic occurrunt, ad sequentes valores reduci

$$\int \frac{yy\, dy}{\sqrt{(1-yy)}} = \frac{1}{2} \cdot \frac{\pi}{2}$$

$$\int \frac{y^4\, dy}{\sqrt{(1-yy)}} = \frac{1.3}{2.4} \cdot \frac{\pi}{2}$$

$$\int \frac{y^6\, dy}{\sqrt{(1-yy)}} = \frac{1.3.5}{2.4.6} \cdot \frac{\pi}{2}$$

$$\int \frac{y^8\, dy}{\sqrt{(1-yy)}} = \frac{1.3.5.7}{2.4.6.8} \cdot \frac{\pi}{2}$$

$$\int \frac{y^{10}\, dy}{\sqrt{(1-yy)}} = \frac{1.3.5.7.9}{2.4.6.8.10} \cdot \frac{\pi}{2} \text{ etc.}$$

ubi nimirum est $\frac{\pi}{2} = \int \frac{dy}{\sqrt{(1-yy)}}$, ita ut $1 : \pi$ exprimat rationem diametri ad peripheriam circuli.

§. 120. Quodsi ergo singulos istos valores introducamus, pro valore quaesito S impetrabimus sequentem seriem infinitam

$$S = -\frac{\pi}{2} \left(\frac{1}{2^2} + \frac{1.3}{2.4^2} + \frac{1.3.5}{2.4.6^2} + \frac{1.3.5.7}{2.4.6.8^2} + \text{etc.} \right.$$

sicque nunc totum negotium eo est reductum, ut istius seriei infinitae summa investigetur; qui labor fortasse haud minus operosus videri potest, quam id ipsum, quod nobis exsequi est propositum. Interim tamen ad cognitionem summae hujus seriei haud difficulter sequenti modo nobis pertingere licebit.

§. 121. Cum sit

$$\frac{1}{\sqrt{(1-zz)}} = 1 + \frac{1}{2} zz + \frac{1.3}{2.4} z^4 + \frac{1.3.5}{2.4.6} z^6 + \text{etc.}$$

si utrinque per $\frac{dz}{z}$ multiplicemus et integremus, obtinebimus

$$\int \frac{dz}{z\sqrt{(1-zz)}} = lz + \frac{1}{2^2} zz + \frac{1.3}{2.4^2} z^4 + \frac{1.3.5}{2.4.6^2} z^6 + \text{etc.}$$

sicque ad ipsam seriem nostram sumus perducti, cujus ergo valor quaeri debet ex hac expressione $\int \frac{dz}{z\sqrt{(1-zz)}} - lz$, integrali scilicet ita sumto, ut evanescat posito $z=0$, quo facto statuatur $z=1$, ac prodibit ipsa series

$$\frac{1}{2^2} + \frac{1.3}{2.4^2} + \frac{1.3.5}{2.4.6^2} + \frac{1.3.5.7}{2.4.6.8^2} + \text{etc.}$$

Hoc igitur modo totum negotium perductum est ad istam formulam integralem $\int \frac{dz}{z\sqrt{(1-zz)}}$, quae posito $\sqrt{(1-zz)} = v$ transit in hanc formam $\frac{-dv}{1-vv}$, cujus integrale constat esse $-\frac{1}{2} l \frac{1+v}{1-v} = -l \frac{1+v}{\sqrt{(1-vv)}}$. Quodsi loco v restituatur valor $\sqrt{(1-zz)}$, tota expressio, qua indigemus, ita se habebit

$$\int \frac{dz}{z\sqrt{(1-zz)}} - lz = -l \frac{[1+\sqrt{(1-zz)}]}{z} - lz + C$$
$$= C - l[1+\sqrt{(1-zz)}],$$

ubi constans ita accipi debet, ut valor evanescat, posito $z=0$, ideoque erit $C = l2$. Quamobrem, posito $z=1$, summa seriei quaesita erit $l2$, hincque valor ipsius formulae integralis propositae erit

$$\int \frac{dx\, lx}{\sqrt{(1-xx)}} = S = -\frac{\pi}{2} l2:$$

prorsus uti longe alia methodo inveneram, ex quo jam satis intelligitur, istam veritatem utique altioris esse indaginis, ideoque attentione Geometrarum maxime dignam.

Alia demonstratio integrationis propositae.

§. 122. Cum sit $\frac{dx}{\sqrt{(1-xx)}}$ elementum arcus circuli cujus sinus $= x$, ponamus istum angulum $= \varphi$, ita ut sit

$$x = \sin.\varphi \text{ et } \frac{dx}{\sqrt{(1-xx)}} = d\varphi,$$

atque facta hac substitutione valor quantitatis S, in quem inquirimus, ita repraesentabitur

$$S = \int d\varphi\, l \sin.\varphi \left[\begin{smallmatrix} a\ \varphi = 0 \\ ad\ \varphi = 90^\circ \end{smallmatrix}\right].$$

Cum enim ante termini fuissent $x = 0$ et $x = 1$, iis nunc respondent $\varphi = 0$ et $\varphi = 90^\circ$, sive $\varphi = \frac{\pi}{2}$. Hic igitur totum negotium eo redit, ut formula $l \sin.\varphi$ commode in seriem infinitam convertatur. Hunc in finem ponamus $l \sin.\varphi = s$ eritque $ds = \frac{d\varphi \cos.\varphi}{\sin.\varphi}$. Novimus autem esse

$$\frac{\cos.\varphi}{\sin.\varphi} = 2\sin.2\varphi + 2\sin.4\varphi + 2\sin.6\varphi + 2\sin.8\varphi + \text{etc.}$$

Si enim utrinque per $\sin.\varphi$ multiplicemus, ob

$$2\sin.n\varphi\sin.\varphi = \cos.(n-1)\varphi - \cos.(n+1)\varphi,$$

utique prodit

$$\begin{aligned}\cos.\varphi = \cos.\varphi &+ \cos.3\varphi + \cos.5\varphi + \cos.7\varphi + \cos.9\varphi + \text{etc.}\\ &- \cos.3\varphi - \cos.5\varphi - \cos.7\varphi + \cos.9\varphi - \text{etc.}\end{aligned}$$

Hac igitur serie pro $\frac{\cos.\varphi}{\sin.\varphi}$ in usum vocata, erit

$$s = C - \cos.2\varphi - \tfrac{1}{2}\cos.4\varphi - \tfrac{1}{3}\cos.6\varphi - \tfrac{1}{4}\cos.8\varphi - \tfrac{1}{5}\cos.10\varphi - \text{etc.}$$

ubi cum sit $s = l\sin.\varphi$, ideoque $s = 0$, quando $\sin.\varphi = 1$, ideoque $\varphi = \frac{\pi}{2}$, constantem C ita definire oportet, ut posito $\varphi = \frac{\pi}{2} = 90^\circ$, evadat $s = 0$, ex quo colligitur fore

$$C = -1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \text{etc.} = -l\,2.$$

§. 123. Cum igitur sit

$$l\sin.\varphi = -l\,2 - \cos.2\varphi - \tfrac{1}{2}\cos.4\varphi - \tfrac{1}{3}\cos.6\varphi - \tfrac{1}{4}\cos.8\varphi - \text{etc.}$$

erit valor formulae propositae

$$\begin{aligned}\int d\varphi\, l\sin.\varphi = C - \varphi\, l\,2 - \tfrac{1}{2}\sin.2\varphi - \tfrac{1}{8}\sin.4\varphi - \tfrac{1}{18}\sin.6\varphi\\ - \tfrac{1}{32}\sin.8\varphi - \tfrac{1}{50}\sin.10\varphi - \text{etc.}\end{aligned}$$

quae expressio cum evanescere debeat posito $\varphi = 0$, constans hic ingressa erit $C = 0$, ita ut jam in genere sit

$$\int d\varphi\, l\sin.\varphi = -\varphi\, l\, 2 - \frac{2\sin.2\varphi}{2^2} - \frac{2\sin.4\varphi}{4^2} - \frac{2\sin.6\varphi}{6^2} - \frac{2\sin.8\varphi}{8^2}$$
$$- \frac{2\sin.10\varphi}{10^2} - \frac{2\sin.12\varphi}{12^2} - \text{etc.}$$

Quodsi jam hic capiatur $\varphi = 90^0 = \frac{\pi}{2}$, omnium angulorum 2φ, 4φ, 6φ, 8φ, etc. qui hic occurrunt sinus evanescunt, ideoque valor quaesitus erit

$$S = \int d\varphi\, l\sin.\varphi \left[\begin{smallmatrix} a\ \varphi = 0 \\ a\ d\varphi = 90^0 \end{smallmatrix}\right] = -\frac{\pi}{2}\, l\, 2;$$

quemadmodum etiam in priore demonstratione ostendimus.

§. 124. Ista autem demonstratio praecedenti ideo longe antecellit, quod nobis non solum valorem formulae propositae exhibeat casu quo $\varphi = 90^0$, sed etiam verum ejus valorem ostendat, quicunque angulus pro φ accipiatur, id quod ad ipsam formulam propositam $\int \frac{dx\, lx}{\sqrt{(1-xx)}}$ transferri poterit, cujus adeo valorem pro quolibet valore ipsius x assignare poterimus. Quodsi enim istius formulae valorem desideremus ab $x = 0$ usque ad $x = a$, quaeratur angulus α cujus sinus sit aequalis ipsi a, atque semper habebitur

$$\int \frac{dx\, lx}{\sqrt{(1-xx)}} \left[\begin{smallmatrix} a b x = 0 \\ a d x = a \end{smallmatrix}\right] = -\alpha\, l\, 2 - \frac{2\sin.2\alpha}{2^2} - \frac{2\sin.4\alpha}{4^2} - \frac{2\sin.6\alpha}{6^2} - \frac{2\sin.8\alpha}{8^2} - \text{etc.}$$

Unde patet, quoties fuerit $\alpha = \frac{i\pi}{2}$, denotante i numerum integrum quemcunque, quoniam omnes sinus evanescunt, valor formulae his casibus finite exprimi per $-\frac{i\pi}{2}\, l\, 2$; aliis vero casibus valor nostrae formulae per seriem infinitam satis concinnam exprimetur. Ita si capiatur $a = \frac{1}{\sqrt{2}}$, ut sit $\alpha = \frac{\pi}{4}$, valor nostrae formulae erit

$$-\frac{\pi}{4}\, l\, 2 - \frac{2}{2^2} + \frac{2}{6^2} - \frac{2}{10^2} + \frac{2}{14^2} - \frac{2}{18^2} + \frac{2}{22^2} - \text{etc.}$$

quae series elegantius ita exprimitur

$$-\frac{\pi}{4}\, l\, 2 - \frac{1}{2}\left(1 - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} + \text{etc.}\right);$$

sicque hic occurrit series satis memorabilis

$$1-\frac{1}{9}+\frac{1}{25}-\frac{1}{49}+\frac{1}{81}-\frac{1}{121}+\text{etc.}$$

cujus summam nullo adhuc modo ad mensuras cognitas revocare licuit.

§. 125. Quoniam tam egregia series hic se quasi praeter exspectationem obtulit, etiam alios casus evolvamus notabiliores, sumamusque $a=\frac{1}{2}$, ut sit $\alpha=30^0=\frac{\pi}{6}$, atque nostrae formulae hoc casu valor erit

$$-\frac{\pi}{6}l\,2-\frac{\sqrt{3}}{2^2}-\frac{\sqrt{3}}{4^2}+\frac{\sqrt{3}}{8^2}+\frac{\sqrt{3}}{10^2}-\frac{\sqrt{3}}{14^2}-\frac{\sqrt{3}}{16^2}+\text{etc.}$$

quae expressio ita exhiberi potest

$$-\frac{\pi}{6}l\,2-\frac{\sqrt{3}}{4}\left(1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\frac{1}{10^2}-\frac{1}{11^2}+\text{etc.}\right)$$

in qua serie quadrata multiplorum ternarii deficiunt. Sumamus nunc simili modo $a=\frac{\sqrt{3}}{2}$, ut sit $\alpha=60^0=\frac{\pi}{3}$, ac valor nostrae formulae hoc casu prodibit

$$-\frac{\pi}{3}l\,2-\frac{\sqrt{3}}{2^2}+\frac{\sqrt{3}}{4^2}-\frac{\sqrt{3}}{8^2}+\frac{\sqrt{3}}{10^2}-\frac{\sqrt{3}}{14^2}+\frac{\sqrt{3}}{16^2}-\text{etc.}$$

sive hoc modo exprimetur

$$-\frac{\pi}{3}l\,2-\frac{\sqrt{3}}{4}\left(1-\frac{1}{2^2}+\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}+\frac{1}{10^2}-\frac{1}{11^2}+\text{etc.}\right)$$

Adhuc alia demonstratio integrationis propositae.

§. 126. Introducatur in formulam nostram angulus φ, cujus cosinus sit $=x$, sive sit $x=\cos.\varphi$, et formula nostra induet hanc formam $-\int d\varphi\, l\cos.\varphi$, quod integrale a $\varphi=90^0$ usque ad $\varphi=0$ erit extendendum. Quodsi autem hos terminos permutemus, valor S, quem quaerimus, ita exprimetur

$$S=\int d\varphi\, l\cos.\varphi\left[\begin{smallmatrix}a\ \varphi=0\\ ad\ \varphi=90^0\end{smallmatrix}\right].$$

Ut hic $l\cos.\varphi$ in seriem idoneam convertamus, statuamus ut ante $s=l\cos.\varphi$,

esse

$$\frac{\sin.\Phi}{\cos.\Phi} = 2\sin.2\Phi - 2\sin.4\Phi + 2\sin.6\Phi - 2\sin.8\Phi + \text{etc.}$$

Cum enim in genere sit

$$2\sin.n\Phi\cos.\Phi = \sin.(n+1)\Phi + \sin.(n-1)\Phi,$$

si utrinque per $\cos.\Phi$ multiplicemus, orietur

$$\begin{aligned}\sin.\Phi = \quad & \sin.3\Phi - \sin.5\Phi + \sin.7.\Phi - \sin.9\Phi + \text{etc.}\\ & + \sin.\Phi - \sin.3\Phi + \sin.5\Phi - \sin.7.\Phi + \sin.9\Phi - \text{etc.}\end{aligned}$$

quare cum sit $\partial s = -\frac{\partial\Phi\sin.\Phi}{\cos.\Phi}$, erit nunc

$$S = C + \frac{\cos.2\Phi}{1} - \frac{\cos.4\Phi}{2} + \frac{\cos.6\Phi}{3} - \frac{\cos.8\Phi}{4} + \frac{\cos.10\Phi}{5} - \text{etc.}$$

Quia igitur est $s = l\cos.\Phi$, evidens est posito $\Phi = 0$, fieri debere $s = 0$, unde colligitur

$$C = -1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \text{etc.} = -l2;$$

sicque erit

$$l\cos.\Phi = -l2 + \frac{\cos.2\Phi}{1} - \frac{\cos.4\Phi}{2} + \frac{\cos.6\Phi}{3} - \frac{\cos.8\Phi}{4} + \text{etc.}$$

quae series ducta in $\partial\Phi$ et integrata praebet

$$\begin{aligned}S = \int\partial\Phi\, l\cos.\Phi = C - \Phi l2 + \frac{\sin.2\Phi}{2} - \frac{\sin.4\Phi}{8} + \frac{\sin.6\Phi}{18} - \frac{\sin.8\Phi}{32}\\ + \frac{\sin.10\Phi}{50} - \text{etc.}\end{aligned}$$

quae expressio quia sponte evanescit posito $\Phi = 0$, inde patet fore $C = 0$, sicque habebimus

$$\int\partial\Phi\, l\cos.\Phi = -\Phi l2 + \tfrac{1}{2}\left(\frac{\sin.2\Phi}{1} - \frac{\sin.4\Phi}{2^2} + \frac{\sin.6\Phi}{3^2} - \frac{\sin.8\Phi}{4^2} + \frac{\sin.10\Phi}{5^2} - \text{etc.}\right)$$

Sumto igitur $\Phi = \frac{\pi}{2} = 90^0$, oritur ut ante $S = -\frac{\pi}{2}l2$. Praeterea vero etiam hinc integrale ad quemvis terminum usque extendere licet.

§. 127. Quodsi formulam posteriorem a praecedente subtrahamus, adipiscemur in genere hanc integrationem

$\int \partial \Phi \, l \, \text{tang.} \, \Phi = - \sin. 2\Phi - \frac{1}{3^2} \sin. 6\Phi - \frac{1}{5^2} \sin. 10\Phi - \text{etc.}$ unde patet hoc integrale evanescere casibus $\Phi = 0^0$ et in genere $\Phi = \frac{i\pi}{2}$. Postquam igitur istam integrationem triplici modo demonstravimus, ipsam Analysin, quae me primum huc perduxit, hic dilucide sum expositurus.

Analysis ad integrationem formulae $\int \frac{\partial x \, lx}{\sqrt{(1-xx)}}$ aliarumque similium perducens.

§. 123. Tota haec Analysis innititur sequenti lemmati a me jam olim demonstrato: Posito brevitatis gratia $(1-x^n)^{\frac{m-n}{n}} = X$, si hinc duae formulae integrales formentur $\int X x^{p-1} \partial x$ et $\int X x^{q-1} \partial x$, quae a termino $x = 0$ usque ad terminum $x = 1$ extendantur, ratio horum valorum sequenti modo ad productum ex infinitis factoribus conflatum reduci potest

$$\frac{\int X x^{p-1} \partial x}{\int X x^{q-1} \partial x} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(q+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(q+2n)}{(p+2n)(m+q+2n)} \text{ etc.}$$

ubi scilicet singuli factores tam numeratoris, quam denominatoris continuo eadem quantitate n augentur. Hic autem probe tenendum est, veritatem istius lemmatis subsistere non posse, nisi singulae litterae m, n, p, et q denotent numeros positivos, quos tamen semper tanquam integros spectare licet.

§. 129. Circa has duas formulas integrales, a termino $x = 0$ usque ad $x = 1$ extensas, duo casus imprimis seorsim notari merentur, quibus integratio actu succedit, verusque valor absolute assignari potest. Prior casus locum habet, si fuerit $p = n$, ita ut formula sit $\int X x^{n-1} \partial x$. Posito enim $x^n = y$ fiet

$$X = (1-y)^{\frac{m-n}{n}}, \text{ et } x^{n-1} \partial x = \frac{1}{n} \partial y$$

sicque ista formula evadet $\frac{1}{n}\int \partial y\,(1-y)^{\frac{m-n}{n}}$, pariter a termino $y=0$ usque ad $y=1$ extendenda, quae porro posito $1-y=z$ abit in hanc formulam $-\frac{1}{n}\int z^{\frac{m-n}{n}}\partial z$, a termino $z=1$ usque ad $z=0$ extendendam; ejus ergo integrale manifesto est $-\frac{1}{m}z^{\frac{m}{n}}+\frac{1}{m}$; unde facto $z=0$ valor erit $=\frac{1}{m}$. Consequenter pro casu $p=n$ habebimus

$$\int X\,x^{n-1}\,\partial x\left[\begin{smallmatrix}ab\,x=0\\ ad\,x=1\end{smallmatrix}\right]=\frac{1}{m};$$

sicque si fuerit vel $p=n$ vel $q=n$, integrale absolute innotescit.

§. 130. Alter casus notatu dignus est, quo $p=n-m$, ita ut formula integranda sit $\int X x^{n-m-1}\partial x$; tum enim, si ponatur $x(1-x^n)^{\frac{-1}{n}}$ sive $\frac{x}{(1-x^n)^{\frac{1}{n}}}=y$, posito $x=0$ fiet $y=0$, at posito $x=1$ fiet $y=\infty$; tum autem erit

$$y^{n-m}=\frac{x^{n-m}}{(1-x^n)^{\frac{n-m}{n}}}=X\,x^{n-m},$$

unde formula integranda erit $\int y^{n-m}\frac{\partial x}{x}$. Cum igitur sit

$$\frac{x}{(1-x^n)^{\frac{1}{n}}}=y,\ \text{erit}\ \frac{x^n}{1-x^n}=y^n,$$

unde colligitur $x^n=\frac{y^n}{1+y^n}$, ideoque $n\,l\,x=n\,l\,y-l\,(1+y^n)$, cujus differentiatio praebet

$$\frac{\partial x}{x}=\frac{\partial y}{y(1+y^n)},$$

quo valore substituto formula nostra integranda erit

$$\int \frac{y^{n-m-1}\partial y}{1+y^n},$$

a termino $y = 0$ usque ad $y = \infty$ extendenda, quae formula ideo est notatu digna, quod ab omni irrationalitate est liberata.

§. 131. Quoniam igitur hoc casu ad formulam rationalem sumus perducti, ex elementis calculi integralis constat, ejus integrationem semper per logarithmos et arcus circulares absolvi posse, tum vero pro hoc casu non ita pridem ostendi, hujus formulae $\int \frac{x^{m-1}\partial x}{1+x^n}$ integrale, ab $x = 0$ usque ad $x = \infty$ extensum, reduci ad valorem $\frac{\pi}{n \sin. \frac{m\pi}{n}}$; facta igitur applicatione pro nostro casu habebimus

$$\int \frac{y^{n-m-1}\partial y}{1+y^n} = \frac{\pi}{n \sin. \frac{(n-m)\pi}{n}} = \frac{\pi}{n \sin. \frac{m\pi}{n}};$$

quamobrem pro casu $p = n - m$ valor integralis sequenti modo absolute exprimi potest, eritque

$$\int X x^{n-m-1}\partial x \left[\begin{smallmatrix} ab\ x=0 \\ ad\ x=1 \end{smallmatrix}\right] = \frac{\pi}{n \sin. \frac{m\pi}{n}},$$

quod idem manifesto tenendum est, si fuerit $q = n - m$.

§. 132. His praemissis, ponamus porro brevitatis gratia

$$\int X x^{p-1}\partial x \left[\begin{smallmatrix} ab\ x=0 \\ ad\ x=1 \end{smallmatrix}\right] = P \text{ et}$$
$$\int X x^{q-1}\partial x \left[\begin{smallmatrix} ab\ x=0 \\ ad\ x=1 \end{smallmatrix}\right] = Q,$$

atque lemma allatum nobis praebet hanc aequationem

$$\frac{P}{Q} = \frac{(m+p)q}{p(m+q)} \cdot \frac{(m+p+n)(p+n)}{(p+n)(m+q+n)} \cdot \frac{(m+p+2n)(p+2n)}{(p+2n)(m+q+2n)} \text{ etc.}$$

Hinc igitur sumendis logarithmis deducimus

$$lP-lQ=l(m+p)-lp+l(m+p+n)-l(p+n)+l(m+p+2n)-l(p+2n)+\text{etc.}$$
$$+lq-l(m+q)+l(q+n)-l(m+q+n)+l(q+2n)-l(m+q+2n)+\text{etc.}$$

haecque aequalitas semper locum habebit, quicunque valores litteris m, n, p et q tribuantur, dummodo fuerint positivi.

§. 133. Cum igitur haec aequalitas in genere subsistat, etiam veritati erit consentanea, quando quaepiam harum litterarum m, n, p et q infinite parum immutantur, sive tanquam variabiles spectantur. Hanc ob rem consideremus solam quantitatem p tanquam variabilem, ita ut reliquae litterae m, n et q maneant constantes, ideoque etiam quantitas Q erit constans dum altera P variabitur; ex quo differentiando nanciscemur hanc aequationem

$$\frac{\partial P}{P}=\frac{\partial p}{m+p}-\frac{\partial p}{p}+\frac{\partial p}{m+p+n}-\frac{\partial p}{p+n}+\frac{\partial p}{m+p+2n}-\frac{\partial p}{p+2n}$$
$$+\frac{\partial p}{m+p+3n}-\frac{\partial p}{p+3n}+\text{etc.}$$

ubi totum negotium eo redit, quemadmodum differentiale formulae P, quae est integralis, exprimi oporteat.

§. 134. Cum igitur p sit formula integralis solam quantitatem x tanquam variabilem involvens, quandoquidem in ejus integratione exponens p ut constans tractari debet, demum post integrationem ipsam quantitatem P tanquam functionem duarum variabilium x et p spectare licebit; unde quaestio huc redit, quomodo valorem, hoc charactere $\left(\frac{\partial P}{\partial p}\right)$ exprimi solitum, investigari oporteat, qui si indicetur littera Π, aequatio ante inventa hanc induet formam

$$\frac{\Pi}{P}=\frac{1}{m+p}-\frac{1}{p}+\frac{1}{m+p+n}-\frac{1}{p+n}+\frac{1}{m+p+2n}-\frac{1}{p+2n}+\text{etc.}$$

Hanc vero seriem infinitam haud difficulter ad expressionem finitam revocare licebit hoc modo: Ponatur

$$s = \frac{v^{m+p}}{m+p} - \frac{v^p}{p} + \frac{v^{m+p+n}}{m+p+n} - \frac{v^{p+n}}{p+n} + \frac{v^{m+p+2n}}{m+p+2n} - \frac{v^{p+2n}}{p+2n} + \text{etc.}$$

ita ut facto $v = 1$ littera s nobis exhibeat valorem quaesitum $\frac{\Pi}{P}$; at vero differentiatio nobis dabit

$$\frac{\partial s}{\partial v} = v^{m+p-1} - v^{p-1} + v^{m+p+n-1} - v^{p+n-1} + v^{m+p+2n-1} - v^{p+2n-1} + \text{etc.}$$

cujus seriei infinitae summa manifesto est

$$\frac{v^{m+p-1} - v^{p-1}}{1 - v^n} = \frac{v^{p-1}(v^m - 1)}{1 - v^n}.$$

Hinc igitur vicissim concludimus fore

$$s = \int \frac{v^{p-1}(v^m - 1)\partial v}{1 - v^n},$$

quae formula integralis a $v = 0$ usque ad $v = 1$ est extendenda; sicque habebimus

$$\frac{\Pi}{P} = \int \frac{v^{p-1}(v^m - 1)\partial v}{1 - v^n} \left[\begin{array}{l} a\ v = 0 \\ ad\ v = 1 \end{array}\right].$$

§. 135. Ad valorem autem $\left(\frac{\partial P}{\partial p}\right)$, quem hic littera Π indicavimus, investigandum, ex principiis calculi integralis ad functiones duarum variabilium applicati jam satis notum est, differentiale formulae integralis $P = \int X x^{p-1} \partial x$ ex sola variabilitate ipsius p oriundum obtineri, si formula post signum integrationis posita $X x^{p-1}$, ex sola variabilitate ipsius p differentietur, atque elementum ∂p signo integrationis praefigatur; at vero quia X non continet p, hic ut constans tractari debet: potestatis vero x^{p-1} differentiale hinc natum erit $x^{p-1} \partial p\, l x$; quam ob rem ex hac differentiatione orietur $\partial P = \partial p \int X x^{p-1} \partial x\, l x$, ita ut tantum post signum integrationis factor $l x$ accesserit, ex quo manifestum est, fore

$$\Pi = \int X x^{p-1} \partial x \, l x \begin{bmatrix} ab\, x = 0 \\ ad\, x = 1 \end{bmatrix},$$

hinc igitur sequens theorema generale constituere licebit.

Theorema generale.

§. 136. Posito brevitatis gratia $X = (1 - x^n)^{\frac{m-n}{n}}$, si sequentes formulae integrales omnes a termino $x = 0$ ad terminum $x = 1$ extendantur, sequens aequalitas semper erit veritati consentanea

$$\int \frac{X x^{p-1} \partial x \, l x}{\int X x^{p-1} \partial x} = \int \frac{x^{p-1} (x^m - 1) \partial x}{1 - x^n}$$

nihil enim obstabat, quo minus loco v scriberemus x, quandoquidem isti valores tantum a terminis integrationis pendent.

§. 137. Hoc igitur modo deducti sumus ad integrationem hujusmodi formularum $\int X x^{p-1} \partial x \, l x$, in quibus quantitas logarithmica $l x$ post signum integrationis tanquam factor inest, quarum valorem exprimere licuit per binas formulas integrales ordinarias, cum sit

$$\int X x^{p-1} \partial x \, l x = \int X x^{p-1} \partial x \, . \int \frac{x^{p-1} (x^m - 1) \partial x}{1 - x^n},$$

integralibus scilicet ab $x = 0$ ad $x = 1$ extensis, ubi brevitatis gratia posuimus $(1 - x^n)^{\frac{m-n}{n}} = X$. Hinc igitur pro binis casibus memorabilibus supra expositis bina theoremata particularia derivemus.

Theorema particulare I, quo $p = n$.

§. 138. Quoniam supra vidimus casu $p = n$ fieri $\int X x^{n-1} \partial x = \frac{1}{m}$, hoc valore substituto habebimus istam aequationem satis elegantem

$$\int X x^{n-1} \partial x \, l x = \frac{1}{m} \int \frac{x^{n-1}(x^m - 1) \partial x}{1 - x^n},$$

dum scilicet ambo integralia ab $x = 0$ ad $x = 1$ extenduntur.

Theorema particulare II, quo $p = n - m$.

§. 139. Quoniam pro hoc casu, quo $p = n - m$ supra ostendimus esse

$$\int X x^{n-m-1} \partial x = \frac{\pi}{n \sin. \frac{m\pi}{n}},$$

nunc deducimur ad sequentem integrationem maxime notatu dignam

$$\int X x^{n-m-1} \partial x \, l x = \frac{\pi}{n \sin. \frac{m\pi}{n}} \int \frac{x^{n-m-1}(x^m - 1) \partial x}{1 - x^n},$$

si quidem haec ambo integralia ab $x = 0$ usque ad $x = 1$ extendantur; ubi meminisse oportet esse

$$X = (1 - x^n)^{\frac{m-n}{n}}.$$

§. 140. Hic probe notetur, theorema generale latissime patere, propterea quod in eo insunt tres exponentes indefiniti, scilicet m, n et p, qui penitus arbitrio nostro relinquuntur, quos ergo infinitis modis pro lubitu definire licet, dummodo singulis valores positivi tribuantur, ita ut semper valor hujus formulae integralis $\int X x^{p-1} \partial x \, l x$, quam ob factorem $l x$ tanquam transcendentem spectari oportet, per formulas integrales ordinarias exprimi queat, quae cum sint generalissima, operae pretium erit nonnullos casus speciales evolvere.

I. Evolutio casus, quo $m = 1$ et $n = 2$.

§. 141. Hoc igitur casu erit $X = \frac{1}{\sqrt{(1-xx)}}$, unde pro hoc casu theorema generale ita se habebit

$$\int \frac{x^{p-1}\,\partial x\, l x}{\sqrt{(1-xx)}} = -\int \frac{x^{p-1}\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^{p-1}\,\partial x}{1+x},$$

siquidem singula haec integralea ab $x = 0$ ad $x = 1$ extendantur. Quoniam igitur hic tantum exponens p arbitrio nostro relinquitur, hinc sequentia exempla perlustremus.

Exemplum I. quo $p = 1$.

§. 142. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{\partial x\, l x}{\sqrt{(1-xx)}} = -\int \frac{\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{\partial x}{1+x}$$

ubi, integralibus ab $x = 0$ ad $x = 1$ extensis, notum est fieri

$$\int \frac{\partial x}{\sqrt{(1-xx)}} = \tfrac{\pi}{2} \text{ et } \int \frac{\partial x}{1+x} = l\,2;$$

ita ut jam habeamus

$$\int \frac{\partial x\, l x}{\sqrt{(1-xx)}} \left[\begin{matrix} ab\ x = 0 \\ ad\ x = 1 \end{matrix}\right] = -\tfrac{\pi}{2}\, l\,2,$$

quae est ea ipsa formula, quam initio hujus dissertationis tractavimus et cujus veritatem jam triplici demonstratione corroboravimus

§. 143. Eundem valorem elicere licet ex theoremate particulari secundo, quo erat $p = n - m$, siquidem nunc ob $n = 2$ et $m = 1$ erit $p = 1$; inde enim ob $X = \frac{1}{\sqrt{(1-xx)}}$, istud theorema praebet

$$\int \frac{\partial x\, l x}{\sqrt{(1-xx)}} = \frac{\pi}{2 \sin. \frac{\pi}{2}} \cdot \int \frac{\partial x}{1+x} = -\tfrac{\pi}{2}\, l\,2.$$

Exemplum II. quo $p = 2$.

§. 144. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{x\,\partial x\, l x}{\sqrt{(1-xx)}} = -\int \frac{x\,\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x\,\partial x}{1+x}$$

Jam vero integralibus ab $x=0$ ad $x=1$ extensis, notum est fore

$$\int\frac{x\,\partial x}{\sqrt{(1-xx)}}=1 \text{ et } \int\frac{x\,\partial x}{1+x}=1-l\,2;$$

ita ut habeamus

$$\int\frac{x\,\partial x\,l\,x}{\sqrt{(1-xx)}}\begin{bmatrix}ab\ x=0\\ ad\ x=1\end{bmatrix}=l\,2-1.$$

§. 145. Quoniam in hac formula integrale $\int\frac{x\,\partial x}{\sqrt{(1-xx)}}$, algebraice exhiberi potest, cum sit $=1-\sqrt{(1-xx)}$, valor quaesitus etiam per reductiones consvetas erui potest, cum sit

$$\int\frac{x\,\partial x\,l\,x}{\sqrt{(1-xx)}}=[1-\sqrt{(1-xx)}]\,l\,x-\int\frac{\partial x}{x}[1-\sqrt{(1-xx)}],$$

positoque $x=1$ erit

$$\int\frac{x\,\partial x\,l\,x}{\sqrt{(1-xx)}}=-\int\frac{\partial x}{x}[1-\sqrt{(1-xx)}],$$

ad quam formam integrandam fiat $1-\sqrt{(1-xx)}=z$, unde colligitur $xx=2z-zz$, ergo $2\,l\,x=l\,z+l\,(2-z)$, sicque fiet $\frac{\partial x}{x}=\frac{\partial z(1-z)}{z(2-z)}$, quibus valoribus substitutis erit

$$+\int\frac{\partial x}{x}[1-\sqrt{(1-xx)}]=+\int\frac{\partial z(1-z)}{2-z},$$

qui ergo valor erit $=C-z-l\,(2-z)$. Quia igitur posito $x=0$ fit $z=0$, constans erit $C=+l\,2$; facto igitur $x=1$, quia tum fit $z=1$, iste valor integralis erit $l\,2-1$, prorsus ut ante.

§. 146. Eundem valorem suppeditat theorema prius supra allatum, quo erat $p=n=2$; inde enim statim fit $\int\frac{x\,\partial x\,l\,x}{\sqrt{(1-xx)}}=\int-\frac{x\,\partial x}{1+x}$. Ante autem vidimus esse $\int\frac{x\,\partial x}{1+x}=1-l\,2$; ita ut etiam hinc prodeat valor quaesitus $l\,2-1$.

Exemplum III. quo $p=3$.

§. 147. Hoc igitur casu aequatio in theoremate generali allata hanc induet formam

$$\int\frac{xx\,\partial x\,l\,x}{\sqrt{(1-xx)}}=-\int\frac{xx\,\partial x}{\sqrt{(1-xx)}}\cdot\int\frac{xx\,\partial x}{1+x}.$$

Per reductiones autem notissimas constat esse

$$\int \frac{xx\,\partial x}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix} = \tfrac{1}{2}\cdot\tfrac{\pi}{2},$$

at vero fractio spuria $\frac{xx}{1+x}$ resolvitur in has partes $x - 1 + \frac{1}{1+x}$, unde erit

$$\int \frac{xx\,\partial x}{1+x} = \tfrac{1}{2}xx - x + l(1+x),$$

quod integrale jam evanescit posito $x=0$; facto ergo $x=1$ ejus valor erit $= -\frac{1}{2} + l\,2$; quamobrem integrale quod quaerimus, erit

$$\int \frac{xx\,\partial x\,lx}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix} = -\tfrac{\pi}{4}\left(l\,2 - \tfrac{1}{2}\right).$$

Exemplum IV. quo $p=4$.

§. 148. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{x^3\,\partial x\,lx}{\sqrt{(1-xx)}} = -\int \frac{x^3\,\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^3\,\partial x}{1+x}.$$

Per reductiones autem notissimas constat esse

$$\int \frac{x^3\,\partial x}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix} = \tfrac{2}{3},$$

tum vero fractio spuria $\frac{x^3}{1+x}$ resolvitur in has partes $xx - x + 1 - \frac{1}{1+1}$, unde integrando fit

$$\int \frac{x^3\,\partial x}{1+x} = \tfrac{1}{3}x^3 - \tfrac{1}{2}xx + x - l(1+x),$$

ox quo valor formulae erit $= \frac{5}{6} - l\,2$. His ergo valoribus substitutis adipiscimur hanc integrationem

$$\int \frac{x^3\,\partial x\,lx}{\sqrt{(1-xx)}} \begin{bmatrix} ab\ x=0 \\ ad\ x=1 \end{bmatrix} = -\tfrac{2}{3}\left(\tfrac{5}{6} - l\,2\right).$$

Exemplum V. quo $p=5$.

§. 149. Hoc igitur casu aequatio superior hanc induet formam

$$\int \frac{x^4\,\partial x\,lx}{\sqrt{(1-xx)}} = -\int \frac{x^4\,\partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^4\,\partial x}{1+x}.$$

Constat autem esse

$$\int \frac{x^4 \partial x}{\sqrt{(1-xx)}} \left[\begin{smallmatrix} ab\ x=0 \\ ad\ x=1 \end{smallmatrix}\right] = \frac{1.3}{2.4} \cdot \frac{\pi}{2},$$

tum vero fractio spuria $\frac{x^4}{1+x}$ manifesto resolvitur in has partes $x^3 - xx + x - 1 + \frac{1}{x+1}$ unde integrando fit

$$\int \frac{x^4 \partial x}{1+x} = \tfrac{1}{4} x^4 - \tfrac{1}{3} x^3 + \tfrac{1}{2} xx - x + l(1+x),$$

ex quo valor formulae erit $= -\frac{7}{12} + l\,2$. His igitur valoribus substitutis prodibit ista integratio

$$\int \frac{x^4 \partial x\, l x}{\sqrt{(1-xx)}} \left[\begin{smallmatrix} ab\ x=0 \\ ad\ x=1 \end{smallmatrix}\right] = -\frac{1.3}{2.4} \cdot \frac{\pi}{2} \left(l\,2 - \tfrac{7}{12}\right).$$

Exemplum VI. quo $p = 6$.

§. 150. Hoc igitur casu aequatio superior induet hanc formam

$$\int \frac{x^6 \partial x\, l x}{\sqrt{(1-xx)}} = -\int \frac{x^6 \partial x}{\sqrt{(1-xx)}} \cdot \int \frac{x^5 \partial x}{1+x}.$$

Constat autem per reductiones notas esse

$$\int \frac{x^6 \partial x}{\sqrt{(1-xx)}} \left[\begin{smallmatrix} ab\ x=0 \\ ad x=1 \end{smallmatrix}\right] = \frac{2.4}{3.5},$$

tum vero fractio spuria $\frac{x^6}{1+x}$ resolvitur in has partes

$$x^4 - x^3 + xx - x + 1 - \frac{1}{x+1},$$

unde integrando nanciscimur

$$\int \frac{x^5 \partial x}{1+x} = \tfrac{1}{5} x^5 - \tfrac{1}{4} x^4 + \tfrac{1}{3} x^3 - \tfrac{1}{2} xx + x - l(1+x),$$

ex quo valor hujus formulae erit $= \frac{47}{60} - l\,2$; quibus valoribus substitutis prodibit ista integratio

$$\int \frac{x^6 \partial x\, l x}{\sqrt{(1-xx)}} \left[\begin{smallmatrix} ab\ x=0 \\ ad\ x=1 \end{smallmatrix}\right] = -\frac{2.4}{3.5} \left(\tfrac{47}{60} - l\,2\right).$$

II. Evolutio casus quo $m = 3$ et $n = 2$.

§. 151. Hic ergo erit $X = \sqrt{(1-xx)}$, unde theorema nostrum generale nobis praebebit hanc aequationem

$$\int x^{p-1}\partial x l x . \sqrt{(1-xx)} = \int x^{p-1}\partial x \sqrt{(1-xx)} . \int \frac{x^{p-1}(x^3-1)\partial x}{1-xx},$$

ubi cum sit

$$\frac{x^3-1}{1-xx} = \frac{-xx-x-1}{x+1} = -x - \frac{1}{x+1},$$

erit postrema formula integralis

$$-\int x^p \partial x - \int \frac{x^{p-1}\partial x}{1+x};$$

quae integrata ab $x=0$ ad $x=1$ dat

$$-\frac{1}{p+1} - \int \frac{x^{p-1}\partial x}{1+x},$$

quamobrem habebimus

$$\int x^{p-1}\partial x l x . \sqrt{(1-xx)} = -\int x^{p-1}\partial x \sqrt{(1-xx}\left(\frac{1}{p+1} + \int \frac{x^{p-1}\partial x}{1+x}\right).$$

Hinc igitur sequentia exempla notasse juvabit.

Exemplum I. quo $p=1$.

§. 152. Pro hoc igitur casu postremus factor evadet, $\frac{1}{2}+l2$, ita ut sit

$$\int \partial x l x . \sqrt{(1-xx)} = -(\tfrac{1}{2}+l2)\int \partial x \sqrt{(1-xx)}.$$

Pro formula autem $\int \partial x \sqrt{(1-xx)}$ statuatur $\sqrt{(1-xx)} = 1-vx$, fietque

$$x = \frac{2v}{1+vv}, \text{ et } \sqrt{(1-xx)} = \frac{1-vv}{1+vv},$$

atque $\partial x = \frac{2\partial v(1-vv)}{(1+vv)^2}$, unde fiet

$$\partial x \sqrt{(1-xx)} = \frac{2\partial v(1-vv)^2}{(1+vv)^3},$$

cujus integrale resolvitur in has partes

$$\frac{2v}{(1+vv)^2} - \frac{v}{1+vv} + \text{Arc. tang. } v;$$

quae expressio, cum extendi debeat ab $x=0$ usque ad $x=1$,

prior terminus erit $v = 0$, alter vero terminus est $v = 1$; ita ut integrale illud a $v = 0$ usque ad $v = 1$ extendi debeat. At vero illa expressio sponte evanescit posito $v = 0$, facto autem $v = 1$, valor integralis erit $= \frac{\pi}{4}$, quamobrem habebimus

$$\int \partial x\, lx \,.\, \sqrt{(1 - xx)} \left[\begin{smallmatrix} ab\, x = 0 \\ ad\, x = 1 \end{smallmatrix}\right] = -\frac{\pi}{4}\left(\frac{1}{2} + l\,2\right).$$

§. 153. Hic quidem calculum per longas ambages evolvimus, prouti reductio ad rationalitatem formulae $\sqrt{(1 - xx)}$ manuduxit; at vero solus aspectus formulae $\int \partial x \sqrt{(1 - xx)}$ statim declarat, eam exprimere aream quadrantis circuli, cujus radius $= 1$, quem novimus esse $= \frac{\pi}{4}$. Caeterum adhiberi potuisset ista reductio

$$\int \partial x \sqrt{(1 - xx)} = \frac{1}{2} x \sqrt{(1 - xx)} + \frac{1}{2} \int \frac{\partial x}{\sqrt{(1 - xx)}}$$

cujus valor ab $x = 0$ ad $x = 1$ extensus manifesto dat $\frac{\pi}{4}$.

Exemplum II. quo $p = 2$.

§. 154. Hoc ergo casu postremus factor fit

$$\frac{1}{3} + \int \frac{x\,\partial x}{1 + x} = \frac{4}{3} - l\,2;$$

sicque habebimus

$$\int x\,\partial x\, lx \,.\, \sqrt{(1 - xx)} = -\left(\frac{4}{3} - l\,2\right) \int x\,\partial x \sqrt{(1 - xx)}:$$

perspicuum autem est, esse

$$\int x\,\partial x \sqrt{(1 - xx)} = C - \frac{1}{3}(1 - xx)^{\frac{3}{2}},$$

qui valor ab $x = 0$ ad $x = 1$ extensus praebet $\frac{1}{3}$, ita ut habeamus

$$\int x\,\partial x\, lx \,.\, \sqrt{(1 - xx)} \left[\begin{smallmatrix} ab\, x = 0 \\ ad\, x = 1 \end{smallmatrix}\right] = -\frac{1}{3}\left(\frac{4}{3} - l\,2\right).$$

III. Evolutio casus quo $m = 1$ et $n = 3$.

§. 155. Hoc igitur casu erit $X = \frac{1}{\sqrt[3]{(1-x^3)^2}}$, unde theorema generale nobis praebet hanc aequationem

$$\int \frac{x^{p-1}\partial x\, l x}{\sqrt[3]{(1-x^3)^2}} = \int \frac{x^{p-1}\partial x}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x^{p-1}(x-1)\partial x}{1-x^3},$$

ubi postrema formula reducitur ad hanc

$$-\int \frac{x^{p-1}\partial x}{xx+x+1},$$

ita ut habeamus

$$\int \frac{x^{p-1}\partial x\, l x}{\sqrt[3]{(1-x^3)^2}} = -\int \frac{x^{p-1}\partial x}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x^{p-1}\partial x}{xx+x+1}:$$

tequentia igitur exempla adiungamus.

Exemplum I. quo $p = 1$.

§. 156. Hoc igitur casu postremus factor evadit $\frac{\partial x}{xx+x+1}$, cujus integrale indefinitum reperitur $\frac{2}{\sqrt{3}}$ Arc. tang. $\frac{x\sqrt{3}}{2+x}$, qui valor posito $x = 1$ abit in $\frac{\pi}{3\sqrt{3}}$; quocirca hoc casu habebimus

$$\int \frac{\partial x \,.\, l x}{\sqrt[3]{(1-x^3)^2}} = -\frac{\pi}{3\sqrt{3}} \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}};$$

at vero formula integralis $\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}}$ peculiarem quantitatem transcendentem involvit, quam neque per logarithmos, neque per arcus circulares explicare licet.

Exemplum II. quo $p = 2$.

§. 157. Hoc igitur casu postremus factor erit $\int \frac{x\partial x}{1+x+xx}$, qui in has partes resolvatur

$$\tfrac{1}{2}\int \frac{2x\partial x+\partial x}{1x+x+x} - \tfrac{1}{2}\int \frac{\partial x}{1+x+xx},$$

ubi partis prioris integrale est

$$\tfrac{1}{2}\,l(1+x+xx) = \tfrac{1}{2}\,l\,3 \text{ (posito scilicet } x = 1\text{);}$$

alterius vero partis integrale est $= \frac{1}{2} \cdot \frac{\pi}{3\sqrt{3}}$, quo valore substituto habebimus

$$\int \frac{x\partial x\, lx}{\sqrt[3]{(1-x^3)^2}} = -\tfrac{1}{2}\left(l\,3 - \tfrac{\pi}{3\sqrt{3}}\right) \int \frac{x\partial x}{\sqrt[3]{(1-x^3)^2}}.$$

Nunc vero istam formulam integralem commode assignare licet per reductionem supra initio indicatam; cum enim hic sit $m = 1$ et $n = 3$, tum vero sumserimus $p = 2$, erit $p = n - m$. Supra autem §. 131. invenimus, hoc casu integrale fore

$$= \frac{\pi}{n \sin. \frac{m\pi}{n}},$$

qui valor nostro casu abit in

$$\frac{\pi}{3 \sin. \frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}}.$$

Hoc igitur valore substituto, nostram formulam per meras quantitates cognitas exprimere poterimus, hoc modo

$$\int \frac{x\partial x\, lx}{\sqrt[3]{(1-x^3)^2}} \left[\begin{smallmatrix} ab\ x=0 \\ ad\ x=1 \end{smallmatrix}\right] = -\frac{\pi}{3\sqrt{3}}\left(l\,3 - \frac{\pi}{3\sqrt{3}}\right).$$

IV. Evolutio casus quo $m=2$ et $n=3$.

§. 158. Hoc igitur casu erit $X=\frac{1}{\sqrt[3]{(1-x^3)}}$, unde theorema generale praebet istam aequationem

$$\int\frac{x^{p-1}\partial x\, lx}{\sqrt[3]{(1-x^3)}}=\int\frac{x^{p-1}\partial x}{\sqrt[3]{(1-x^3)}}.\int\frac{x^{p-1}(xx-1)\partial x}{1-x^3},$$

ubi forma postrema transmutatur in hanc

$$-\int\frac{x^{p-1}\partial x(1+x)}{1+x+xx};$$

unde fiet

$$\int\frac{x^{p-1}\partial x\, lx}{\sqrt[3]{(1-x^3)}}=-\int\frac{x^{p-1}\partial x}{\sqrt[3]{(1-x^3)}}.\int\frac{x^{p-1}\partial x(1+x)}{1+x+xx}:$$

unde sequentia exempla expediamus.

Exemplum I. quo $p=1$.

§. 159. Hoc ergo casu membrum postremum erit $\int\frac{\partial x(1+x)}{1+x+xx}$, cujus integrale in has partes distribuatur.

$$\tfrac{1}{2}\int\frac{2x\partial x+\partial x}{1+x+xx}+\tfrac{1}{2}\int\frac{\partial x}{1+x+xx},$$

unde manifesto pro casu $x=1$ prodit $\frac{1}{2}\left(l\,3+\frac{\pi}{3\sqrt{3}}\right)$; quamobrem nostra aequatio erit

$$\int\frac{\partial x\, lx}{\sqrt[3]{(1-x^3)}}=-\tfrac{1}{2}\left(l\,3+\frac{\pi}{3\sqrt{3}}\right)\int\frac{\partial x}{\sqrt[3]{(1-x^3)}}.$$

In hac autem formula integrali, ob $m=2$ et $n=3$, quia sumsimus $p=1$, erit $p=n-m$; pro hoc ergo casu per §. 131. valor istius formulae absolute exprimi poterit, eritque

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)}} = \frac{2\pi}{3\sqrt{3}};$$

consequenter etiam hoc casu per quantitates absolutas consequimur hanc formam

$$\frac{\partial x\, l x}{\sqrt[3]{(1-x^3)}} \begin{bmatrix} ab\, x=0 \\ ad\, x=1 \end{bmatrix} = -\frac{\pi}{3\sqrt{3}}\left(l\,3+\frac{\pi}{3\sqrt{3}}\right).$$

§. 160. Quodsi hanc formam cum postrema casus praecedentis, quae itidem absolute prodiit expressa, combinemus, earum summa primo dabit

$$\int \frac{x\,\partial x\, l x}{\sqrt[3]{(1-x^3)^2}} + \int \frac{\partial x\, l x}{\sqrt[3]{(1-x^3)}} = -\frac{2\pi l 3}{3\sqrt{3}}:$$

sin autem posterior a priore subtrahatur, orietur ista aequatio

$$\int \frac{x\,\partial x\, l x}{\sqrt[3]{(1-x^3)^2}} - \int \frac{\partial x\, l x}{\sqrt[3]{(1-x^3)}} = \frac{2\pi\pi}{27}.$$

Quoniam hoc modo ad expressiones satis simplices sumus perducti, operae pretium erit ambas aequationes sub alia forma repraesentare, qua binae partes integrales commode in unam conjungi queant; statuamus scilicet $\frac{x}{\sqrt[3]{(1-x^3)}} = z$, unde fit $\frac{xx}{\sqrt[3]{(1-x^3)^2}} = zz$, sicque prior formula induet hanc speciem $\int \frac{zz\,\partial x\, l x}{x}$, posterior vero istam $\int \frac{z\,\partial x\, l x}{x}$; tum vero habebimus $\frac{x^3}{1-x^3} = z^3$, unde fit $x^3 = \frac{z^3}{1+z^3}$, ideoque

$$l x = l z - \tfrac{1}{3} l (1+z^3) = l \frac{z}{\sqrt[3]{(1+z^3)}}.$$

hincque porro

$$\frac{\partial x}{x} = \frac{\partial z}{z} - \frac{zz\partial z}{1+z^3} = \frac{\partial z}{z(1+z^3)};$$

quare his valoribus adhibitis, prior formula integralis evadit

$$\int \frac{z\partial z}{1+z^3} \cdot l \frac{z}{\sqrt[3]{(1+z^3)}};$$

altera vero formula erit

$$\int \frac{\partial z}{1+z^3} \cdot l \frac{z}{\sqrt[3]{(1+z^3)}}.$$

§. 161. Quoniam autem integralia ab $x = 0$ ad $x = 1$ extendi debeat, notandum est, casu $x = 0$ fieri $z = 0$, at vero casu $x = 1$ prodire $z = \infty$, ita ut novas istas formas a $z = 0$ ad $z = \infty$ extendi oporteat. Quo observato prior harum formularum dabit

$$\int \frac{z\partial z}{1+z^3} \cdot l \frac{z}{\sqrt[3]{(1+z^3)}} \begin{bmatrix} a\ z = 0 \\ ad\ z = \infty \end{bmatrix} = -\frac{\pi l 3}{3\sqrt{3}} + \frac{\pi\pi}{27},$$

posterior vero

$$\int \frac{\partial z}{1+z^3} \cdot l \frac{z}{\sqrt[3]{(1+z^3)}} \begin{bmatrix} a\ z = 0 \\ ad\ z = \infty \end{bmatrix} = -\frac{\pi l 3}{3\sqrt{3}} - \frac{\pi\pi}{27}.$$

Hinc igitur summa harum formularum erit

$$\int \frac{\partial z(1+z)}{1+z^3} \cdot l \frac{z}{\sqrt[3]{(1+z^3)}} = -\frac{2\pi l 3}{3\sqrt{3}},$$

at vero differentia

$$\int \frac{\partial z(z-1)}{1+z^3} \cdot l \frac{z}{\sqrt[3]{(1+z^3)}} = \frac{2\pi\pi}{27}.$$

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§. 162. Hic non inutile erit observasse, istum logarithmum $l\frac{z}{\sqrt[3]{(1+z^3)}}$ commode in seriem infinitam satis simplicem converti posse; cum enim sit

$$l\frac{z}{\sqrt[3]{(1+z^3)}}=\tfrac{1}{3}l\frac{z^3}{1+z^3}=-\tfrac{1}{3}l\frac{1+z^3}{z^3},$$

erit per seriem

$$l\frac{z}{\sqrt[3]{(1+z^3)}}=-\tfrac{1}{3}\left(\frac{1}{z^3}-\frac{1}{2z^6}+\frac{1}{3z^9}-\frac{1}{4z^{12}}+\frac{1}{5z^{15}}-\text{etc.}\right)$$

verum ista resolutio nullum usum praestare potest ad integralia haec per series evolvenda, propterea quod potestates ipsius z in denominatoribus occurrunt, ideoque singulae partes non ita integrari possunt, ut evanescant posito $z=0$.

Exemplum II. quo $p=2$.

§. 163. Hoc igitur casu factor postremus evadit $\int\frac{x\partial x(1+x)}{1+x+xx}$, qui in has duas partes discerpitur $\int\partial x-\int\frac{\partial x}{1+x+xx}$, cujus ergo integrale ab $x=0$ ad $x=1$ extensum est $=1-\frac{\pi}{3\sqrt{3}}$. Hinc igitur deducimur ad hanc aequationem

$$\int\frac{x\partial x\,lx}{\sqrt[3]{(1-x^3)}}=-\left(1-\frac{\pi}{3\sqrt{3}}\right)\int\frac{x\partial x}{\sqrt[3]{(1-x^3)}}.$$

Hic autem notandum, istam formulam integralem nullo modo absolute exhiberi posse, sed peculiarem quandam quantitatem transcendentem involvere.

V. Evolutio casus, quo $m=2$ et $n=4$.

§. 164. Hoc igitur casu erit $X=\frac{1}{\sqrt{(1-x^4)}}$, unde theorema nostrum generale nobis dabit hanc aequationem

$$\int \frac{x^{p-1}\partial x\, l x}{\sqrt{(1-x^4)}} = -\int \frac{x^{p-1}\partial x}{\sqrt{(1-x^4)}} \cdot \int \frac{x^{p-1}\partial x}{1+xx};$$

at vero problema particulare prius pro hoc casu praebet

$$\int \frac{x^3\partial x\, l x}{\sqrt{(1-x^4)}} = -\tfrac{1}{2}\int \frac{x^3\partial x}{1+xx}.$$

Cum autem sit

$$\int \frac{x^3\partial x}{1+xx} = \tfrac{1}{2} - \tfrac{1}{2} l\, 2,$$

erit absolute

$$\int \frac{x^3\partial x\, l x}{\sqrt{(1-x^4)}} \left[\begin{matrix} ab\ x=0 \\ ad\ x=1 \end{matrix}\right] = -\tfrac{1}{4}(1 - l\, 2),$$

at vero hic casus congruit cum supra §. 144. tractato. Si enim hic ponamus $xx = y$, quo facto termini integrationis manent $y = 0$ et $y = 1$, erit $l x = \frac{1}{2} l y$ et $x\partial x = \frac{1}{2}\partial y$; quibus valoribus substitutis nostra aequatio abibit in hanc formam

$$\tfrac{1}{4}\int \frac{y\partial y\, l y}{\sqrt{(1-yy)}} = -\tfrac{1}{4}(1 - l\, 2), \text{ sive } \int \frac{y\partial y}{\sqrt{(1-yy)}} = l\, 2 - 1,$$

prorsus ut supra.

§. 165. Alterum vero theorema particulare ad praesentem casum accommodatum dabit

$$\int \frac{x\partial x\, l x}{\sqrt{(1-x^4)}} = -\tfrac{\pi}{4}\int \frac{x\partial x}{1+xx};$$

est vero

$$\int \frac{x\partial x}{1+xx} = l\sqrt{(1+xx)} = \tfrac{1}{2} l\, 2,$$

ita ut habeamus

$$\int \frac{x\partial x\, l x}{\sqrt{(1-x^4)}} \left[\begin{matrix} ab\ x=0 \\ ad\ x=1 \end{matrix}\right] = -\tfrac{\pi}{8} l\, 2.$$

Quodsi vero hic ut ante statuamus $xx = y$, obtinebitur

$$\int \frac{\partial y\, l y}{\sqrt{(1-yy)}} = -\tfrac{\pi}{2} l\, 2,$$

qui est casus supra §. 142. tractatus. His duobus casibus exponens p erat numerus par, unde casus impares evolvi conveniet.

Exemplum I. quo $p = 1$.

§. 166. Hoc igitur casu formula integralis postrema fiet $\int \frac{\partial x}{1+xx} =$ Arc. tang. x, ita ut posito $x = 1$ prodeat Arc. tang. $x = \frac{\pi}{4}$; tum vero aequatio nostra erit

$$\int \frac{\partial x\, lx}{\sqrt{(1-x^4)}} = -\frac{\pi}{4} \int \frac{\partial x}{\sqrt{(1-x^4)}},$$

integralibus scilicet ab $x = 0$ ad $x = 1$ extensis; ubi formula $\int \frac{\partial x}{\sqrt{(1-x^4)}}$ arcum curvae elasticae rectangulae exprimit, ideoque absolute exhiberi nequit.

Exemplum II. quo $p = 3$.

§. 167. Hoc ergo casu formula integralis postrema erit

$$\int \frac{xx\partial x}{1+xx} = \int \partial x - \int \frac{\partial x}{1+xx},$$

cujus integrale posito $x = 1$ fit $= 1 - \frac{\pi}{4}$, ita ut nunc aequatio nostra evadat

$$\int \frac{xx\partial x\, lx}{\sqrt{(1-x^4)}} = -\left(1 - \frac{\pi}{4}\right) \int \frac{xx\partial x}{\sqrt{(1-x^4)}},$$

quae formula integralis pariter absolute exhiberi nequit; exprimit enim applicatam curvae elasticae rectangulae.

§. 168. Quanquam autem haec duo exempla ad formulas inextricabiles perduxerunt, tamen jam pridem demonstravi, productum horum duorum integralium

$$\int \frac{\partial x}{\sqrt{(1-x^4)}} \cdot \int \frac{xx\partial x}{\sqrt{(1-x^4)}}$$

aequari areae circuli, cujus diameter $= 1$, sive esse $= \frac{\pi}{4}$; quamobrem, binis exemplis conjungendis, hoc insigne theorema adipiscimur

$$\int \frac{\partial x\, lx}{\sqrt{(1-x^4)}} \cdot \int \frac{xx\partial x\, lx}{\sqrt{(1-x^4)}} = \frac{\pi^2}{16}\left(1 - \frac{\pi}{4}\right).$$

Facile autem patet, innumera alia hujusmodi theoremata ex hoc fonte hauriri posse, quae, per se spectata, profundissimae indaginis sunt censenda.

SUPPLEMENTUM IV.

AD TOM. I. CAP. V.

DE

INTEGRATIONE FORMULARUM ANGULOS SINUSVE ANGULORUM IMPLICANTIUM.

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1) De formulis differentialibus angularibus maxime irrationalibus, quas tamen per logarithmos et arcus circulares integrare licet. *M. S. Academiae exhibit. die* 5. *Maii* 1777.

§. 1. Quae jam saepius sum commentatus de formulis differentialibus irrationalibus, quae nulla substitutione ad rationalitatem revocari possunt, nihilo vero minus integrationem per logarithmos et arcus circulares admittunt: etiam transferri possunt ad ejusmodi formulas angulares, quae sinus et cosinus cujuspiam anguli involvunt. Forma autem generalis hujusmodi differentialium, quae hoc modo tractari possunt, sequenti modo repraesentari potest: denotante Φ angulum quemcunque, designet $\varPhi$ functionem quamcunque rationalem ipsius tang. $n\Phi$, atque inveni istam formulam

$$\frac{\varPhi \partial \Phi (f \sin. \lambda \Phi + g \cos. \lambda \Phi)}{\sqrt[n]{(a \sin. n \Phi + b \cos. n \Phi)^{\lambda}}}$$

semper per logarithmos et arcus circulares integrari posse, id quod a casibus simplicioribus inchoando in sequentibus problematibus ostendere constitui.

Problema 1.

§. 2. *Proposita formula differentiali* $\frac{\partial \phi \cos. \phi}{\sqrt[n]{\cos. n \phi}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Quoniam mihi quidem alia adhuc via non patet istud praestandi, nisi per imaginaria procedendo, formulam $\sqrt{-1}$ littera i in posterum designabo, ita ut sit $i i = -1$, ideoque $\frac{1}{i} = -i$. Jam ante omnia in numeratore nostrae formulae loco $\cos. \phi$ has duas partes substituamus

$$\tfrac{1}{2}(\cos. \phi + i \sin. \phi) + \tfrac{1}{2}(\cos. \phi - i \sin. \phi),$$

atque ipsam formulam propositam per duas hujusmodi partes repraesentemus, quae sint

$$\partial p = \frac{\partial \phi (\cos. \phi + i \sin. \phi)}{\sqrt[n]{\cos. n \phi}} \text{ et } \partial q = \frac{\partial \phi (\cos. \phi - i \sin. \phi)}{\sqrt[n]{\cos. n \phi}},$$

ita ut ipsa formula nostra proposita sit $\frac{1}{2}\partial p + \frac{1}{2}\partial q$, ideoque ejus integrale $\frac{p+q}{2}$.

§. 3. Nunc ambas istas partes seorsim sequenti modo tractemus. Pro formula scilicet priore

$$\partial p = \frac{\partial \phi (\cos. \phi + i \sin. \phi)}{\sqrt[n]{\cos. n \phi}} \text{ statuamus } \frac{\cos. \phi + i \sin. \phi}{\sqrt[n]{\cos. n \phi}} = x,$$

ut sit $\partial p = x \partial \phi$, ac sumtis potestatibus exponentis n habebimus

$$x^n = \frac{(\cos.\,\Phi + i\sin.\,\Phi)^n}{\cos.\,n\,\Phi}.$$

Constat autem esse

$$(\cos.\,\Phi + i\sin.\,\Phi)^n = \cos.\,n\,\Phi + i\sin.\,n\,\Phi,$$

sicque erit $x^n = 1 + i\,\mathrm{tang.}\,n\,\Phi$, unde colligitur

$$\mathrm{tang.}\,n\,\Phi = \frac{x^n - 1}{i} = i(1 - x^n):$$

hinc cum posito in genere tang. $\omega = Z$, sit $\partial\,\omega = \frac{\partial Z}{1+ZZ}$, erit pro nostro casu

$$n\,\partial\,\Phi = \frac{-n\,i\,x^{n-1}\,\partial\,x}{1 + i\,i - 2\,i\,i\,x^n + i\,i\,x^{2n}},$$

quae formula ob $i\,i = -1$ transmutatur in hanc

$$\partial\,\Phi = \frac{-i\,x^{n-1}\,\partial\,x}{2\,x^n - x^{2n}},$$

hincque ipsa formula

$$\partial\,p = x\,\partial\,\Phi = \frac{-i\,\partial\,x}{2 - x^n},$$

quae cum sit rationalis, ejus integratio nulli difficultati est subjecta.

§. 4. Quodsi jam simili modo pro altera formula

$$\partial\,q = \frac{\partial\,\Phi(\cos.\,\Phi - i\sin.\,\Phi)}{\sqrt[n]{\cos.\,n\,\Phi}}, \text{ statuatur } \frac{\cos.\,\Phi - i\sin.\,\Phi}{\sqrt[n]{\cos.\,n\,\Phi}} = y,$$

ut sit $\partial\,q = y\,\partial\,\Phi$, per similes operationes, quae a praecedentibus in hoc solo discrepabunt, quod littera i negative sit accipienda, resultabit ista transformatio $\partial\,q = \frac{i\,\partial\,y}{2 - y^n}$, quae cum priori prorsus

sit similis, eadem integratione totum negotium conficietur, et pro ipso integrali quaesito habebimus

$$p + q = -i\int\frac{\partial x}{2 - x^n} + i\int\frac{\partial y}{2 - y^n}.$$

§. 5. Constat autem integralia talium formularum ex duplicis generis partibus, scilicet logarithmicis et arcubus circularibus constare, ita ut illarum forma generalis sit $f\,l(\alpha + \beta x + \gamma xx)$, harum vero g Arc. tang. $(\delta + \varepsilon x)$. Quare cum hic differentia inter binas formulas integrales similes occurrat, ex singulis partibus logarithmicis orietur talis forma $- i f\,l\frac{\alpha + \beta x + \gamma xx}{\alpha + \beta y + \gamma yy}$, ubi tam x quam y imaginaria involvit, hanc ob rem ponamus brevitatis gratia $x = r + is$ et $y = r - is$, ubi erit

$$r = \frac{\cos.\,\Phi}{\sqrt[n]{\cos.\,n\,\Phi}} \text{ et } s = \frac{\sin.\,\Phi}{\sqrt[n]{\cos.\,n\,\Phi}};$$

his igitur valoribus substitutis, quaelibet pars logarithmica erit

$$- i f\,l\frac{\alpha + \beta r + \gamma rr - \gamma ss + i(\beta s + 2\gamma rs)}{\alpha + \beta r + \gamma rr - \gamma ss - i(\beta s + 2\gamma rs)}.$$

§. 6. Loco hujus expressionis prolixioris scribamus brevitatis gratia $- i f\,l\frac{t + iu}{t - iu}$, ita ut sit

$$t = \alpha + \beta r + \gamma rr - \gamma ss \text{ et } u = \beta s + 2\gamma rs,$$

sicque etiam hi valores per angulum Φ innotescunt. Quoniam igitur jam saepius est demonstratum, esse

$$l\frac{t + u\sqrt{-1}}{t - u\sqrt{-1}} = 2\sqrt{-1}.\ \text{Arc. tang.}\ \frac{u}{t},$$

ista portio integralis erit $= + 2 f$ Arc. tang. $\frac{u}{t}$, quae ergo penitus est realis, dum imaginaria se mutuo sustulerunt, ita ut quaelibet portio logarithmica imaginaria producat arcum circularem realem.

§. 7. Simili modo conjungamus in genere binos arcus circulares per integrationem prodeuntes, qui ex forma assumta erunt

$$- i g \text{ Arc. tang. } (\delta + \varepsilon x) + i g \text{ Arc. tang. } (\delta + \varepsilon y),$$

quae forma ita in unum arcum contrahetur, qui erit

$$- i g \text{ Arc. tang. } \frac{\varepsilon (x - y)}{1 + (\delta + \varepsilon x)(\delta + \varepsilon y)}:$$

quae introductis valoribus assumtis $x = r + i s$ et $y = r - i s$, induet hanc formam

$$- i g \text{ Arc. tang. } \frac{2 i \varepsilon s}{1 + \delta\delta + 2 \varepsilon \delta r + \varepsilon\varepsilon (rr + ss)}.$$

Cum igitur in genere sit

$$\text{Arc. tang. } v \sqrt{-1} = \frac{\sqrt{-1}}{2} l \frac{1 + v}{1 - v},$$

ista pars circularis transformabitur in sequentem logarithmum realem

$$\frac{g}{2} l \frac{1 + \delta\delta + 2 \delta \varepsilon r + \varepsilon\varepsilon (rr + ss) + 2 \varepsilon s}{1 + \delta\delta + 2 \delta \varepsilon r + \varepsilon\varepsilon (rr + ss) - 2 \varepsilon s}:$$

hoc ergo modo sumendis omnium integralium partibus, tandem obtinebitur integrale quaesitum per meros logarithmos et arcus circulares realiter expressum.

Problema 2.

§. 8. *Proposita formula differentiali* $\frac{\partial \Phi \sin. \Phi}{\sqrt[n]{\cos. n \Phi}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Hic loco sin. Φ scribatur haec forma duabus constans partibus

$$\frac{1}{2i} (\cos. \Phi + i \sin. \Phi) - \frac{1}{2i} (\cos. \Phi - i \sin. \Phi),$$

ac formula proposita resolvatur in has partes

$$\partial p = \frac{\partial \Phi (\cos. \Phi + i \sin. \Phi)}{\sqrt[n]{\cos. n \Phi}} \text{ et } \partial q = \frac{\partial \Phi (\cos. \Phi - i \sin. \Phi)}{\sqrt[n]{\cos. n \Phi}},$$

ita ut ipsa formula proposita jam fiat $\frac{\partial p - \partial q}{2 i}$, ideoque ipsum integrale quaesitum $\frac{p - q}{2 i}$.

§. 9. Quodsi jam rursus ut ante statuamus

$$\frac{\cos. \Phi + i \sin. \Phi}{\sqrt[n]{\cos. n \Phi}} = x \text{ et } \frac{\cos. \Phi - i \sin. \Phi}{\sqrt[n]{\cos. n \Phi}} = y,$$

reperietur ut supra

$$\partial p = -\frac{i \partial x}{2 - x^n} \text{ et } \partial q = \frac{i \partial y}{2 - y^n};$$

unde ergo fiet ipsum integrale quaesitum

$$\frac{p - q}{2 i} = -\tfrac{1}{2} \int \frac{\partial x}{2 - x^n} - \tfrac{1}{2} \int \frac{\partial y}{2 - y^n},$$

ubi coefficientes evaserunt reales.

§. 10. Consideremus nunc ex forma integrali utriusque partis quamlibet portionem logarithmicam, quae sit $f l (\alpha + \beta x + \gamma x x)$, hincque pro integrali quaesito ex utraque parte orietur

$$-\tfrac{1}{2} f l (\alpha + \beta x + \gamma x x) - \tfrac{1}{2} f l (\alpha + \beta y + \gamma y y).$$

Quodsi jam ut supra ponamus brevitatis gratia $x = r + i s$ et $y = r - i s$, tum vero

$$t = \alpha + \beta r + \gamma r r - \gamma s s \text{ et } u = \beta s + 2 \gamma r s,$$

hi ambo logarithmi evadunt

$$= -\tfrac{1}{2} f l (t + i u) - \tfrac{1}{2} f l (t - i u),$$

qui contrahuntur in $-\tfrac{1}{2} f l (t t + u u$, quae expressio jam est realis, neque ulla ulteriori reductione indiget.

§. 11. Eodem modo binae partes circulares ex integratione oriundae

$$-\tfrac{1}{2}g \text{ Arc. tang. } (\delta + \varepsilon x) - \tfrac{1}{2}g \text{ Arc. tang. } (\delta + \varepsilon y),$$

quae per r et s ita repraesentantur

$$-\tfrac{1}{2}g\,[\text{Arc. tang. } (\delta + \varepsilon r + i\varepsilon s) + \text{Arc. tang. } (\delta + \varepsilon r - i\varepsilon s)],$$

qui duo arcus ita in unum contrahuntur

$$-\tfrac{1}{2}g \text{ Arc. tang. } \frac{2\delta + 2\varepsilon r}{1-(\delta+\varepsilon r)^2 - \varepsilon\varepsilon ss},$$

quae expressio jam ultro prodiit realis.

Problema 3.

§. 12. *Proposita formula differentiali* $\frac{\partial\Phi \cos.\lambda\Phi}{\sqrt[n]{\cos. n\Phi^\lambda}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Cum sit

$$\cos.\lambda\Phi = \tfrac{1}{2}(\cos.\Phi + i\sin.\Phi)^\lambda + \tfrac{1}{2}(\cos.\Phi - i\sin.\Phi)^\lambda,$$

formula proposita in has duas partes discerpatur

$$\partial p = \frac{\partial\Phi(\cos.\Phi + i\sin.\Phi)^\lambda}{\sqrt[n]{\cos. n\Phi^\lambda}} \text{ et } \partial q = \frac{\partial\Phi(\cos.\Phi - i\sin.\Phi)^\lambda}{\sqrt[n]{\cos. n\Phi^\lambda}},$$

ita ut integrale quaesitum fiat $\frac{p+q}{2}$.

§. 13. Jam statuamus, ut ante fecimus,

$$\frac{\cos.\Phi + i\sin.\Phi}{\sqrt[n]{\cos. n\Phi}} = x \text{ et } \frac{\cos.\Phi - i\sin.\Phi}{\sqrt[n]{\cos. n\Phi}} = y,$$

quo facto fiet $\partial p = x^\lambda\,\partial\Phi$ et $\partial q = y^\lambda\,\partial\Phi$. Calculo autem ut supra expedito obtinebimus

$$\partial\Phi = -\frac{i x^{n-1}\partial x}{2x^n - x^{2n}},\ \text{hincque}\ \partial p = -\frac{i x^{\lambda-1}\partial x}{2 - x^n};$$

similique modo erit $\partial q = \frac{i y^{\lambda-1}\partial y}{2 - y^n}$, sicque totum integrale quaesitum erit

$$= -\tfrac{i}{2}\int\frac{x^{\lambda-1}\partial x}{2 - x^n} + \tfrac{i}{2}\int\frac{y^{\lambda-1}\partial y}{2 - y^n}$$

§. 14. Quoniam haec duo integralia sibi sunt similia, ideoque similes partes tam logarithmicas quam circulares complectuntur, ex parte logarithmica, quae sit $fl(\alpha + \beta x + \gamma xx)$, ponendo ut supra $x = r + is$ et $y = r - is$, tum vero

$$t = \alpha + \beta r + \gamma rr - \gamma ss \text{ et } u = \beta s + 2\gamma rs,$$

hinc primo ista pars logarithmica colligitur $-ifl\frac{t+iu}{t-iu}$, quae cum sit imaginaria reducitur ad hunc arcum circularem realem $= 2f$ Arc. tang. $\frac{u}{t}$: simili modo si forma arcus circularis ex integratione oriunda fuerit $-g$ Arc. tang. $(\delta + \varepsilon x)$, ex partibus circularibus primo oritur sequens arcus imaginarius

$$-ig \text{ Arc. tang. } \frac{2i\varepsilon s}{1 + \delta\delta + 2\varepsilon\delta r + \varepsilon\varepsilon(rr + ss)},$$

qui denique ad hunc logarithmum realem revocatur

$$\frac{g}{2}\, l\, \frac{1 + \delta\delta + 2\delta\varepsilon r + \varepsilon\varepsilon(rr + ss) + 2\varepsilon s}{1 + \delta\delta + 2\delta\varepsilon r + \varepsilon\varepsilon(rr + ss) - 2\varepsilon s}.$$

Problema 4.

§. 15. *Proposita formula differentiali* $\frac{\partial\Phi \sin.\lambda\Phi}{\sqrt[n]{\cos. n\Phi^{\lambda}}}$, *ejus integrale per logarithmos et arcus circulares investigare.*

Solutio.

Cum sit

$$\sin.\lambda\Phi = \frac{1}{2i}(\cos.\Phi + i\sin.\Phi)^\lambda - \frac{1}{2i}(\cos.\Phi - i\sin.\Phi)^\lambda,$$

constituamus ut hactenus has duas partes

$$\partial p = \frac{\partial\Phi(\cos.\Phi + i\sin.\Phi)^\lambda}{\sqrt[n]{\cos.n\Phi^\lambda}} \text{ et } \partial q = \frac{\partial\Phi(\cos.\Phi - i\sin.\Phi)^\lambda}{\sqrt[n]{\cos.n\Phi^\lambda}},$$

ita ut integrale quaesitum sit $\frac{p-q}{2i}$. Statuamus nunc iterum

$$\frac{\cos.\Phi + i\sin.\Phi}{\sqrt[n]{\cos.n\Phi}} = x \text{ et } \frac{\cos.\Phi - i\sin.\Phi}{\sqrt[n]{\cos.n\Phi}} = y,$$

ut fiat $\partial p = x^\lambda\partial\Phi$ et $\partial q = y^\lambda\partial\Phi$, hincque calculo ut supra instituto, fiet

$$\partial p = -\frac{ix^{\lambda-1}\partial x}{2-x^n} \text{ et } \partial q = \frac{iy^{\lambda-1}\partial y}{2-y^n},$$

sicque integrale quaesitum erit

$$-\tfrac{1}{2}\int\frac{x^{\lambda-1}\partial x}{2-x^n} - \tfrac{1}{2}\int\frac{y^{\lambda-1}\partial y}{2-y^n}.$$

§. 16. Quodsi jam ut hactenus est factum, ponamus $x = r + is$ et $y = r - is$, et pro partibus logarithmicis, quarum forma sit $fl(\alpha + \beta x + \gamma xx)$, ponamus

$$t = \alpha + \beta r + \gamma rr - \gamma ss \text{ et } u = \beta\alpha + 2\gamma rs,$$

binae partes logarithmicae imaginariae uti in problemate secundo in unum logarithmum realem contrahentur, qui erit $-\frac{1}{2}fl(tt + uu)$. At si pro partibus circularibus, quarum forma sit g Arc. tang. $(\delta + \varepsilon x)$, bini tales arcus imaginarii jungantur, illi coalescent in unum arcum realem

$$-\tfrac{1}{2}g \text{ Arc. tang } \frac{2\delta + 2\varepsilon r}{1 - (\delta + \varepsilon r)^2 - \varepsilon\varepsilon ss}.$$

Problema generale.

§. 17. *Si Φ denotet functionem quamcunque rationalem ipsius tang. $n\,\phi$, ac proposita fuerit haec formula differentialis*

$$\frac{\Phi\,\partial\phi\,(F\sin.\lambda\phi + G\cos.\lambda\phi)}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)^{\lambda}}},$$

ejus integrationem ad logarithmos et arcus circulares reducere.

Solutio.

Ex praecedentibus jam facile intelligitur, formulam numeratoris F sin. $\lambda\,\phi$ + G cos. $\lambda\,\phi$ semper ad talem formam revocari posse

$$F'(\cos.\phi + i\sin.\phi)^{\lambda} + G'(\cos.\phi - i\sin.\phi)^{\lambda},$$

atque hinc ipsa forma proposita discerpatur in has duas partes

$$\partial p = \frac{\Phi\,\partial\phi\,(\cos.\phi + i\sin.\phi)^{\lambda}}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)^{\lambda}}} \text{ et}$$

$$\partial q = \frac{\Phi\,\partial\phi\,(\cos.\phi + i\sin.\phi)^{\lambda}}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)^{\lambda}}};$$

ita ut integrale quaesitum jam futurum sit $F'\,p + G'\,q$.

§. 18. Jam pro formula priori $\partial\,p$ statuatur

$$\frac{\cos\phi + i\sin.\phi}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)}} = x, \text{ et pro posteriori}$$

$$\frac{\cos.\phi - i\sin.\phi}{\sqrt[n]{(a\cos.n\phi + b\sin.n\phi)}} = y$$

ita ut hinc futurum sit

$$\partial p = \Phi\,x^{\lambda}\,\partial\phi \text{ et } \partial q = \Phi\,y^{\lambda}\,\partial\phi;$$

inde autem fiet

$$x^n = \frac{\cos. n\,\Phi + i \sin. n\,\Phi}{a \cos. n\,\Phi + i \sin. n\,\Phi},$$

unde colligitur

$$\text{tang.}\ n\,\Phi = \frac{1 - a x^n}{b x^n - i};$$

quare cum Φ denotet functionem rationalem ipsius tang. $n\,\Phi$, evadet quoque functio rationalis ipsius x, atque adeo ipsius x^n, quae designetur per **X**. Praeterea vero etiam differentiale $\partial\,\Phi$ rationaliter determinabitur; cum fiat

$$\partial\,\Phi = \frac{(a - b)\, x^{n-1}\, \partial x}{(aa + bb)\, x^{2n} - 2(a - ib)\, x^n},$$

hoc ergo modo habebimus

$$\partial p = \frac{(ia - b)\, \mathrm{X}\, x^{\lambda - 1}\, \partial x}{(aa + bb)\, x^n - 2(a + ib)},$$

quae cum sit penitus rationalis, certum est, ejus integrale, quantumcunque etiam laborem postulaverit, semper per logarithmos et arcus circulares expediri posse.

§. 19. Simili modo res se habet in altera formula ∂q, quae ab ista tantum ratione signi litterae i differet, et quoniam hic omnia rationaliter per y prodibunt expressa, quo pacto Φ abeat in Y, atque obtinebitur

$$\partial q = - \frac{(b + ia)\, \mathrm{Y}\, y^{\lambda - 1}\, \partial y}{(aa + bb) y^n - 2a + 2ib},$$

cujus integratio omnino similis erit praecedenti, et quasi eodem labore absolvetur.

§. 20. Manifestum autem est, in hujusmodi calculo imaginaria cum realibus multo arctius commisceri, quam in praecedentibus problematibus usu venit, quandoquidem jam statim ab initio coëfficientes derivati F′ et G′ jam imaginaria involvunt; deinde vero

etiam utrinque tang. $n\,\phi$ imaginariis inquinatur, unde etiam in valores X et Y imaginaria ingredientur; quamobrem reductio ad realitatem plerumque maximum laborem exigere poterit, proque autem negotio praecepta necessaria jam satis sunt cognita.

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2) Theorema maxime memorabile circa formulam integralem $\int \frac{\partial \phi \cos.\lambda\,\phi}{(1 + aa - 2a\cos.\phi)^{n+1}}$. *M. S. Academiae exhib. die* 13. *Augusti* 1778.

§. 21. Haec formula aliam restrictionem non postulat nisi quod littera λ numeros tantum integros designat sive positivos sive negativos. Evidens autem est valores negativos non discrepare a positivis, cum semper sit $\cos. -\phi = \cos. +\phi$. Hoc notato si istius formulae integrale a termino $\phi = 0$ usque ad terminum $\phi = 180^\circ$ sive $\phi = \pi$ porrigatur, ejus valor semper sequenti formula exprimetur $\frac{\pi a}{(1 - aa)^{2n+1}} \cdot V$, existente

$$V = \left(\frac{n-\lambda}{0}\right)\left(\frac{n+\lambda}{\lambda}\right) + \left(\frac{n-\lambda}{1}\right)\left(\frac{n+\lambda}{\lambda+1}\right) aa$$
$$+ \left(\frac{n-\lambda}{2}\right)\left(\frac{n+\lambda}{\lambda+2}\right) a^4 + \left(\frac{n-\lambda}{3}\right)\left(\frac{n+\lambda}{\lambda+3}\right) a^6$$
$$+ \left(\frac{n-\lambda}{4}\right)\left(\frac{n+\lambda}{\lambda+4}\right) a^8 + \left(\frac{n-\lambda}{5}\right)\left(\frac{n+\lambda}{\lambda+5}\right) a^{10} \text{ etc.}$$

Ubi formulae uncinulis inclusae non fractiones, sed eos characteres designant, quibus unciae potestatum Binomii designari solent, ita ut sit

$$\left(\frac{\alpha}{\beta}\right) = \frac{\alpha}{1} \cdot \frac{\alpha - 1}{2} \cdot \frac{\alpha - 2}{3} \quad \cdot \; \cdot \; \cdot \; \cdot \; \cdot \; \cdot \quad \frac{\alpha - \beta + 1}{\beta},$$

quae expressio quoniam nostro casu β ubique est numerus integer, determinatum valorem facile quovis casu exhibendam declarat, ubi notasse sufficiet, quoties fuerit $\beta = 0$ semper fore $(\frac{\alpha}{0}) = 1$; sin autem fuerit β numerus negativus, valorem hujus characteris in nihilum abire; tum vero etiam observari convenit, si fuerit $\beta = \alpha$ fore $(\frac{\alpha}{\alpha}) = 1$, et si $\beta > \alpha$ pariter valores evanescere. Cum semper sit $(\frac{\alpha}{\beta}) = (\frac{\alpha}{\alpha - \beta})$.

§. 22. His explicatis evolvamus praecipuos casus quibus exponenti n valores simpliciores 0, 1, 2, 3, 4 etc. tribuuntur.

Casus I.

quo $n = 0$, et formula integralis haec proponitur

$$\int \frac{\partial \Phi \cos. \lambda \Phi}{1 + aa - 2a \cos. \Phi} \begin{bmatrix} b\,x = 0 \\ ad\,x = \pi \end{bmatrix}.$$

Quia hic $n = 0$, pro prioribus factoribus quantitatis V habebimus

$$(\frac{0-\lambda}{0}) = 1; \ (\frac{0-\lambda}{1}) = -\lambda; \ (\frac{0-\lambda}{2}) = \frac{\lambda}{1} \cdot \frac{\lambda+1}{2};$$

$$(\frac{0-\lambda}{3}) = -\frac{\lambda}{1} \cdot \frac{\lambda+1}{2} \cdot \frac{\lambda+2}{3}; \ (\frac{0-\lambda}{4}) = \frac{\lambda}{1} \cdot \frac{\lambda+1}{2} \cdot \frac{\lambda+2}{3} \cdot \frac{\lambda+3}{4}; \text{ etc.}$$

Pro posterioribus vero faetoribus babebimus

$$(\frac{0+\lambda}{\lambda}) = 1; \ (\frac{0+\lambda}{\lambda+1}) = 0; \ (\frac{0+\lambda}{\lambda+2}) = 0 \text{ etc.}$$

hic scilicet omnes isti factores praeter primum evanescunt; unde colligitur valor quantitatis $V = 1$, ideoque integrale quaesitum hujus casus erit $= \frac{\pi a^\lambda}{1 - aa}$.

Hinc ergo si fuerit $n = 0$, erit $\int \frac{\partial \Phi}{1 + aa - 2a \cos. \Phi} = \frac{\pi}{1 - aa}$ quod egregie consentit cum integratione satis cognita

$$\int \frac{\partial \Phi}{\alpha + \beta \cos. \Phi} = \frac{1}{\sqrt{(\alpha\alpha - \beta\beta)}} \text{ Arc. cos. } \frac{\alpha \cos. \Phi + \beta}{\alpha + \beta \cos. \Phi},$$

quod integrale jam sponte evanescit sumto $\Phi = 0$. Statuatur igitur, ut hic perpetuo assumimus, $\Phi = 180^\circ = \pi$, atque ob cos. $\Phi = -1$, erit istud integrale

$$\frac{1}{\sqrt{(\alpha\alpha - \beta\beta)}} \text{Arc. cos.} -1 = \frac{\pi}{\sqrt{(\alpha\alpha - \beta\beta)}}.$$

Jam nostro casu est $\alpha = 1 + a\,a$ et $\beta = -2\,a$, unde fit $\sqrt{(\alpha\alpha - \beta\beta)} = 1 - a\,a$.

Casus II.

quo $n = 1$, et formula integralis haec proponitur

$$\int \frac{\partial\Phi \cos.\lambda\Phi}{(1 + aa - 2a\cos.\Phi)^2} \begin{bmatrix} a\ \Phi = 0 \\ ad\ \Phi = \pi \end{bmatrix}.$$

Quia hic est $n = 1$, erit pro prioribus factoribus quantitatis V

$$\left(\frac{1-\lambda}{0}\right) = 1;\ \left(\frac{1-\lambda}{1}\right) = -(\lambda - 1);$$
$$\left(\frac{1-\lambda}{2}\right) = \frac{\lambda(\lambda-1)}{1.\ \ 2}.$$

Pro posterioribus vero factoribus habebimus

$$\left(\frac{1+\lambda}{\lambda}\right) = \lambda + 1;\ \left(\frac{1+\lambda}{\lambda+1}\right) = 1;$$

sequentes vero formulae evanescunt, sicque erit

$$V = \lambda + 1 - (\lambda - 1)\,a\,a;$$

quocirca valor integralis propositi erit

$$\frac{\pi\,a^\lambda}{(1 - a\,a)^3}\,[(\lambda + 1) - (\lambda - 1)\,a\,a]:$$

hinc ergo sequentes casus speciales apposuisse juvabit, ubi brevitatis gratia loco formulae $1 + a\,a - 2\,a$ cos. Φ characterem Δ scribamus

$$\int \frac{\partial\Phi}{\Delta^2} = \frac{\pi(1+aa)}{(1-aa)^3},$$
$$\int \frac{\partial\Phi \cos.\Phi}{\Delta^2} = \frac{2\pi a}{(1-aa)^3},$$

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^2} = \frac{\pi a^2 (3 - a a)}{(1 - a a)^3},$$
$$\int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} = \frac{\pi a^3 (4 - 2 a a)}{(1 - a a)^3},$$
$$\int \frac{\partial \Phi \cos. 4\Phi}{\Delta^2} = \frac{\pi a^4 (5 - 3 a a)}{(1 - a a)^3},$$
$$\int \frac{\partial \Phi \cos. 5\Phi}{\Delta^2} = \frac{\pi a^5 (6 - 4 a a)}{(1 - a a)^3},$$
$$\int \frac{\partial \Phi \cos. 6\Phi}{\Delta^2} = \frac{\pi a^6 (7 - 5 a a)}{(1 - a a)^3},$$
$$\text{etc.} \qquad \text{etc.}$$

Casus III.

quo $n = 2$, et formula integralis haec proponitur

$$\int \frac{\partial \Phi \cos. \lambda \Phi}{(1 + a a - 2 a \cos. \Phi)^3} \left[\begin{array}{l} a\ \Phi = 0 \\ ad\ \Phi = \pi \end{array} \right].$$

Hic factores priores, qui in valore quantitatis V occurrunt, erunt

$$\left(\frac{2-\lambda}{0}\right) = 1; \ \left(\frac{2-\lambda}{1}\right) = -(\lambda - 2); \ \left(\frac{2-\lambda}{2}\right) = \frac{(\lambda-2)(\lambda-1)}{1. \quad 2};$$
$$\left(\frac{2-\lambda}{3}\right) = \frac{\lambda - 2 . \lambda - 1 . \lambda}{1. \quad 2. \quad 3} \text{ etc.}$$

factores autem posteriores erunt

$$\left(\frac{2+\lambda}{\lambda}\right) = \frac{\lambda+2}{1} . \frac{\lambda+1}{2}; \ \left(\frac{2+\lambda}{\lambda+1}\right) = \lambda + 2; \ \left(\frac{2+\lambda}{\lambda+2}\right) = 1;$$

et sequentes omnes evanescunt; hinc ergo colligimus

$$V = \frac{(\lambda+2)(\lambda+1)}{1. \quad 2} - (\lambda\lambda - 4) a a + \frac{(\lambda-2)(\lambda-1)}{1. \quad 2} a^4,$$

hocque valore invento erit integrale quaesitum $\frac{\pi a^\lambda}{(1 - aa)^5} . V$, unde sequentes casus speciales, statuendo ut ante $1 + aa - 2a \cos. \Phi = \Delta$, evolvamus

$$\int \frac{\partial \Phi}{\Delta^3} = \frac{\pi}{(1 - a a)^5} (1 + 4 a a + a^4),$$
$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^3} = \frac{3 \pi a}{(1 - a a)^5} (1 + a a),$$
$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^3} = \frac{6 \pi a^2}{(1 - a a)^5},$$

$$\int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^3} = \frac{\pi a^3}{(1-aa)^5} (10 - 5 aa + a^4),$$
$$\int \frac{\partial \Phi \cos. 4 \Phi}{\Delta^3} = \frac{3 \pi a^4}{(1-aa)^5} (5 - 4 aa + a^4),$$
$$\int \frac{\partial \Phi \cos. 5 \Phi}{\Delta^3} = \frac{3 \pi a^5}{(1-aa)^5} (7 - 7 aa + 2 a^4),$$
$$\int \frac{\partial \Phi \cos. 6 \Phi}{\Delta^3} = \frac{2 \pi a^6}{(1-aa)^5} (14 - 16 aa + 5 a^4),$$
etc. etc.

Casus IV.

quo $n = 3$, et formula integralis haec proponitur

$$\int \frac{\partial \Phi \cos. \lambda \Phi}{(1 + aa - 2 a \cos. \Phi)^4} \left[\begin{matrix} a \Phi = 0 \\ ad \Phi = \pi \end{matrix}\right].$$

Hic pro prioribus factoribus quantitatis V habebimus

$$\left(\frac{3-\lambda}{0}\right) = 1; \left(\frac{3-\lambda}{1}\right) = -(\lambda - 3); \left(\frac{3-\lambda}{2}\right) = \frac{3-\lambda}{1} \cdot \frac{2-\lambda}{2};$$
$$\left(\frac{3-\lambda}{3}\right) = \frac{3-\lambda}{1} \cdot \frac{2-\lambda}{2} \cdot \frac{1-\lambda}{3}; \quad \left(\frac{3-\lambda}{4}\right) = \frac{3-\lambda}{1} \cdot \frac{2-\lambda}{2} \cdot \frac{1-\lambda}{3} \cdot \frac{-\lambda}{4};$$

factores autem posteriores erunt

$$\left(\frac{3+\lambda}{\lambda}\right) = \frac{3+\lambda}{1} \cdot \frac{2+\lambda}{2} \cdot \frac{1+\lambda}{3}; \quad \left(\frac{3+\lambda}{\lambda+1}\right) = \frac{3+\lambda}{1} \cdot \frac{2+\lambda}{2};$$
$$\left(\frac{3+\lambda}{\lambda+2}\right) = 3 + \lambda; \left(\frac{3+\lambda}{\lambda+3}\right) = 1;$$

et sequentes omnes evanescunt, hinc ergo colligimus

$$V = \frac{(\lambda+1)(\lambda+2)(\lambda+3)}{1. \quad 2. \quad 3} - \frac{(\lambda+2)(\lambda\lambda-9)}{1. \quad 2} aa + \frac{(\lambda-2)(\lambda\lambda-9)}{1. \quad 2} a^4$$
$$- \frac{(\lambda-1)(\lambda-2)(\lambda-3)}{1. \quad 2. \quad 3} a^6.$$

Quo valore invento colligimus integrale quaesitum $= \frac{\pi a^\lambda}{(1-aa)^7} . V$, hincque sequentes casus speciales, ponendo ut hactenus $1 + aa - 2 a \cos. \Phi = \Delta$, evolvamus

$$\int \frac{\partial \Phi}{\Delta^4} = \frac{\pi}{(1-aa)^7} (1 + 9 aa + 9 a^4 + a^6),$$
$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^4} = \frac{4 \pi a}{(1-aa)^7} (1 + 3 aa + a^4),$$
$$\int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^4} = \frac{10 \pi a^2}{(1-aa)^7} (1 + aa),$$

$$\int \frac{\partial \Phi \cos. 3\Phi}{\Delta^4} = \frac{20\pi a^3}{(1-aa)^7},$$
$$\int \frac{\partial \Phi \cos. 4\Phi}{\Delta^4} = \frac{\pi a^4}{(1-aa)^7} (35 - 21\, aa + 7\, a^4 - a^6),$$
etc. etc.

§. 23. Hic longius progredi superfluum foret, cum forma generalis pro V inventa totum negotium facillime conficiat; verum haud inutile erit, litterae n etiam valores negativos tribuere, quibus casibus tota integratio per methodos consuetas haud difficulter expeditur, unde jucundum erit pulcherrimum consensum nostrae formae generalis perspicere.

Casus I.

quo $n = -1$, et formula integralis haec proponitur
$$\int \partial \Phi \cos. \lambda \Phi \begin{bmatrix} a\,\Phi = 0 \\ ad\,\Phi = \pi \end{bmatrix}.$$
Haec formula absolute est integrabilis, cum sit
$$\int \partial \Phi \cos. \lambda \Phi = \frac{1}{\lambda} \sin. \lambda \Phi,$$
quae formula cum jam evanescat posito $\Phi = 0$; sumendo $\Phi = \pi$, ob λ numerum integrum iste valor semper erit $= 0$, solo casu excepto $\lambda = 0$. Spectato enim λ tanquam infinite parvo, erit $\sin. \lambda\pi = \lambda\pi$, ideoque hoc casu valor erit $= \pi$. Nunc autem forma generalis pro quantitate V data erit
$$V = \left(\frac{-1-\lambda}{0}\right)\left(\frac{-1+\lambda}{\lambda}\right) + \left(\frac{-1-\lambda}{1}\right)\left(\frac{-1+\lambda}{\lambda+1}\right) a^2$$
$$+ \left(\frac{-1-\lambda}{2}\right)\left(\frac{-1+\lambda}{\lambda+2}\right) a^4 + \left(\frac{-1-\lambda}{3}\right)\left(\frac{-1+\lambda}{\lambda+3}\right) a^6$$
$$+ \left(\frac{-1-\lambda}{4}\right)\left(\frac{-1+\lambda}{\lambda+4}\right) a^8 + \left(\frac{-1-\lambda}{5}\right)\left(\frac{-1+\lambda}{\lambda+5}\right) a^{10}$$
etc. etc.

Cujus expressionis factores posteriores omnes evanescunt, quoties fuerit vel $\lambda = 1$ vel $\lambda > 1$, propterea quod numeri inferiores majores, quam superiores, utrique vero positivi; quae conclusio autem

non valet, quando superior numerus evadit negativus, uti evenit casu $\lambda = 0$, quem ergo solum perpendisse necesse est; hoc autem casu factores priores evadent

$$(\frac{-1}{0}) = 1;\ (\frac{-1}{1}) = -1;\ (\frac{-1}{2}) = +1;$$
$$(\frac{-1}{3}) = -1;\ (\frac{-1}{4}) = +1;\ \text{etc.}$$

at vero valores posteriores eosdem determinationes recipiunt; sicque habebimus

$$V = 1 + aa + a^4 + a^6 + a^8 + a^{10} + \text{etc.}$$

quae series cum sit geometrica, erit $V = \frac{1}{1-aa}$ quare cum, ob $n = -1$ et $\lambda = 0$, valor quaesitus per nostram formam generalem sit $\pi(1-aa)V$, iste valor nunc ob $V = \frac{1}{1-aa}$, abit in π, uti supra.

Casus II.

quo $n = -2$, et formula integralis haec proponitur

$$\int \partial\Phi \cos.\lambda\Phi\,(1 + aa - 2a\cos.\Phi) \left[\begin{smallmatrix} a\ \Phi = 0 \\ ad\ \Phi = \pi \end{smallmatrix}\right].$$

Per formam nostram generalem integrale quaesitum erit $\pi a^\lambda (1 - aa)^3 V$, existente

$$V = (\frac{-2-\lambda}{0})(\frac{-2+\lambda}{\lambda}) + (\frac{-2-\lambda}{1})(\frac{-2+\lambda}{\lambda+1})aa + (\frac{-2-\lambda}{2})(\frac{-2+\lambda}{\lambda+2})a^4$$
$$+ (\frac{-2-\lambda}{3})(\frac{-2+\lambda}{\lambda+3})a^6 + (\frac{-2-\lambda}{4})(\frac{-2+\lambda}{\lambda+4})a^8 + (\frac{-2-\lambda}{5})(\frac{-2+\lambda}{\lambda+5})a^{10}$$
$$\text{etc.} \qquad \text{etc.}$$

Ubi iterum evidens est, si fuerit vel $\lambda = 2$ vel $\lambda > 2$, omnes factores posteriores evanescere, ideoque fieri $V = 0$, ita ut etiam valor integralis quaesitus semper evanescat, id quod ex ipsa natura formulae sponte sequitur, quippe cujus integrale, ob

$$\cos.\Phi\cos.\lambda\Phi = \tfrac{1}{2}\cos.(\lambda - 1)\Phi + \tfrac{1}{2}\cos.(\lambda + 1)\Phi,$$

in genere erit

$$\frac{1+aa}{\lambda}\sin.\lambda\Phi - \frac{a}{\lambda-1}\sin.(\lambda-1)\Phi - \frac{a}{\lambda+1}\sin.(\lambda+1)\Phi,$$

quod quia $\lambda > 1$ casu $\Phi = \pi$ manifesto evanescit; unde duos casus perpendere superest, alterum quo $\lambda = 0$, et alterum quo $\lambda = 1$.

I°. Sit $\lambda = o$, et integrale $\pi (1 - aa)^3 V$, ubi pro V factores posteriores evadunt

$$\left(\frac{-2}{0}\right) = 1; \left(\frac{-2}{1}\right) = -2; \left(\frac{-2}{2}\right) = 3; \left(\frac{-2}{3}\right) = -4;$$
$$\left(\frac{-2}{4}\right) = +5; \left(\frac{-2}{5}\right) = -6; \text{ etc.}$$

simili modo priores factores erunt

$$\left(\frac{-2}{0}\right) = 1; \left(\frac{-2}{1}\right) = -2; \left(\frac{-2}{2}\right) = 3; \text{ etc.}$$

unde colligitur fore

$$V = 1 + 4aa + 9a^4 + 16a^6 + 25a^8 + 36a^{10} + \text{ etc.}$$

Pro qua serie summanda, inde subtrahatur series Vaa, et remanebit

$$V(1 - aa) = 1 + 3aa + 5a^4 + 7a^6 + 9a^8 + \text{ etc.}$$

Multiplicitur denuo utrinque per $1 - aa$, ac prodibit

$$V(1 - aa)^2 = 1 + 2aa + 2a^4 + 2a^6 + 2a^8 + \text{ etc}$$

quae denuo ducta in $1 - aa$ praebet

$$V(1 - aa)^3 = 1 + aa, \text{ ideoque } V = \frac{1 + aa}{(1 - aa)^3}.$$

Consequenter integrale quaesitum erit $= \pi(1 + aa)$, id quod utique oritur ex integratione actuali, cum sit

$$\int \partial\Phi (1 + aa - 2a\cos.\Phi) = (1 + aa)\Phi - 2a\sin.\Phi,$$

quod facto $\Phi = \pi$ abit in $(1 + aa)\pi$.

II°. Sit $\lambda = 1$, et integrale quaesitum $\pi a (1 - aa)^3 V$; ubi pro factoribus posterioribus est

$$\left(\frac{-1}{1}\right) = -1; \left(\frac{-1}{2}\right) = +1; \left(\frac{-1}{3}\right) = -1;$$
$$\left(\frac{-1}{4}\right) = +1; \left(\frac{-1}{5}\right) = -1; \text{ etc.}$$

Factores vero priores evadunt

$$(\frac{-3}{0}) = 1;\ (\frac{-3}{1}) = -3;\ (\frac{-3}{2}) = 6;\ (\frac{-3}{3}) = -10;$$
$$(\frac{-3}{4}) = 15;\ (\frac{-3}{5}) = -21;\ (\frac{-3}{6}) = 28;$$
$$(\frac{-3}{7}) = -36;\ \text{etc.}$$

hinc igitur habebimus

$$V = -1 - 3aa - 6a^4 - 10a^6 - 15a^8 - 21a^{10} - 28a^{12} - 36a^{14} - \text{etc.}$$

Pro cujus summatione multiplicetur utrinque per $1 - aa$, et prodibit

$$V(1 - aa) = -1 - 2aa - 3a^4 - 4a^6 - 5a^8 - 6a^{10} - 7a^{12} - 8a^{14} - \text{etc.}$$

multiplicando denuo per $1 - aa$, prodit

$$V(1 - aa)^2 = -1 - aa - a^4 - a^6 - a^8 - a^{10} - a^{12} - a^{14} - \text{etc.}$$

et multiplicando rursus per $1 - aa$, erit

$$V(1 - aa)^3 = -1,\ \text{ita ut sit}\ V = -\frac{1}{(1 - aa)^3},$$

consequenter integrale quaesitum $= -\pi a$. Ipsa autem integratio ob $\cos.\Phi^2 = \frac{1}{2} + \frac{1}{2}\cos.2\Phi$ praebet

$$\int \partial\Phi \cos.\Phi (1 + aa - 2a\cos.\Phi) = (1 + aa)\sin.\Phi$$
$$- a\Phi - \tfrac{1}{2} a \sin.2\Phi,$$

unde statuendo $\Phi = \pi$, oritur integrale $= -a\pi$.

Casus III.

quo $n = -3$, et formula integralis haec proponitur

$$\int \partial\Phi \cos.\lambda\Phi (1 + aa - 2a\cos.\Phi)^2 \begin{bmatrix} a\,\Phi = 0 \\ ad\,\Phi = \pi \end{bmatrix}.$$

Hoc ergo casu ex forma generali erit integrale

$$\pi a^\lambda (1 - aa)^5 V,\ \text{existente}$$

$$V = (\frac{-3-\lambda}{0})(\frac{-3+\lambda}{\lambda}) + (\frac{-3-\lambda}{1})(\frac{-3+\lambda}{\lambda+1})a^2$$
$$+ (\frac{-3-\lambda}{2})(\frac{-3+\lambda}{\lambda+2})a^4 + (\frac{-3-\lambda}{3})(\frac{-3+\lambda}{\lambda+3})a^6$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

ubi factores posteriores manifesto omnes evanescunt, quando fuerit vel $\lambda = 3$ vel $\lambda > 3$, quibus ergo casibus totum integrale evanescit, ut cuilibet calculum instituenti facile patebit: tres autem casus considerandi restant, quibus $\lambda < 3$.

I. Sit $\lambda = 0$, atque tam priores quam posteriores factores convenient, eruntque

$$(\tfrac{-3}{0}) = 1;\ (\tfrac{-3}{1}) = -3;\ (\tfrac{-3}{2}) = 6;\ (\tfrac{-3}{3}) = -10;$$
$$(\tfrac{-3}{4}) = 15;\ (\tfrac{-3}{5}) = -21;\ (\tfrac{-3}{6}) = 28;\ \text{etc.}$$

unde colligitur

$$V = 1 + 9aa + 36a^4 + 100a^6 + 225a^8 + 441a^{10} + \text{etc.}$$

quae series cum tandem perducat ad differentias constantes, simili modo ut hactenus summari poterit, prima enim multiplicatio per $1 - aa$ praebet

$$V(1-aa) = 1 + 8aa + 27a^4 + 64a^6 + 125a^8 + 216a^{10} + 343a^{12} + \text{etc.}$$

Secunda multiplicatio per $1 - aa$ praebet

$$V(1-aa)^2 = 1 + 7aa + 19a^4 + 37a^6 + 61a^8 + 91a^{10} + 127a^{12} + \text{etc.}$$

Tertia multiplicatio dat

$$V(1-aa)^3 = 1 + 6aa + 12a^4 + 18a^6 + 24a^8 + 30a^{10} + \text{etc.}$$

Quarta multiplicatio dat

$$V(1-aa)^4 = 1 + 5aa + 6a^4 + 6a^6 + 6a^8 + 6a^{10} + \text{etc.}$$

ac denique

$V(1-aa)^5 = 1 + 4aa + a^4$, ita ut sit $V = \frac{1+4aa+a^4}{(1-aa)^5}$;

consequenter valor integralis quaesitus hoc casu erit $\pi(1 + 4aa + a^4)$, quod egregie cum integrali more solito invento congruit.

II. Sit $\lambda = 1$, quo casu priores factores ipsius V erunt

$$\left(\frac{-4}{0}\right) = 1;\ \left(\frac{-4}{1}\right) = -4;\ \left(\frac{-4}{2}\right) = 10;\ \left(\frac{-4}{3}\right) = -20;$$
$$\left(\frac{-4}{4}\right) = 35;\ \left(\frac{-4}{5}\right) = -56;\ \left(\frac{-4}{6}\right) = 84;\ \left(\frac{-4}{7}\right) = -120 \text{ etc.}$$

posteriores vero ita se habent

$$\left(\frac{-2}{1}\right) = -2;\ \left(\frac{-2}{2}\right) = +3;\ \left(\frac{-2}{3}\right) = -4;\ \left(\frac{-2}{4}\right) = +5;$$
$$\left(\frac{-2}{5}\right) = -6;\ \left(\frac{-2}{6}\right) = +7;\ \left(\frac{-2}{7}\right) = -8:\ \left(\frac{-2}{8}\right) = +9; \text{ etc.}$$

ideoque

$$V = -2 - 12\,a^2 - 40\,a^4 - 100\,a^6 - 210\,a^8 - 392\,a^{10}$$
$$- 672\,a^{12} - 1080\,a^{14} - \text{etc.}$$

quae series cum tandem perducat ad differentias constantes, simili modo ut ante summari poterit; prima enim multiplicatio per $1 - aa$ dat

$$V(1 - aa) = -2 - 10\,a^2 - 28\,a^4 - 60\,a^6 - 110\,a^8$$
$$- 182\,a^{10} - 280\,a^{12} - \text{etc.}$$

Secunda multiplicatio per $1 - aa$ praebet

$$V(1 - aa)^2 = -2 - 8\,a^2 - 18\,a^4 - 32\,a^6 - 50\,a^8$$
$$- 72\,a^{10} - 98\,a^{12} - \text{etc.}$$

Tertia multiplicatio dat

$$V(1 - aa)^3 = -2 - 6\,a^2 - 10\,a^4 - 14\,a^6 - 18\,a^8$$
$$- 22\,a^{10} - 26\,a^{12} - \text{etc.}$$

Quarta multiplicatio dat

$$V(1 - aa)^4 = -2 - 4a^2 - 4\,a^4 - 4\,a^6 - 4\,a^8$$
$$- 4\,a^{10} - 4\,a^{12} - \text{etc.}$$

ac denique quinta multiplicatio per $1 - aa$ praebet

$$V(1 - aa)^5 = -2 - 2\,aa = -2(1 + aa);$$

unde colligitur $V = -\frac{2(1+aa)}{(1-aa)^5}$, ideoque valor integralis quaesitus erit $= -2\pi a(1 + aa)$, qui egregie cum integrali more solito invento congruit.

III. Sit $\lambda = 2$, atque factores priores ipsius V erunt

$$\left(\tfrac{-5}{0}\right) = 1;\ \left(\tfrac{-5}{1}\right) = -5;\ \left(\tfrac{-5}{2}\right) = 15;\ \left(\tfrac{-5}{3}\right) = -35;$$
$$\left(\tfrac{-5}{4}\right) = 70;\ \left(\tfrac{-5}{5}\right) = -126;\ \left(\tfrac{-5}{6}\right) = 210;$$
$$\left(\tfrac{-5}{7}\right) = -330 \text{ etc.}$$

posteriores vero factores ita se habebunt

$$\left(\tfrac{-1}{2}\right) = 1;\ \left(\tfrac{-1}{3}\right) = -1;\ \left(\tfrac{-1}{4}\right) = 1;\ \left(\tfrac{-1}{5}\right) = -1;$$
$$\left(\tfrac{-1}{6}\right) = 1;\ \left(\tfrac{-1}{7}\right) = -1;\ \left(\tfrac{-1}{8}\right) = 1;\ \left(\tfrac{-1}{9}\right) = -1 \text{ etc.}$$

unde colligitur

$$V = 1 + 5a^2 + 15a^4 + 35a^6 + 70a^8 + 126a^{10} + 210a^{12} + 330a^{14} + \text{etc.}$$

quae series eodem modo ut ante summata praebet $V = + \frac{1}{(1-aa)^5}$, unde colligitur valor integralis quaesitus $= \pi aa$, qui cum integrali more solito invento utique egregie congruit.

§. 24. Quodsi haec integralia quibus n est numerus negativus cum iis comparemus, quibus n est numerus positivus, insignis analogia deprehenditur inter valores harum formularum

$$\int \Delta^n \partial \Phi \cos. \lambda \Phi \text{ et } \int \frac{\partial \Phi \cos. \lambda \Phi}{\Delta^{n+1}},$$

quae affinitas, si per plures casus exploretur, sequens nobis suppeditat theorema maxime notabile.

Theorema.

§. 25. Posito brevitatis gratia $\Delta = 1 + aa - 2a \cos. \Phi$, atque integralia a termino $\Phi = 0$ usque ad terminum $\Phi = 180°$ extendantur, semper locum habebit sequens proportio

$$\int \Delta^n \partial \Phi \cos. \lambda \Phi : \int \frac{\partial \Phi \cos. \lambda \Phi}{\Delta^{n+1}} = \left(\tfrac{n}{\lambda}\right)(1-aa)^n : \left(\tfrac{-n-1}{\lambda}\right)(1-aa)^{-n-1},$$

vel si statuamus

$$\frac{\Delta}{1-aa} = \frac{1+aa-2a\cos.\Phi}{1-aa} = \Gamma,$$

simplicius erit

$$\int \Gamma^n \partial\Phi \cos.\lambda\Phi : \int \frac{\partial\Phi \cos.\lambda\Phi}{\Gamma^{n+1}} = (\tfrac{n}{\lambda}) : (\tfrac{-n-1}{\lambda}).$$

§. 26. Ita exempla gratia si ponamus $n = 2$, erit ex priore proportione

$$\int \Delta^2 \partial\Phi \cos.\lambda\Phi : \int \frac{\partial\Phi \cos.\lambda\Phi}{\Delta^3} = (\tfrac{2}{\lambda})(1-aa)^2 : (\tfrac{-3}{\lambda})(1-aa)^{-3}$$

unde si $\lambda = 0$, ob $(\frac{n}{0}) = 1$ et $(\frac{-3}{0}) = 1$, erit

$$\int \Delta^2 \partial\Phi : \int \frac{\partial\Phi}{\Delta^3} = (1-aa)^2 : \frac{1}{(1-aa)^3} = 1 : \frac{1}{(1-aa)^5},$$

ideoque erit

$$\int \frac{\partial\Phi}{\Delta^3} = \frac{1}{(1-aa)^5} \int \Delta^2 \partial\Phi.$$

Cum igitur sit

$$\int \Delta^2 \partial\Phi = \pi(1+4aa+a^4), \text{ erit}$$

$$\int \frac{\partial\Phi}{\Delta^3} = \frac{\pi}{(1-aa)^5}(1+4aa+a^4).$$

§. 27. Manente $n = 2$, sit $\lambda = 1$, ob $(\frac{2}{1}) = 1$ et $(\frac{-3}{1}) = -3$, erit

$$\int \Delta^2 \partial\Phi \cos.\Phi : \int \frac{\partial\Phi \cos.\Phi}{\Delta^3} = 2(1-aa)^2 : -3(1-aa)^{-3} = 1 : \frac{-3}{2(1-aa)^5}$$

unde fit

$$\int \frac{\partial\Phi \cos.\Phi}{\Delta^3} = \frac{-3}{2(1-aa)^5} \int \Delta^2 \partial\Phi \cos.\Phi;$$

cum igitur sit

$$\int \Delta^2 \partial\Phi \cos.\Phi = -2\pi a(1+aa), \text{ erit}$$

$$\int \frac{\partial\Phi \cos.\Phi}{\Delta^3} = \frac{+3\pi a(1+aa)}{(1-aa)^5}.$$

§. 28. Simili modo sumatur $\lambda = 2$, et ob $(\frac{2}{2}) = 1$ et $(\frac{-3}{2}) = 6$, erit

$\int \Delta^2 \partial \Phi \cos. 2 \Phi : \int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^3} = (1 - aa)^2 : 6(1 - aa)^{-3} = 1 : \frac{6}{(1 - aa)^5}$,

unde fit

$$\int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^3} = \frac{6}{(1 - aa)^5} \int \Delta^2 \partial \Phi \cos. 2 \Phi.$$

Erat autem

$$\int \Delta^2 \partial \Phi \cos. 2 \Phi = \pi a a,$$

consequenter

$$\int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^3} = \frac{6 \pi a a}{(1 - aa)^5}.$$

§. 29. Cum character $(\frac{n}{\lambda})$ fiat $= 1$ casu $\lambda = n$, casibus vero quibus $\lambda > n$ semper sit $(\frac{n}{\lambda}) = 0$, siquidem λ fuerit numerus integer, uti hic perpetuo assumimus, evidens est istis casibus, quibus $\lambda > n$, semper valorem formulae $\int \Delta^n \partial \Phi \cos. \lambda \Phi$ in nihilum abire.

§. 30. Theorema, quod hic proposuimus, non solum ob simplicitatem rationis omni attentione est dignum, sed etiam quod id tantum per plures casus sola inductione conclusimus, neque adhuc ulla via patere videtur, qua ejus veritas directe demonstrari queat; hujusmodi autem theoremata summam Geometrarum attentionem merentur. Evolvamus autem adhuc alios quosdam casus memorabiles nostri theorematis initio propositi.

Evolutio casus
quo $\lambda = n$, et formula integralis proposita

$$\int \frac{\partial \Phi \cos. n \Phi}{\Delta^{n+1}}.$$

Ex forma generali hoc casu integrale erit $\frac{\pi a^n}{(1 - aa)^{2n+1}} V$, existente

$$V = \left(\frac{0}{0}\right)\left(\frac{2n}{n}\right) + \left(\frac{0}{1}\right)\left(\frac{2n}{n+1}\right) aa + \left(\frac{0}{2}\right)\left(\frac{2n}{n+2}\right) a^4 + \text{etc.}$$

ubi manifesto omnes termini praeter primum evanescunt, ita ut sit $V = \left(\frac{2n}{n}\right)$, ideoque nostrum integrale

$$\int \frac{\partial \Phi \cos. n \Phi}{\Delta^{n+1}} = \frac{\pi a^n}{(1-aa)^{2n+1}} \cdot \left(\frac{2n}{n}\right);$$

ubi notetur, valores characteris $\left(\frac{2n}{n}\right)$ pro variis valoribus numeri n sequenti modo se habere

n	0,	1,	2,	3,	4,	5,	6,	7	etc.
$\left(\frac{2n}{n}\right)$	1,	2,	6,	20,	70,	252,	924,	3432	

quae series facillime per hos factores continuatur

$$\frac{2}{1} \cdot \frac{6}{2} \cdot \frac{10}{3} \cdot \frac{14}{4} \cdot \frac{18}{5} \cdot \frac{22}{6} \cdot \frac{26}{7} \text{ etc.}$$

Postremum vero theorema inventum ad hunc casum applicatum praebebit hanc proportionem

$$\int \Delta^n \partial \Phi \cos. n\Phi : \int \frac{\partial \Phi \cos. n\Phi}{\Delta^{n+1}} = (1-aa)^n : \left(\frac{-1-n}{n}\right)(1-aa)^{-n-1},$$

unde fit

$$\int \Delta^n \partial \Phi \cos. n \Phi = \frac{\pi a^n}{\left(\frac{-n-1}{n}\right)} \cdot \left(\frac{2n}{n}\right) = \left(\frac{2n}{n}\right) \pi a^n : \left(\frac{-n-1}{n}\right);$$

ubi notetur valores characteris $\left(\frac{-n-1}{n}\right)$ pro variis valoribus ipsius n esse

n	0,	1,	2,	3,	4,	5,	6	etc.
$\left(\frac{-n-1}{n}\right)$	-1,	-2,	6,	-20,	70,	-252,	924	

unde patet esse $\left(\frac{-n-n}{n}\right) = \pm \left(\frac{2n}{n}\right)$, dum signum superius valet, quando n est numerus par, contra vero signum inferius, quando n est numerus impar; hinc ergo erit

$$\int \Delta^n \partial \Phi \cos. n \Phi = \pm \pi a^n.$$

His notatis evolvamus casus simpliciores pro utraque formula integrali

$n = 0$	$\int \frac{\partial \Phi}{\Delta} = \frac{\pi}{1 - aa}$	$\int \partial \Phi = + \pi$
$n = 1$	$\int \frac{\partial \Phi \cos. \Phi}{\Delta^2} = \frac{2 \pi a}{(1 - aa)^3}$	$\int \Delta \partial \Phi \cos. \Phi = - \pi a$
$n = 2$	$\int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^3} = \frac{2 \pi a^2}{(1 - aa)^6}$	$\int \Delta^2 \partial \Phi \cos. 2 \Phi = + \pi a^2$
$u = 3$	$\int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^4} = \frac{20 \pi a^3}{(1 - aa)^7}$	$\int \Delta^3 \partial \Phi \cos. 3 \Phi = - \pi a^3$
$n = 4$	$\int \frac{\partial \Phi \cos. 4 \Phi}{\Delta^5} = \frac{70 \pi a^4}{(1 - aa)^9}$	$\int \Delta^4 \partial \Phi \cos. 4 \Phi = + \pi a^4$
$n = 5$	$\int \frac{\partial \Phi \cos. 5 \Phi}{\Delta^6} = \frac{252 \pi a^6}{(1 - aa)^{11}}$	$\int \Delta^5 \partial \Phi \cos. 5 \Phi = - \pi a^5$
$n = 6$	$\int \frac{\partial \Phi \cos. 6 \Phi}{\Delta^7} = \frac{924 \pi a^6}{(1 - aa)^{13}}$	$\int \Delta^6 \partial \Phi \cos. 6 \Phi = + \pi a^6$
	etc.	etc.

Hic imprimis notatu dignum occurrit, quod his casibus $\lambda = n$ integralia tam succincte exprimuntur; nunc autem alios perpendamus casus, quibus litterae λ successive valores 0, 1, 2, 3 etc. tribuantur.

Evolutio casus

quo $\lambda = 0$, et formula integralis proposita

$$\int \frac{\partial \Phi}{\Delta^{n+1}}.$$

§. 31. Cum hic sit $\lambda = 0$, integrale quaesitum ex nostra formula erit $\frac{\pi}{(1 - aa)^{2n+1}}$ V, existente

$$V = (\tfrac{n}{0})^2 + (\tfrac{n}{1})^2 aa + (\tfrac{n}{2})^2 a^4 + (\tfrac{n}{3})^2 a^6 + (\tfrac{n}{4})^2 a^8 + \text{etc.}$$

simul vero hinc etiam assignari poterit valor hujus formulae $\int \Delta^n \partial \Phi$, cum sit

$$\int \Delta^n \partial \Phi : \int \frac{\partial \Phi}{\Delta^{n+1}} = (1-aa)^n : (1-aa)^{-n-1} = (1-aa)^{2n+1} : 1,$$

ex qua proportione colligitur

$$\int \Delta^n \partial \Phi = \pi \,.\, \text{V}.$$

Percurramus igitur simpliciores casus pro exponente n, quos sequenti tabula subjungamus

$$n = 0 \begin{cases} \int \frac{\partial \Phi}{\Delta} = \frac{\pi}{1-aa} \\ \int \partial \Phi = \pi \end{cases}$$

$$n = 1 \begin{cases} \int \frac{\partial \Phi}{\Delta^2} = \frac{\pi}{(1-aa)^3} (1+aa) \\ \int \Delta \partial \Phi = \pi \, (1+aa) \end{cases}$$

$$n = 2 \begin{cases} \int \frac{\partial \Phi}{\Delta^3} = \frac{\pi}{(1-aa)^5} (1+2^2 aa + a^4) \\ \int \Delta^2 \partial \Phi = \pi \, (1+2^2 aa + a^4) \end{cases}$$

$$n = 3 \begin{cases} \int \frac{\partial \Phi}{\Delta^4} = \frac{\pi}{(1-aa)^7} (1+3^2 aa + 3^2 a^4 + a^6) \\ \int \Delta^3 \partial \Phi = \pi \, (1+3^2 aa + 3^2 a^4 + a^6) \end{cases}$$

$$n = 4 \begin{cases} \int \frac{\partial \Phi}{\Delta^5} = \frac{\pi}{(1-aa)^9} (1+4^2 aa + 6^2 a^4 + 4^2 a^6 + a^8) \\ \int \Delta^4 \partial \Phi = \pi \, (1+4^2 aa + 6^2 a^4 + 4^2 a^6 + a^8) \end{cases}$$

etc. etc.

Evolutio casuum

quibus $\lambda = 1$, et formula integralis proposita

$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^{n+1}}$$

§. 32. Hoc igitur casu integrale quaesitum erit

$$\frac{\pi a}{(1-aa)^{2n+1}} \,.\, \text{V}$$

existente

$$V = \left(\frac{n-1}{0}\right)\left(\frac{n+1}{1}\right) + \left(\frac{n-1}{1}\right)\left(\frac{n+1}{2}\right) aa$$
$$+ \left(\frac{n-1}{2}\right)\left(\frac{n+1}{3}\right) a^4 + \left(\frac{n-1}{3}\right)\left(\frac{n+1}{4}\right) a^6$$
$$+ \left(\frac{n-1}{4}\right)\left(\frac{n+1}{5}\right) a^6 + \left(\frac{n-1}{5}\right)\left(\frac{n+1}{6}\right) a^8 + \text{etc.}$$

Tum vero cum ob $\lambda = 1$ fit

$$\int \Delta^n \partial\Phi \cos.\Phi : \int \frac{\partial\Phi \cos.\Phi}{\Delta^{n+1}} = n(1-aa)^n : -(n+1)(1-aa)^{-n-1}$$

unde fit

$$\int \Delta^n \partial\Phi \cos.\Phi = -\frac{n}{n+1} \cdot \pi a V.$$

Pro casibus ergo simplicioribus ipsius n sequentem tabulam subjungamus

$$n=0 \begin{cases} \int \frac{\partial\Phi \cos.\Phi}{\Delta} = \frac{\pi a}{1-aa} \\ \int \partial\Phi \cos.\Phi = 0 \end{cases}$$

$$n=1 \begin{cases} \int \frac{\partial\Phi \cos.\Phi}{\Delta^2} = \frac{2\pi a}{(1-aa)^3} \\ \int \Delta \partial\Phi \cos.\Phi = -\pi a \end{cases}$$

$$n=2 \begin{cases} \int \frac{\partial\Phi \cos.\Phi}{\Delta^3} = \frac{\pi a}{(1-aa)^6}(1.3+1.3aa) \\ \int \Delta^2 \partial\Phi \cos.\Phi = -\frac{2}{3}\pi a(1.3+1.3aa) \end{cases}$$

$$n=3 \begin{cases} \int \frac{\partial\Phi \cos.\Phi}{\Delta^4} = \frac{\pi a}{(1-aa)^7}(1.4+26aa+1.4a^4) \\ \int \Delta^3 \partial\Phi \cos.\Phi = -\frac{3}{4}\pi a(1.4+2.6aa+1.4a^4) \end{cases}$$

$$n=4 \begin{cases} \int \frac{\partial\Phi \cos.\Phi}{\Delta^5} = \frac{\pi a}{(1-aa)^9}(1.5+3.10aa+3.10a^4+1.5.a^6) \\ \int \Delta^4 \partial\Phi \cos.\Phi = -\frac{4}{5}\pi a(1.5+3.10aa+3.10a^4+1.5.a^6) \end{cases}$$

$$n=5 \begin{cases} \int \frac{\partial\Phi \cos.\Phi}{\Delta^6} = \frac{\pi a}{(1-aa)^{11}}(1.6+4.15aa+6.20a^4+4.15a^6+1.6a^8) \\ \int \Delta^5 \partial\Phi \cos.\Phi = -\frac{5}{6}\pi(1.6+\text{etc.}) \end{cases}$$

$$n=6 \begin{cases} \int \frac{\partial\Phi \cos.\Phi}{\Delta^7} = \frac{\pi a}{(1-aa)^{13}}(1.7+5.21aa+10.35a^4+10.35a^6+\text{etc.}) \\ \int \Delta^6 \partial\Phi \cos.\Phi = -\frac{6}{7}\pi a(1.7+\text{etc.}) \end{cases}$$

Evolutio casuum
quibus $\lambda = 2$, et formula integralis proposita
$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^{n+1}}.$$

§. 33. Hoc ergo casu integrale quaesitum erit
$$\frac{\pi a^2}{(1-aa)^{2n+1}} . V$$
existente
$$V = \left(\frac{n-2}{0}\right)\left(\frac{n+2}{2}\right) + \left(\frac{n-2}{1}\right)\left(\frac{n+2}{3}\right) aa + \left(\frac{n-2}{2}\right)\left(\frac{n+2}{4}\right) a^4$$
$$+ \left(\frac{n-2}{3}\right)\left(\frac{n+2}{5}\right) a^6 + \left(\frac{n-2}{4}\right)\left(\frac{n+2}{6}\right) a^8 + \text{etc.}$$
tum vero erit altera forma
$$\int \Delta^n \partial \Phi \cos. 2\Phi = \frac{n(n-1)}{(n+1)(n+2)} \pi aa V.$$
Percurramus ergo ut hactenus casus simpliciores, et quia integratio formulae $\int \Delta^n \partial \Phi \cos. 2\Phi$ sponte patet ex ultima formula, superfluum foret haec integralia allegare

$$n = 0 : \int \frac{\partial \Phi \cos. 2\Phi}{\Delta} = \frac{\pi aa}{1-aa}$$
$$n = 1 : \int \frac{\partial \Phi \cos. 2\Phi}{\Delta^2} = \frac{\pi aa}{(1-aa)^3} (1.3 - 1.1.aa)$$
$$n = 2 : \int \frac{\partial \Phi \cos. 2\Phi}{\Delta^3} = \frac{\pi aa}{(1-aa)^5} (1.6)$$
$$n = 3 : \int \frac{\partial \Phi \cos. 2\Phi}{\Delta^4} = \frac{\pi aa}{(1-aa)^7} (1.10 + 1.10\, aa)$$
$$n = 4 : \int \frac{\partial \Phi \cos. 2\Phi}{\Delta^5} = \frac{\pi aa}{(1-aa)^9} (1.15 + 2.20\, aa + 1.15\, a^4)$$
$$n = 5 : \int \frac{\partial \Phi \cos. 2\Phi}{\Delta^6} = \frac{\pi aa}{(1-aa)^{11}} (1.21 + 3.35\, a^2 + 3.35\, a^4 + 1.21\, a^6)$$
$$n = 6 : \int \frac{\partial \Phi \cos. 2\Phi}{\Delta^7} = \frac{\pi aa}{(1-aa)^{13}} (1.28 + 4.56\, aa + 6.70\, a^4 + 4.56\, a^6 - 1.28\, a^8)$$
etc. etc.

Evolutio casuum
quibus $\lambda = 3$ et formula integralis proposita

$$\int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^{n+1}}.$$

§. 34. Hoc ergo casu integrale erit

$$\frac{\pi a^3}{(1-aa)^{2n+1}}. \mathrm{V},$$

existente

$$\mathrm{V} = \left(\frac{n-3}{0}\right)\left(\frac{n+3}{3}\right) + \left(\frac{n-3}{1}\right)\left(\frac{n+3}{4}\right) aa + \left(\frac{n-3}{2}\right)\left(\frac{n+3}{5}\right) a^4$$
$$+ \left(\frac{n-3}{3}\right)\left(\frac{n+3}{6}\right) a^6 + \text{etc.}$$

pro altera autem formula habebimus

$$\int \Delta^n \partial \Phi \cos. 3 \Phi = - \frac{n(n-1)(n-2)}{(n+1)(n+2)(n+3)} \pi a^3 \mathrm{V}.$$

Pro praecipuis igitur casibus habebimus sequentem tabellam

$$n = 0 : \int \frac{\partial \Phi \cos. 3 \Phi}{\Delta} = \frac{\pi a^3}{1-aa}$$
$$n = 1 : \int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^2} = \frac{\pi a^3}{(1-aa)^3}(1.4 - 2.1aa)$$
$$n = 2 : \int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^3} = \frac{\pi a^3}{(1-aa)^5}(1.10 - 1.5aa)$$
$$n = 3 : \int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^4} = \frac{\pi a^3}{(1-aa)^7}(1.20)$$
$$n = 4 : \int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^5} = \frac{\pi a^3}{(1-aa)^9}(1.35 + 1.35aa)$$
$$n = 5 : \int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^6} = \frac{\pi a^3}{(1-aa)^{11}}(1.56 + 2.70aa + 1.56a^4)$$
$$n = 6 : \int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^7} = \frac{\pi a^3}{(1-aa)^{13}}(1.84 + 3.126aa + 3.126a^4 + 1.84a^6)$$

etc. etc.

Observatio circa valores negativos ipsius λ.

§. 35. Jam initio monuimus, pro littera λ tantum numeros integros positivos sumi oportere, qua conditione generalitas no-

strae quaestionis non restringitur cum semper sit cos. $-\lambda\Phi =$ cos. $\lambda\Phi$. Interim tamen hic ingens paradoxon se offert, quod solutiones supra inventae evadant falsae, quando ipsi λ valores negativi tribuuntur; quod quo clarius pateat consideremus casum $n = 0$; pro quo supra invenimus

$$\int \frac{\partial\Phi \cos.\lambda\Phi}{\Delta} = \frac{\pi a^\lambda}{1-aa},$$

unde videtur sequi debere, casu $\lambda = -i$ fore

$$\int \frac{\partial\Phi \cos. i\Phi}{\Delta} = \frac{\pi}{a^i(1-aa)},$$

quod autem manifesto est falsum, cum verum integrale utique sit $\frac{\pi a^i}{1-aa}$, perinde ac si esset $\lambda = +i$. At vero ista restrictio tantum est apparens, atque solutio nostra generalis nihilo minus veritati est consentanea, etiamsi litterae λ valores negativi tribuantur, dummodo fuerint integri; quandoquidem perpetuo assumimus, casu $\Phi = \pi$ semper esse sin. $\lambda\Phi = 0$; hoc igitur maxime operae erit pretium clarius ostendisse.

§. 36. Sufficiet autem, casum quo $n = 0$ perpendisse, pro quo nostra solutio generalis praebet

$$\int \frac{\partial\Phi \cos.\lambda\Phi}{\Delta} = \frac{\pi a^\lambda}{1-aa} V,$$

existente

$$V = \left(\frac{-\lambda}{0}\right)\left(\frac{\lambda}{\lambda}\right) + \left(\frac{-\lambda}{1}\right)\left(\frac{\lambda}{\lambda+1}\right) aa + \left(\frac{-\lambda}{2}\right)\left(\frac{\lambda}{\lambda+2}\right) a^4$$
$$+ \left(\frac{-\lambda}{3}\right)\left(\frac{\lambda}{\lambda+3}\right) a^6 + \text{etc.}$$

Cujus expressionis tantum prima pars remanet, quando λ est numerus positivus integer, propterea quod tum formulae $\left(\frac{\lambda}{\lambda+1}\right)$, $\left(\frac{\lambda}{\lambda+2}\right)$, $\left(\frac{\lambda}{\lambda+3}\right)$, etc. evanescunt; longe secus autem se res habet, quando

pro λ assumitur numerus negativus, veluti si ponamus $\lambda = -i$ tum erit

$$V = (\tfrac{i}{0})(\tfrac{-i}{-i}) + (\tfrac{i}{1})(\tfrac{-i}{1-i})\,aa + (\tfrac{i}{2})(\tfrac{-i}{2-i})\,a^4$$
$$+ (\tfrac{i}{3})(\tfrac{-i}{3-i})\,a^6 + \text{etc.}$$

ubi notetur, omnium horum characterum, quamdiu denominator est negativus, valores evanescere; quoniam vero denominatores continuo crescunt, tandem evadent positivi, atque adeo valores determinatos exhibebunt. Ad hoc ostendendum ponamus primo $\lambda = -1$ sive $i = +1$, eritque $V = -aa$ ubi primum membrum sine dubio est $= 0$, secundum vero

$$(\tfrac{1}{1})(\tfrac{+1}{0})\,aa = aa,$$

Cum igitur sit $V = aa$ casu $\lambda = -1$, nostra formula praebet hoc integrale

$$\int \frac{\partial\Phi \cos. -\Phi}{\Delta} = \frac{\pi a^{-1}}{1-aa} \cdot aa = \frac{\pi a}{1-aa},$$

id quod prorsus convenit.

§. 37. Sumamus nunc $\lambda = -2$ sive $i = 2$, manente $n = 0$, eritque

$$V = (\tfrac{2}{0})(\tfrac{-2}{-2}) + (\tfrac{2}{1})(\tfrac{-2}{-1})\,aa + (\tfrac{2}{2})(\tfrac{-2}{0})\,a^4,$$

ubi sequentes termini manifesto evanescunt; ob factores priores autem bini termini initiales etiam evanescunt ob denominatores negativos; tertius autem terminus ob $(\tfrac{-2}{0}) = 1$ praebet $V = a^4$, consequenter casu $\lambda = -2$ habebimus

$$\int \frac{\partial\Phi \cos. -2\Phi}{\Delta} = \frac{\pi a^{-2}}{1-aa} \cdot a^4 = \frac{\pi aa}{1-aa},$$

prorsus atque invenimus pro $\int \frac{\partial\Phi \cos. 2\Phi}{\Delta}$.

§. 38. Simili modo facile intelligitur, casu $\lambda = -3$ proditurum esse $V = a^6$, eodemque modo casu $\lambda = -4$ reperietur $V = a^8$, atque adeo in genere casu $\lambda = -i$ obtinebitur $V = a^{2i}$, sicque hujus formulae $\int \frac{\partial \Phi \cos. -i\Phi}{\Delta}$ integrale erit

$$\frac{\pi a^{-i}}{1-aa} \cdot a^{2i} = \frac{\pi a^i}{1-aa},$$

quod ipsum est integrale formulae $\int \frac{\partial \Phi \cos. i\Phi}{\Delta}$, uti natura rei postulat.

§. 39. Talis autem egregius consensus locum habebit pro omnibus valoribus ipsius n. Sit enim verbi gratia $n = 2$, et integratio nostra

$$\int \frac{\partial. \Phi \cos. \lambda \Phi}{\Delta^3} = \frac{\pi a^\lambda}{(1-aa)^5} \cdot V$$

existente

$$V = (\tfrac{2-\lambda}{0})(\tfrac{2+\lambda}{\lambda}) + (\tfrac{2-\lambda}{1})(\tfrac{2+\lambda}{\lambda+1})\, aa + (\tfrac{2-\lambda}{2})(\tfrac{2+\lambda}{\lambda+2})\, a^4 + \text{etc.}$$

quare sumto $\lambda = -3$, ut forma nostra sit

$$\int \frac{\partial \Phi \cos. -3\Phi}{\Delta^3} = \frac{\pi a^{-3}}{(1-aa)^5} \cdot V,$$

existente

$$V = (\tfrac{5}{0})(\tfrac{-1}{-3}) + (\tfrac{5}{1})(\tfrac{-1}{-2})\, aa + (\tfrac{5}{2})(\tfrac{-1}{-1})\, a^4 + (\tfrac{5}{3})(\tfrac{-1}{0})\, a^6$$
$$+ (\tfrac{5}{4})(\tfrac{-1}{1})\, a^8 + (\tfrac{5}{5})(\tfrac{-1}{2})\, a^{10},$$

ubi tria priora membra evanescunt, sequentia autem ob

$$(\tfrac{-1}{0}) = 1, \quad (\tfrac{-1}{1}) = -1, \quad (\tfrac{-1}{2}) = 1, \text{ erit}$$

$$V = 10a^6 - 5a^8 + a^{10} = a^6(10 - 5aa + a^4),$$

consequenter nostrum integrale fit

$$\int \frac{\partial \Phi \cos. -3\Phi}{\Delta^3} = \frac{\pi a^3}{(1-aa)^5}(10 - 5aa + a^4),$$

prorsus uti supra invenimus pro casu $\int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^3}$; talis autem consensus perpetuo deprehendi debebit.

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3) Disquisitio conjecturalis super formula integrali

$$\int \frac{\partial \Phi \cos. i \Phi}{(a + \beta \cos. \Phi)^n}.$$

M. S. Academiae exhib. die 31. *Augusti* 1778.

§. 40. Incipiamus a casu simplicissimo quo $i = 0$ et $n = 1$, et formula integranda proponitur haec $\int \frac{\partial \Phi}{\alpha + \beta \cos. \Phi}$, ad quod praestandum commodissime in subsidium vocatur haec substitutio tang. $\frac{1}{2} \Phi = t$, unde statim fit $\partial \Phi = \frac{2 \partial t}{1 + tt}$: porro vero cum hinc sit

$$\sin. \tfrac{1}{2} \Phi = \frac{t}{\sqrt{(1+tt)}} \text{ et } \cos. \tfrac{1}{2} \Phi = \frac{1}{\sqrt{(1+tt)}},$$

erit $\cos. \Phi = \frac{1 - tt}{1 + tt}$, ideoque denominator nostrae formulae

$$a + \beta \cos. \Phi = \frac{\alpha + \beta + (\alpha - \beta) tt}{1 + tt},$$

sicque nostra formula integranda erit

$$\int \frac{2 \partial t}{\alpha + \beta + (\alpha - \beta) tt}.$$

§. 41. Constat autem ex elementis esse

$$\int \frac{\partial t}{f + gtt} = \frac{1}{\sqrt{fg}} \text{ Arc. tang. } t \sqrt{\frac{g}{f}}.$$

Quare cum pro nostro casu sit $f = \alpha + \beta$ et $g = \alpha - \beta$, habebimus hanc integrationem

$$\int \frac{\partial \Phi}{\alpha + \beta \cos. \Phi} = \frac{2}{\sqrt{(\alpha\alpha - \beta\beta)}} \text{ Arc. tang. } t \sqrt{\frac{\alpha - \beta}{\alpha + \beta}},$$

existente $t = \text{tang. } \frac{1}{2} \Phi$; quod ergo integrale evanescit casu $t = 0$, ideoque casu $\Phi = 0$. Quodsi ergo hoc integrale extendere velimus

a termino $\Phi = 0$ usque ad terminum $\Phi = 180^\circ$, ubi fit $t = \infty$, istud integrale erit $\frac{2}{(\alpha\alpha - \beta\beta)} \cdot \frac{\pi}{2}$, denotante π semiperipheriam circuli, cujus radius $= 1$.

§. 42. Quoniam igitur integrale nostrae formulae a termino $\Phi = 0$ usque ad terminum $\Phi = 180^\circ$ tam concinne et simpliciter exprimitur, etiam generatim in hac dissertatione in ea tantum integralia formulae generalis propositae

$$\int \frac{\partial \Phi \cos. i \Phi}{(\alpha + \beta \cos. \Phi)^n},$$

sum inquisiturus, quae comprehenduntur inter terminos $\Phi = 0$ et $\Phi = 180^\circ$. Quia autem in casu tractato formula inest irrationalis $\sqrt{(\alpha\alpha - \beta\beta)}$, ad hoc incommodum tollendum, in sequentibus perpetuo assumemus $\alpha = 1 + aa$ et $\beta = -2a$, unde fit $\sqrt{(\alpha\alpha - \beta\beta)} = 1 - aa$, sicque nostrae disquisitiones versabuntur circa integrationem hujus formulae generalis

$$\int \frac{\partial \Phi \cos\ i \Phi}{(1 + aa - 2a \cos. \Phi)^n},$$

pro qua brevitatis gratia ubique statuamus

$$1 + aa - 2a \cos. \Phi = \Delta,$$

ut nostra formula generalis jam sit

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^n},$$

ubi ut jam notatum, eum tantum integralis valorem explorare nobis est propositum, qui intra terminos $\Phi = 0$ et $\Phi = 180^\circ$ contineatur, quem valorem ex casibus particularibus concludere conabimur. Praeterea vero hic in genere notetur, litteram i nobis perpetuo alios numeros non designare praeter integros, et quidem positivos, quandoquidem semper est

$$\cos. -i \Phi = \cos. +i \Phi.$$

I. De integratione formulae $\int\frac{\partial\Phi\cos. i\Phi}{\Delta}\left[\begin{smallmatrix}a\,\Phi=0\\ ad\,\Phi=180^\circ\end{smallmatrix}\right]$.

§. 43. Hic ergo casus in generali continetur, ponendo exponentem $n=1$, quem casum ut simplicissimum spectamus, siquidem casus $n=0$ nulla prorsus laborat difficultate, cum sit

$$\int\partial\Phi\cos. i\Phi=\frac{1}{i}\sin. i\Phi,$$

quod integrale jam evanescit casu $i=0$, et quoniam i numeros tantum integros significat, sumto $\Phi=180^\circ$ hoc integrale iterum evanescit, solo casu excepto quo $i=0$, quippe quo casu integrale fiet $=\Phi$, ideoque sumto $\Phi=180^\circ$. erit pro terminis integrationis constitutis $\int\partial\Phi=\pi$.

§. 44. Iste postremus casus fundamentum continet, unde integralia formae hic propositae haurire conveniet; cum enim sit

$$\partial\Phi=\frac{(1+aa)\partial\Phi}{\Delta}-\frac{2a\partial\Phi\cos.\Phi}{\Delta},$$

erit integrando pro terminis praescriptis

$$\pi=(1+aa)\int\frac{\partial\Phi}{\Delta}-2a\int\frac{\partial\Phi\cos.\Phi}{\Delta};$$

supra autem invenimus esse $\int\frac{\partial\Phi}{\Delta}=\frac{\pi}{1-aa}$, quo valore substituto adipiscimur integrationem casus $i=1$, cum enim sit

$$\pi=\frac{(1+aa)\pi}{1-aa}-2a\int\frac{\partial\Phi\cos.\Phi}{\Delta},\text{ erit }\int\frac{\partial\Phi\cos.\Phi}{\Delta}=\frac{\pi a}{1-aa};$$

sicque jam duos casus sumus adepti, qui sunt

$$\int\frac{\partial\Phi}{\Delta}=\frac{\pi}{1-aa}\text{ et }\int\frac{\partial\Phi\cos.\Phi}{\Delta}=\frac{\pi a}{1-aa}.$$

§. 45. Ex his autem duobus casibus $i=0$ et $i=1$ sequentes omnes haud difficulter derivare licet ope hujus lemmatis; cum sit ut vidimus $\int\partial\Phi\cos. i\Phi=0$, erit

$$0=(1+aa)\int\frac{\partial\Phi\cos. i\Phi}{\Delta}-2a\int\frac{\partial\Phi\cos.\Phi\cos. i\Phi}{\Delta}.$$

Constat autem esse

$$2 \cos.\Phi \cos. i\Phi = \cos.(i-1)\Phi + \cos.(i+1)\Phi,$$

unde habebimus hanc aequationem

$$\frac{1+aa}{a}\int\frac{\partial\Phi \cos. i\Phi}{\Delta} = \int\frac{\partial\Phi \cos.(i-1)\Phi}{\Delta} + \int\frac{\partial\Phi \cos.(i+1)\Phi}{\Delta},$$

unde oritur istud lemma

$$\int\frac{\partial\Phi \cos.(i+1)\Phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\Phi \cos. i\Phi}{\Delta} - \int\frac{\partial\Phi \cos.(i-1)\Phi}{\Delta}.$$

Sumto nunc $i=1$, istud lemma nobis suppeditat hunc casum

$$\int\frac{\partial\Phi \cos. 2\Phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\Phi \cos.\Phi}{\Delta} - \int\frac{\partial\Phi}{\Delta},$$

qui ergo per binos praecedentes expeditur; fiet enim

$$\int\frac{\partial\Phi \cos. 2\Phi}{\Delta} = \frac{\pi aa}{1-aa}.$$

Sumatur nunc $i=2$, et lemma nobis dabit

$$\int\frac{\partial\Phi \cos. 3\Phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\Phi \cos. 2\Phi}{\Delta} - \int\frac{\partial\Phi \cos.\Phi}{\Delta}, \text{ sive}$$

$$\int\frac{\partial\Phi \cos. 3\Phi}{\Delta} = \frac{\pi a^3}{1-aa}:$$

simili modo sumto $i=3$, lemma dabit

$$\int\frac{\partial\Phi \cos. 4\Phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\Phi \cos. 3\Phi}{\Delta} - \int\frac{\partial\Phi \cos. 2\Phi}{\Delta}, \text{ sive}$$

$$\int\frac{\partial\Phi \cos. 4\Phi}{\Delta} = \frac{\pi a^4}{1-aa}.$$

Porro casus $i=4$ praebet

$$\int\frac{\partial\Phi \cos. 5\Phi}{\Delta} = \frac{1+aa}{a}\int\frac{\partial\Phi \cos. 4\Phi}{\Delta} - \int\frac{\partial\Phi \cos. 3\Phi}{\Delta}, \text{ sive}$$

$$\int\frac{\partial\Phi \cos. 5\Phi}{\Delta} = \frac{\pi a^5}{1-aa}, \text{ atque ita porro.}$$

§. 46. Hinc igitur patet, singulos istos casus ex binis praecedentibus determinari ope scalae relationis $\frac{1+aa}{a}, -1$, atque seriem recurrentem hinc oriundam abire in geometricam: quodsi enim bini termini postremi jam inventi fuerint

$$\frac{\pi a^\lambda}{1-aa} \text{ et } \frac{\pi a^{\lambda+1}}{1-aa},$$

sequens reperitur $= \frac{\pi a^{\lambda+2}}{1-aa}$, ex quo ergo sine ullo dubio sequitur, pro casu particulari hoc loco tractati in genere fore

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta} = \frac{\pi a^i}{1-aa},$$

ubi autem probe est notandum, loco i non nisi numeros integros positivos assumi debere.

II. De integratione formulae.

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^2} \left[\begin{array}{c} a \Phi = 0 \\ ad \Phi = 180^\circ \end{array}\right].$$

§. 47. Casus simplicissimus hic occurret $\int \frac{\partial \Phi}{\Delta^2}$, cujus ergo integrale ante omnia perscrutari oportet; hunc in finem consideremus hanc formulam finitam $\frac{\sin. \Phi}{\Delta} = V$, quae pro utroque termino $\Phi = 0$ et $\Phi = 180^\circ$ evanescit; hinc autem erit

$$\partial V = \frac{\partial \Phi \cos. \Phi}{\Delta} - \frac{2a \partial \Phi \sin. \Phi^2}{\Delta^2}, \text{ sive}$$

$$\partial V = \frac{(1+aa)\partial \Phi \cos. \Phi - 2a\partial \Phi}{\Delta^2};$$

unde integrando jam novimus esse

$$0 = (1+aa)\int \frac{\partial \Phi \cos. \Phi}{\Delta^2} - 2a \int \frac{\partial \Phi}{\Delta^2}.$$

Porro vero quoniam ante habuimus $\int \frac{\partial \Phi}{\Delta} = \frac{\pi}{1-aa}$, hanc formulam integralem supra et infra per Δ multiplicando, erit quoque

$$\frac{\pi}{1-aa} = (1+aa)\int \frac{\partial \Phi}{\Delta^2} - 2a \int \frac{\partial \Phi \cos. \Phi}{\Delta^2}.$$

Ex praecedente autem colligitur

$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^2} = \frac{2a}{1+aa} \cdot \int \frac{\partial \Phi}{\Delta^2},$$

quo valore substituto habebimus

$$\frac{\pi}{1-aa} = (1+aa)\int \frac{\partial \Phi}{\Delta^2} - \frac{4aa}{1+aa}\int \frac{\partial \Phi}{\Delta^2} = \frac{(1-aa)^2}{1+aa}\int \frac{\partial \Phi}{\Delta^2},$$

quamobrem hinc adipiscimur hanc integrationem principalem

$$\int \frac{\partial \Phi}{\Delta^3} = \frac{\pi(1+aa)}{(1-aa)^3},$$

ex quo immediate deducitur casus sequens

$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^3} = \frac{2\pi a}{(1-aa)^3}.$$

§. 48. Pro sequentibus casibus consideremus integrationem in articulo praecedente inventam

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta} = \frac{\pi a^i}{1-aa},$$

quae formula integralis supra et infra per Δ multiplicando discerpitur in sequentes duas partes

$$\frac{\pi a^i}{1-aa} = (1+aa)\int \frac{\partial \Phi \cos. i \Phi}{\Delta^2} - 2a\int \frac{\partial \Phi \cos. \Phi \cos. i \Phi}{\Delta^2},$$

quae aequatio porro evolvitur in hanc formam

$$\frac{\pi a^i}{1-aa} = (1+aa)\int \frac{\partial \Phi \cos. i \Phi}{\Delta^2} - a\int \frac{\partial \Phi \cos. (i-1)\Phi}{\Delta^2}$$
$$- a\int \frac{\partial \Phi \cos. (i+1)\Phi}{\Delta^2};$$

unde deducitur hoc quasi lemma

$$\int \frac{\partial \Phi \cos. (i+1)\Phi}{\Delta^2} = \frac{1+aa}{a}\int \frac{\partial \Phi \cos. i \Phi}{\Delta^2}$$
$$- \int \frac{\partial \Phi \cos. (i-1)\Phi}{\Delta^2} - \frac{\pi a^{i-1}}{1-aa}.$$

§. 49. Sumamus nunc statim $i=1$, atque istud lemma nobis praebet

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^2} = \frac{1+aa}{a}\int \frac{\partial \Phi \cos. \Phi}{\Delta^2} - \int \frac{\partial \Phi}{\Delta^2} - \frac{\pi}{1-aa};$$

hic jam bini valores jam inventi substituantur, atque reperietur

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^2} = \frac{\pi(1+aa) - \pi(1-aa)^2}{(1-aa)^3},$$

hinc ergo sequitur fore

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^2} = \frac{\pi(3aa - a^4)}{(1-aa)^5} = \frac{\pi aa(3-aa)}{(1-aa)^3}.$$

Sumatur nunc pro lemmate praemisso $i = 2$, eritque

$$\int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} = \frac{1+aa}{a} \int \frac{\partial \Phi \cos. 2\Phi}{\Delta^2} - \int \frac{\partial \Phi \cos. \Phi}{\Delta^2} - \frac{\pi a}{1-aa}, \text{ sive}$$

$$\int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} = \frac{(1+aa)\pi a(3-aa) - 2\pi a - \pi a(1-aa)^2}{(1-aa)^3},$$

quae expressio contrahitur in hanc

$$\int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} = \frac{\pi a^3(4-2aa)}{(1-aa)^3}.$$

Sit nunc in lemmate praemisso $i = 3$, eritque.

$$\int \frac{\partial \Phi \cos. 4\Phi}{\Delta^2} = \frac{1+aa}{a} \int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} - \int \frac{\partial \Phi \cos. 2\Phi}{\Delta^2} - \frac{\pi aa}{1-aa}, \text{ sive}$$

$$\int \frac{\partial \Phi \cos. 4\Phi}{\Delta^2} = \frac{(1+aa)\pi aa(4-2aa) - \pi aa(3-aa) - \pi aa(1-aa)^2}{(1-aa)^3},$$

quae expressio contrahitur in hanc

$$\int \frac{\partial \Phi \cos. 4\Phi}{\Delta^2} = \frac{\pi a^4(5-3aa)}{(1-aa)^3}.$$

Sit nunc in lemmate nostro $i = 4$, eritque

$$\int \frac{\partial \Phi \cos. 5\Phi}{\Delta^2} = \frac{1+aa}{a} \int \frac{\partial \Phi \cos. 4\Phi}{\Delta^2} - \int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} = \frac{\pi a^3}{1-aa}, \text{ sive}$$

$$\int \frac{\partial \Phi \cos. 5\Phi}{\Delta^2} = \frac{(1+aa)\pi a^3(5-3aa) - \pi a^3(4-2aa) - \pi a^3(1-aa)^2}{(1-aa)^3},$$

quae expressio contrahitur in hanc

$$\int \frac{\partial \Phi \cos. 5\Phi}{\Delta^2} = \frac{\pi a^5(6-4aa)}{(1-aa)^3}.$$

Sit nunc in lemmate nostro $i = 5$, eritque

$$\int \frac{\partial \Phi \cos. 3\Phi}{\Delta^2} = \frac{1+aa}{a} \int \frac{\partial \Phi \cos. 5\Phi}{\Delta^2} - \int \frac{\partial \Phi \cos. 4\Phi}{\Delta^2} - \frac{\pi a^4}{1-aa}, \text{ sive}$$

$$\int \frac{\partial \Phi \cos. 6\Phi}{\Delta^2} = \frac{(1+aa)\pi a^4(6-4aa) - \pi a^4(5-3aa) - \pi a^4(1-aa)^2}{(1-aa)^3},$$

quae expressio contrahitur in hanc

$$\int \frac{\partial \Phi \cos. 6\Phi}{\Delta^2} = \frac{\pi a^6(7-5aa)}{(1-aa)^3}.$$

§. 50. Qui has formulas earumque generationem attentius perpendet, nullo certe modo dubitabit, inde hanc conclusionem deducere, quin in genere pro casu hic proposito futurum sit

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^2} = \frac{\pi a^i [i + 1 - (i - 1) a a]}{(1 - a a)^3}$$

cujus lex cum non sit tam manifesta, quam in casu praecedente, omnes formulas inventas junctim ante oculos ponamus

$$\int \frac{\partial \Phi}{\Delta^2} = \frac{\pi (1 + aa)}{(1 - aa)^3}$$

$$\int \frac{\partial \Phi \cos. \Phi}{\Delta^2} = \frac{\pi a (2 - 0 aa)}{(1 - aa)^3}$$

$$\int \frac{\partial \Phi \cos. 2 \Phi}{\Delta^2} = \frac{\pi aa (3 - aa)}{(1 - aa)^3}$$

$$\int \frac{\partial \Phi \cos. 3 \Phi}{\Delta^2} = \frac{\pi a^3 (4 - 2aa)}{(1 - aa)^3}$$

$$\int \frac{\partial \Phi \cos. 4 \Phi}{\Delta^2} = \frac{\pi a^4 (5 - 3aa)}{(1 - aa)^3}$$

$$\int \frac{\partial \Phi \cos. 5 \Phi}{\Delta^2} = \frac{\pi a^5 (6 - 4aa)}{(1 - aa)^3}$$

$$\int \frac{\partial \Phi \cos. 6 \Phi}{\Delta^2} = \frac{\pi a^6 (7 - 5aa)}{(1 - aa)^3}$$

III. De integratione formulae.

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^3} \begin{bmatrix} a\ \Phi = 0 \\ ad\ \Phi = 180 \end{bmatrix}.$$

§. 51. Pro casu simplicissimo $\int \frac{\partial \Phi}{\Delta^3}$ eruendo, utamur hac formula

$$V = \frac{\sin. \Phi}{\Delta^2}, \text{ eritque } \partial V = \frac{\partial \Phi \cos. \Phi}{\Delta^2} - \frac{2 \partial \Phi \sin. \Phi^2}{\Delta^3}, \text{ sive}$$

$$\partial V = \frac{(1 + aa) \partial \Phi \cos. \Phi - 2a \partial \Phi \cos. \Phi^2 - 4a \partial \Phi \sin. \Phi^2}{\Delta^3}.$$

Hic loco $\sin. \Phi^2$ scribatur $1 - \cos. \Phi^2$, atque integrando, ob $V = 0$ habebimus hanc aequationem

$$0 = (1 + a a) \int \frac{\partial \Phi \cos. \Phi}{\Delta^3} - 4 a \int \frac{\partial \Phi}{\Delta^3} + 2 a \int \frac{\partial \Phi \cos. \Phi^2}{\Delta^3}.$$

§. 52. Huc addamus hanc formam indefinitam

$$s = A\int\frac{\partial\Phi}{\Delta} + B\int\frac{\partial\Phi}{\Delta^2}$$

cujus differentiale ad denominationem Δ^3 perducatur, litterae vero A et B ita definiantur, ut membra $\partial\Phi\cos.\Phi$ et $\partial\Phi\cos.\Phi^2$ evanescant, eritque formulis differentialibus additis

$$\begin{aligned}\frac{\Delta^3(\partial V+\partial s)}{\partial\Phi} = &-4a \qquad\quad + (1+aa)\cos.\Phi \qquad + 2a\cos.\Phi^2\\ &+ A(1+aa)^2 - 4Aa(1+aa)\cos.\Phi + 4Aaa\cos.\Phi^2\\ &+ B(1+aa) - 2Ba\cos.\Phi.\end{aligned}$$

Nunc igitur ut termini $\cos.\Phi^2$ abigantur, statuatur

$$2a + 4Aaa = 0, \text{ ideoque } A = \frac{-1}{2a}.$$

Nunc etiam termini $\cos.\Phi$ e medio tollantur, eritque

$$1 + aa - 4Aa(1+aa) - 2Ba = 0, \text{ unde fit}$$

$$B = \frac{3(1+aa)}{2a}.$$

Ex quibus valoribus nanciscimur

$$\frac{\Delta^3(\partial V+\partial s)}{\partial\Phi} = \frac{(1-aa)^2}{a};$$

hinc ergo vicissim integrando habebimus

$$V + s = \frac{(1-aa)^2}{a}\int\frac{\partial\Phi}{\Delta^3}.$$

§. 53. Cum igitur, ut jam notavimus, sit $V = 0$, atque ex casibus jam tractatis

$$s = \frac{-1}{2a}\cdot\frac{\pi}{1-aa} + \frac{3(1+aa)}{2a}\cdot\frac{\pi(1+aa)}{(1-aa)^3},$$

habebimus hanc aequationem

$$\frac{(1-aa)^2}{a}\int\frac{\partial\Phi}{\Delta^3} = \frac{3\pi(1+aa)^2 - \pi(1-aa)^2}{2a(1-aa)^3},$$

unde colligitur

$$\int\frac{\partial\Phi}{\Delta^3} = \frac{\pi(1+4aa+a^4)}{(1-aa)^5}.$$

§. 54. Cum sit $\int\frac{\partial\Phi}{\Delta^2}=\frac{\pi(1+aa)}{(1-aa)^3}$, erit per reductionem hactenus usitatam

$$\frac{\pi(1+aa)}{(1-aa)^3}=(1+aa)\int\frac{\partial\Phi}{\Delta^3}-2a\int\frac{\partial\Phi\cos.\Phi}{\Delta^3},$$

unde concludimus

$$\int\frac{\partial\Phi\cos.\Phi}{\Delta^3}=\frac{1+aa}{2a}\int\frac{\partial\Phi}{\Delta^3}-\frac{\pi(1+aa)}{2a(1-aa)^3}, \text{ ideoque}$$

$$\int\frac{\partial\Phi\cos.\Phi}{\Delta^3}=\frac{1+aa}{2a}\cdot\frac{\pi(1+4aa+a^4)}{(1-aa)^5}-\frac{\pi(1+aa)}{2a(1-aa)^3}$$

$$=\frac{3\pi a(1+aa)}{(1-aa)^5}=\frac{\pi a(3+3aa)}{(1-aa)^5}.$$

§. 55. Cum igitur in articulo praecedente invenimus

$$\int\frac{\partial\Phi\cos.i\Phi}{\Delta^2}=\frac{\pi a^i[i+1-(i-1)aa]}{(1-aa)^3},$$

hanc formulam integralem supra et infra per Δ multiplicando habebimus

$$\frac{\pi a^i[i+1-(i-1)aa]}{(1-aa)^3}=(1+aa)\int\frac{\partial\Phi\cos.i\Phi}{\Delta^3}$$

$$-2a\int\frac{\partial\Phi\cos.i\Phi\cos.\Phi}{\Delta^3}, \text{ sive}$$

$$\frac{\pi a^i[i+1-(i-1)aa]}{(1-aa)^3}=(1+aa)\int\frac{\partial\Phi\cos.i\Phi}{\Delta^3}$$

$$-a\int\frac{\partial\Phi\cos.(i-1)\Phi}{\Delta^3}-a\int\frac{\partial\Phi\cos.(i+1)\Phi}{\Delta^3};$$

unde deducitur hoc quasi lemma

$$\int\frac{\partial\Phi\cos.(i+1)\Phi}{\Delta^3}=\frac{1+aa}{a}\int\frac{\partial\Phi\cos.i\Phi}{\Delta^3}$$

$$-\int\frac{\partial\Phi\cos.(i-1)\Phi}{\Delta^3}-\frac{\pi a^{i-1}[i+1-(i-1)aa]}{(1-aa)^3}$$

§. 56. Sumamus nunc statim $i = 1$, atque istud lemma nobis praebet

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^3} = \frac{1+aa}{2a} \int \frac{\partial \Phi \cos. \Phi}{\Delta^3} - \int \frac{\partial \Phi}{\Delta^3} - \frac{2\pi}{2(1-aa)^3};$$

hic jam bini valores jam inventi substituantur, reperieturque

$$\int \frac{\partial \Phi \cos. 2\Phi}{\Delta^3} = \frac{1+aa}{a} \cdot \frac{\pi a(3+3aa)}{(1-aa)^5} - \frac{\pi(1+4aa+a^4)}{(1-aa)^5}$$
$$- \frac{\pi(1-aa)^2}{(1-aa)^5} = \frac{\pi aa(6)}{(1-aa)^5};$$

sumto $i = 2$, erit

$$\int \frac{\partial \Phi \cos. 3\Phi}{\Delta^3} = \frac{\pi a^3(10-5aa+a^4)}{(1-aa)^5};$$

sumto $i = 3$, nanciscimur

$$\int \frac{\partial \Phi \cos. 4\Phi}{\Delta^3} = \frac{\pi a^4(15-12aa+3a^4)}{(1-aa)^5};$$

sumto $i = 4$, prodit

$$\int \frac{\partial \Phi \cos. 5\Phi}{\Delta^3} = \frac{\pi a^5(21-21aa+6a^4)}{(1-aa)^5};$$

posito $i = 5$, erit

$$\int \frac{\partial \Phi \cos. 6\Phi}{\Delta^3} = \frac{\pi a^6(28-32aa+10a^4)}{(1-aa)^5};$$

et in genere

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta^3} = \pi a^i \left[\frac{i(i+3)+2}{2} - 2(ii-4)aa + \left[\frac{i(i-3)+2}{2}\right] a^4\right],$$

quae forma facile transformatur in hanc

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta^3} = \frac{\pi a^i}{(1-aa)^5} \left[\frac{(i+1)(i+2)}{2} - (i+2)(i-2)aa + \frac{(i-1)(i-2)}{2} a^4\right].$$

§. 57. Hoc modo procedere liceret ad sequentes formulas, in quibus denominator est Δ^4, Δ^5, Δ^6, etc. verum integralium formae ita continuo magis fierent complicatae, ut vix ullus ordo in iis observari posset, quamobrem aliam viam inire conveniet, qua numerum i pro dato assumimus, et continuo a minoribus ad majores

numeros n procedemus. Primo igitur sumamus $i = 0$, et investigemus valorem integralem formulae $\int \frac{\partial \Phi}{\Delta^{n+1}}$.

Integratio formulae.

$$\int \frac{\partial \Phi}{\Delta^{n+1}} \left[\begin{array}{l} a\,\Phi = 0 \\ ad\,\Phi = 180 \end{array}\right]$$

existente $\Delta = 1 + aa - 2\,a \cos. \Phi$.

§. 58. Ex praecedentibus colligere licet, quemlibet casum exponentis $n + 1$ a duobus praecedentibus pendere, ita ut sit sub terminis integrationis praescriptis

$$\int \frac{\partial \Phi}{\Delta^{n+1}} = \alpha \int \frac{\partial \Phi}{\Delta^{n}} + \beta \int \frac{\partial \Phi}{\Delta^{n-1}};$$

ubi totum negotium eo redit, ut coefficientes α et β rite determinentur: hunc in finem statuamus in genere esse

$$\int \frac{\partial \Phi}{\Delta^{n+1}} = \alpha \int \frac{\partial \Phi}{\Delta^{n}} + \beta \int \frac{\partial \Phi}{\Delta^{n-1}} + \gamma \frac{\sin. \Phi}{\Delta^{n}},$$

quippe qui postremus terminus pro utroque integrationis termino evanescit.

§. 59. Differentietur nunc ista aequatio, et facta divisione per $\partial \Phi$, orietur sequens aequatio

$$\frac{1}{\Delta^{n+1}} = \frac{\alpha}{\Delta^{n}} + \frac{\beta}{\Delta^{n-1}} + \frac{\gamma \cos. \Phi (1 + aa - 2 a \cos. \Phi) - 2 \gamma\, a n \sin. \Phi^2}{\Delta^{n+1}},$$

haecque aequatio multiplicata per Δ^{n+1} abibit in hanc formam

$$1 = \alpha (1 + aa - 2 a \cos. \Phi) + \beta (1 + aa)^2 - 2 \beta\, a \cos. \Phi (1 + aa)$$
$$+ 4 \beta\, a a \cos. \Phi^2 + \gamma \cos. \Phi (1 + aa - 2 a \cos. \Phi) - 2 \gamma\, a n \sin. \Phi^2.$$

Cum nunc sit

$$2 \cos. \Phi^2 = 1 + \cos. 2 \Phi \text{ et } 2 \sin. \Phi^2 = 1 - \cos. 2 \Phi,$$

hac reductione adhibita pervenietur ad sequentem aequationem

$$\begin{array}{lll} 1 = \alpha(1+aa) & - 2\alpha a \cos.\Phi & + 2\beta a a \cos. 2\Phi \\ +\beta(1+aa)^2 & - 4\beta a(1+aa)\cos.\Phi & - \gamma a \cos. 2\Phi \\ + 2\beta a a & + \gamma(1+aa)\cos.\Phi & + \gamma n a \cos. 2\Phi \\ - \gamma a & & \\ - \gamma n a. & & \end{array}$$

§. 60. Ut nunc hanc aequationem resolvamus, necesse est, ut tam termini involventes cos. Φ, quam cos. $2\,\Phi$, seorsim ad nihilum redigantur; unde ex postremo termino deducimus

$$2\beta a a - \gamma a + \gamma n a = 0;$$

ideoque

$$\beta = \frac{\gamma(1-n)}{2a} = -\frac{\gamma(n-1)}{2a},$$

qui valor in terminis cos. Φ affectis substitutus perducit ad hanc aequationem

$$-2\alpha a + 2\gamma(n-1)(1+aa) + \gamma(1+aa) = 0,$$

unde fit

$$2\alpha a = 2\gamma n(1+aa) - \gamma(1+aa);$$

ideoque erit

$$\alpha = \frac{\gamma(1+aa)(2n-1)}{2a}.$$

Jam hic valores loco α et β inventi substituantur in prima parte, atque deducemur ad hanc aequationem

$$1 = \frac{\gamma n(1+aa)^2}{a} - \frac{\gamma(n-1)(1+aa)^2}{2a} - \gamma a(n-1) - \gamma a - \gamma n a, \text{ sive}$$

$$2a = 2\gamma n(1+aa)^2 - \gamma(n-1)(1+aa)^2 - 2\gamma aa(n-1) - 2\gamma aa - 2\gamma n aa,$$

$$\text{vel } 2a = \gamma(n+1)(1+aa)^2 - 4\gamma n aa,$$

unde fit

$$\gamma = \frac{2a}{n(1-aa)^2}.$$

§. 61. Invento jam isto valore γ, hinc eliciemus

$$\alpha = \frac{(2n-1)(1+aa)}{n(1-aa)^2} \text{ et } \beta = \frac{-(n-1)}{n(1-aa)^2},$$

hincque per $n(1-aa)^2$ multiplicando, adipiscimur

$$n(1-aa)^2 \int \frac{\partial \Phi}{\Delta^{n+1}} = (2n-1)(1+aa) \int \frac{\partial \Phi}{\Delta^n} - (n-1) \int \frac{\partial \Phi}{\Delta^{n-1}},$$

cujus beneficio ex cognitis jam duobus casibus assignari poterit casus sequens.

§. 62. Jam ante autem invenimus esse $\int \frac{\partial \Phi}{\Delta} = \frac{\pi}{1-aa}$. Pro sequentibus vero ponamus

$$\int \frac{\partial \Phi}{\Delta^2} = \frac{\pi A}{(1-aa)^3}; \quad \int \frac{\partial \Phi}{\Delta^3} = \frac{\pi B}{(1-aa)^5}; \quad \int \frac{\partial \Phi}{\Delta^4} = \frac{\pi C}{(1-aa)^7};$$

$$\int \frac{\partial \Phi}{\Delta^5} = \frac{\pi D}{(1-aa)^9}; \quad \int \frac{\partial \Phi}{\Delta^6} = \frac{\pi E}{(1-aa)^{11}}; \text{ etc.}$$

Ubi jam ante invenimus $A = 1 + aa$ et $B = 1 + 4aa + a^4$, unde sequentes valores omnes C, D, E, etc. ope reductionis inventae definiri poterunt.

§. 63. Introducamus ergo istos valores, atque sequentes nanciscemur aequationes

$$\begin{array}{rl}
\text{I.} & A = 1 + aa, \\
\text{II.} & 2B = 3(1+aa)A - (1-aa)^2, \\
\text{III.} & 3C = 5(1+aa)B - 2(1-aa)^2 . A, \\
\text{IV.} & 4D = 7(1+aa)C - 3(1-aa)^2 B, \\
\text{V.} & 5E = 9(1+aa)D - 4(1-aa)^2 C, \\
\text{VI.} & 6F = 11(1+aa)E - 5(1-aa)^2 D, \\
\text{VII.} & 7G = 13(1+aa)F - 6(1-aa)^2 E, \\
\text{VIII.} & 8H = 15(1+aa)G - 7(1-aa)^2 F,
\end{array}$$

etc.

§. 64 Harum aequationum prima statim dat valorem ante inventum $A = 1 + aa$; secunda vero praebet

$$2B = \begin{cases} 3 + 6aa + 3a^4 \\ -1 + 2aa + a^4 \end{cases}$$

unde fit

$$B = 1 + 4aa + a^4.$$

Deinde vero tertia aequatio praebet

$$3C = \begin{cases} 5 + 25aa + 25a^4 + 5a^6 \\ -2 + 2aa + 2a^4 - 2a^6 \end{cases}$$

unde elicitur

$$C = 1 + 9aa + 9a^4 + a^6.$$

Porro quarta aequatio

$$4D = \begin{cases} 7 + 70aa + 126a^4 + 70a^6 + 7a^8 \\ -3 - 6aa + 18a^4 - 6a^6 - 3a^8 \end{cases}$$

unde colligitur

$$D = 1 + 16aa + 36a^4 + 16a^6 + a^8.$$

Simili modo ex aequatione quinta colligimus

$$5E = \begin{cases} 9 + 153aa + 468a^4 + 468a^6 + 153a^8 + 9a^{10} \\ -4 - 28aa + 32a^4 + 32a^6 - 28a^8 - 4a^{10} \end{cases}$$

unde colligitur

$$E = 1 + 25aa + 100a^4 + 100a^6 + 25a^4 + a^{10}.$$

Evolvamus etiam sextam aequationem quae praebet

$$6F = \begin{cases} 11 + 286aa + 1375a^4 + 2200a^6 + 1375a^8 + 286a^{10} + 11a^{12} \\ -5 - 70aa - 25a^4 + 200a^6 - 25a^8 - 70a^{10} - 5a^{12}, \end{cases}$$

hincque concluditur

$$F = 1 + 36aa + 225a^4 + 400a^6 + 225a^5 + 36a^{10} + a^{12}$$

§. 65. Hic non sine admiratione deprehendimus, omnes coefficientes harum formarum esse numeros quadratos, quorum radices occurrunt in potestatibus respondentibus binomii $1+aa$, sicque pro littera sequente habebimus

$$G = 1 + 7^2 aa + 21^2 a^4 + 35^2 a^6 + 35^2 a^8 + 21^2 a^{10} + 7^2 a^{12} + a^{14},$$

quae littera respondet formulae integrali $\int \frac{\partial \Phi}{\Delta^{7+1}}$, ita ut hic sit $n = 7$. Quodsi ergo formae generalis $\int \frac{\partial \Phi}{\Delta^{n+1}}$ integrale statuamus $= \frac{\pi V}{(1-aa)^{in+1}}$, erit valor litterae

$$V = 1 + \left(\frac{n}{1}\right)^2 aa + \left(\frac{n}{2}\right)^2 a^4 + \left(\frac{n}{3}\right)^2 a^6 + \left(\frac{n}{4}\right)^2 a^8 + \left(\frac{n}{5}\right)^2 a^{10} + \text{etc.}$$

adhibitis scilicet characteribus, quibus coefficientes potestatum binomii designare consuevimus, dum scilicet est

$$\left(\frac{n}{1}\right) = n;\ \left(\frac{n}{2}\right) = \frac{n}{1}\cdot\frac{n-1}{2};\ \left(\frac{n}{3}\right) = \frac{n}{1}\cdot\frac{n-1}{2}\cdot\frac{n-2}{3} \text{ etc.}$$

§. 66. Haec quidem conclusio tantum per inductionem quasi conjectura est deducta; vix enim quisquam reperietur, cui ista conjectura suspecta videatur, quamquam rigorosa demonstratione nondum sit corroborata; casu enim fortuito neutiquam evenire certe potest, ut omnes istos coefficientes prodierint numeri quadrati, atque adeo ipsorum coefficientium qui in evolutione potestatis $(1+aa)^n$ occurrunt, interim tamen deinceps vidi pro hac veritate solidam demonstrationem adornari posse.

§. 67. Hac igitur lege stabilita, valores litterarum A, B, C, D etc., quas in expressiones integralium induximus, sequenti modo se habebunt

$$A = 1^2 + 1^2 aa,$$
$$B = 1^2 + 2^2 aa + 1^2 a^4,$$
$$C = 1^2 + 3^2 aa + 3^2 a^4 + 1^2 a^6,$$

$$D = 1^2 + 4^2 aa + 6^2 a^4 + 4^2 a^6 + 1^2 a^8,$$
$$E = 1^2 + 5^2 aa + 10^2 a^4 + 10^2 a^6 + 5^2 a^8 + 1^2 a^8,$$
$$F = 1^2 + 6^2 aa + 15^2 a^4 + 20^2 a^6 + 15^2 a^8 + 6^2 a^{10} + 1^2 a^{12},$$
$$G = 1^2 + 7^2 aa + 21^2 a^4 + 35^2 a^6 + 35^2 a^8 + 21^2 a^{10} + 7^2 a^{12} + 1^2 a 14,$$
etc. etc.

Integratio formulae generalis

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^{n+1}} \left[\begin{array}{l} \text{a } \Phi = 0 \\ \text{ad } \Phi = 180 \end{array} \right]$$

existente $\Delta = 1 + aa - 2a \cos. \Phi$.

§. 68. Haec formula generalis perinde tractari potest ac praecedens, dum valor integralis cujusque casus etiam a duobus casibus praecedentibus pendet, ita ut ponere queamus

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^{n+1}} = \alpha \int \frac{\partial \Phi \cos. i \Phi}{\Delta^n} + \beta \int \frac{\partial \Phi \cos. i \Phi}{\Delta^{n-1}},$$

quatenus scilicet integralia ad binos terminos integrationis stabilitos referuntur; quia autem necesse est, ut aequationem generalem ob ista conditione liberam constituamus, aliquot membra adjungi oportet, quae pro utroque termino evanescant, neque enim hic sufficit, ut ante unicum terminum adjunxisse, verum adeo ternos hujusmodi terminos adjungi debebunt, cujus ratio mox ex ipso calculo elucebit; hanc ob rem constituamus sequentem aequationem

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^{n+1}} = \alpha \int \frac{\partial \Phi \cos. i \Phi}{\Delta^n} + \beta \int \frac{\partial \Phi \cos. i \Phi}{\Delta^{n-1}}$$
$$+ \gamma \frac{\sin i \Phi}{\Delta^n} + \delta \frac{\sin. (i-1) \Phi}{\Delta^n} + \varepsilon \frac{\sin. (i+1) \Phi}{\Delta^n},$$

quae postrema membra, quoniam i est numerus integer, pro utroque termino integrationis evanescunt.

§. 69. Differentietur igitur nunc ista aequatio, ac posito brevitatis gratia $1 + aa = b$, ut sit $\Delta = b - 2a \cos. \Phi$, negligantur denominatores, qui erunt Δ^{n+1} una cum elemento $\partial \Phi$. Primo notetur esse

$$\Delta \cos. i\Phi = b \cos. i\Phi - a \cos. (i-1)\Phi - a \cos. (i+1)\Phi,$$

tum vero ob

$$\Delta^2 = bb - 4ab \cos. \Phi + 4aa \cos. \Phi^2 = 2aa + bb - 4ab \cos. \Phi + 2aa \cos. 2\Phi, \text{ erit}$$

$$\Delta^2 \cos. i\Phi = (bb + 2aa) \cos. i\Phi - 2ab \cos. (i-1)\Phi - 2ab \cos. (i+1)\Phi + aa \cos. (i-2)\Phi + aa \cos. (i+2)\Phi.$$

Deinde vero habebitur

$$\partial . \frac{\sin. i\Phi}{\Delta^n} = i\Delta \cos. i\Phi - 2na \sin. i\Phi \sin. \Phi = ib \cos. i\Phi + ia \cos. (i-1)\Phi - ia \cos. (i+1)\Phi - na \cos. (i-1)\Phi + na \cos. (i+1)\Phi.$$

Simili modo erit

$$\partial . \frac{\sin. (i-1)\Phi}{\Delta^n} = (i-1) b \cos. (i-1)\Phi - (i-1) a \cos. (i-2)\Phi - (i-1) a \cos. i\Phi - na \cos. (i-2)\Phi + na \cos. i\Phi,$$

ac denique

$$\partial . \frac{\sin. (i+1)\Phi}{\Delta^n} = (i+1) b \cos. (i+1)\Phi - (i+1) a \cos. i\Phi - (i+1) a \cos. (i+2)\Phi - na \cos. i\Phi + na \cos. (i+2)\Phi.$$

§. 70. Hic igitur occurrunt quinque anguli scilicet

$$i\Phi, \; (i-1)\Phi, \; (i+1)\Phi, \; (i-2)\Phi \text{ et } (i+2)\Phi,$$

unde patet ratio, cur terni termini absoluti sint supra adjuncti; diffe-

erentiale ergo facta evolutione singulorum terminorum, per quinque columnas sequenti modo repraesentetur, ita ut membrum sinistrum, quod est cos. $i\,\Phi$, aequetur sequenti expressioni

cos. $i\,\Phi$	cos. $(i-1)\,\Phi$	cos. $(i+1)\,\Phi$	cos. $(i-2)\,\Phi$	cos. $(i+2)\,\Phi$
$+ab$	$-aa$	$-aa$		
$+\beta(bb+2aa)$	$-2\beta ab$	$-2\beta ab$	$+\beta aa$	$+\beta aa$
$+\gamma ib$	$-\gamma ia$	$-\gamma ia$		
	$-\gamma na$	$+\gamma na$		
$-\delta(i-1)a$	$+\delta(i-1)b$	$+\varepsilon(i+1)b$	$-\delta(i-1)a$	$-\varepsilon(i+1)a$
$+\delta na$			$-\delta na$	$+\varepsilon na$
$-\varepsilon(i+1)a$				
$-\varepsilon na$				

§. 71. Hic igitur omnes quatuor posteriores columnae ad nihilum redigi debent, propterea quod sola prima columna membro sinistro aequari potest; incipiamus igitur a binis columnis ultimis, unde deducimus

$$\delta = \frac{\beta a}{i+n-1} \text{ et } \varepsilon = \frac{\beta a}{i-n+1}.$$

His valoribus introductis, pro secunda columna erit

$$-2\beta ab + \delta(i-1)b = \frac{\beta ab(1-i-2n)}{i+n-1} = -\frac{\beta ab(i+2n-1)}{i+n-1}.$$

Pro tertia vero columna erit

$$-2\beta ab + \varepsilon(i+1)b = -\frac{\beta ab(i-2n+1)}{i-n+1};$$

unde haec binae columnae nobis praebent has duas aequationes

$$-aa - \gamma(i+n)a - \frac{\beta ab(i+2n-1)}{i+n-1} = 0,$$

$$-aa - \gamma(i-n)a - \frac{\beta ab(i-2n+1)}{i-n+1} = 0$$

§. 72. Harum duarum aequationum subtrahatur posterior a priore, ac prodibit

$$-2\gamma na - \frac{2\beta inab}{ii-(n-1)^2} = 0,$$

unde colligimus

$$\gamma = -\frac{\beta i b}{ii-(n-1)^2}.$$

Atque hinc porro ex secunda deduci potest valor ipsius α, cum sit

$$\alpha a = -\gamma(i+n)a - \frac{\beta a b(i+2n-1)}{i+n-1},$$

erit

$$\alpha = \frac{\beta i(i+n)b}{ii-(n-1)^2} - \frac{\beta(i+2n-1)b}{i+n-1} = \frac{\beta(2nn-3n+1)b}{ii-(n-1)^2}$$
$$= \frac{\beta(n-1)(2n-1)b}{ii-(n-1)^2}.$$

§. 73. Hi jam valores substituantur in prima columna, atque orietur sequens aequatio

$$\left.\begin{array}{ll} \frac{\beta(n-1)(2n-1)bb}{ii-(n-1)^2} & +2\beta aa \\ +\beta bb & -\frac{\beta(i-n-1)aa}{i+n-1} \\ -\frac{\beta iibb}{ii-(n-1)^2} & -\frac{\beta(i+n+1)aa}{i-n+1} \end{array}\right\} = 1.$$

Multiplicando igitur per $ii-(n-1)^2$, prodibit haec aequatio

$$ii-(n-1)^2 = 2\beta aa[ii-(n-1)^2] + \beta bb(n-1)(2n-1)$$
$$-\beta aa(i-n-1)(i-n+1) + \beta bb[ii-(n-1)^2]$$
$$-\beta aa(i+n+1)(i+n-1) - \beta iibb.$$

Facta autem reductione, terminus βaa multiplicabitur per

$$2[ii-(n-1)^2] - (i-n)^2 + 1 - (i+n)^2 + 1,$$

sive per $-4n(n-1)$; at vero βbb multiplicabitur per

$$(n-1)(2n-1) + ii - (n-1)^2 - ii,$$

sive per $n(n-1)$, sicque erit

$$ii(n-1)^2 = -4\beta n(n-1)aa + \beta n(n-1)bb$$
$$= \beta n(n-1)(bb-4aa).$$

Cum igitur posuerimus $b = 1 + aa$, erit

$$bb - 4aa = (1 - aa)^2,$$

consequenter hinc elicimus

$$\beta = \frac{ii - (n-1)^2}{n(n-1)(1-aa)^2}.$$

§. 74. Invento jam valore litterae β, ex eo deducimus valorem $\alpha = \frac{(2n-1)b}{n(1-aa)^2}$: valores autem litterarum γ, δ, et ε non amplius in censum veniunt, et reductio quam quaerimus erit

$$\int \frac{\partial\Phi \cos. i\Phi}{\Delta^{n+1}} = \alpha \int \frac{\partial\Phi \cos. \Phi}{\Delta^n} + \beta \int \frac{\partial\Phi \cos. i\Phi}{\Delta^{n-1}},$$

sive sublatis fractionibus habebitur ista aequatio

$$n(n-1)(1-aa)^2 \int \frac{\partial\Phi \cos. i\Phi}{\Delta^{n+1}} = (n-1)(2n-1)(1+aa) \int \frac{\partial\Phi \cos. i\Phi}{\Delta^n}$$
$$+ [ii - (n-1)^2] \int \frac{\partial\Phi \cos. i\Phi}{\Delta^{n-1}},$$

quae aequatio casu $i = 0$ redit ad reductionem praecedentis sectionis.

§. 75. Inventa hac reductione generali, pro ejus applicatione cum sit

$$\int \frac{\partial\Phi \cos. i\Phi}{\Delta} = \frac{\pi a^i}{1-aa}, \text{ ubi } n = 0,$$

ponamus pro sequentibus

$$\int \frac{\partial\Phi \cos. i\Phi}{\Delta^2} = \frac{\pi a^i}{(1-aa)^3} \text{ A, ubi } n = 1$$

$$\int \frac{\partial\Phi \cos. i\Phi}{\Delta^3} = \frac{\pi a^i}{(1-aa)^5} \text{ B, ubi } n = 2$$

$$\int \frac{\partial\Phi \cos. i\Phi}{\Delta^4} = \frac{\pi a^i}{(1-aa)^7} \text{ C, ubi } n = 3$$

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^5} = \frac{\pi a^i}{(1-aa)^9} \text{ D, ubi } n = 4$$

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^6} = \frac{\pi a^i}{(1-aa)^{11}} \text{ E, ubi } n = 5,$$

atque adeo in genere sit

$$\int \frac{\partial \Phi \cos. i \Phi}{\Delta^{n+1}} = \frac{\pi a^i}{(1-aa)^{2n+1}} \text{ V:}$$

supra autem jam i invenimus esse

$$A = i + 1 - (i - 1) a a,$$

sive terminos positive repraesentando

$$A = 1 + i + (1 - i) a a.$$

§. 76. Quodsi in reductione nostra inventa poneremus $n = 1$, ea daret $i i \int \partial \Phi \cos. i \Phi = 0$, quod primo verum est casu $i = 0$, tum vero ob $\int \partial \Phi \cos. i \Phi = \frac{1}{i} \sin. i \Phi = 0$, quod quidem per se patet. Incipiamus igitur a casu $n = 2$, et procedendo per sequentes valores $n = 3$, $n = 4$, $n = 5$, etc. nanciscemur sequentes aequationes

I. Si $n = 2$, erit
$$2 . 1 \text{ B} = 1 . 3 (1 + aa) \text{ A} + (ii - 1)(1 - aa)^2.$$

II. Si $n = 3$, erit
$$3 . 2 \text{ C} = 2 . 5 (1 + aa) \text{ B} + (ii - 4)(1 - aa)^2 \text{ A}.$$

III. Si $n = 4$, erit
$$4 . 3 \text{ D} = 3 . 7 (1 + aa) \text{ C} + (ii - 9)(1 - aa)^2 \text{ B}.$$

IV. Si $n = 5$, erit
$$5 . 4 \text{ E} = 4 . 9 (1 + aa) \text{ D} + (ii - 16)(1 - aa)^2 \text{ C}.$$

V. Si $n = 6$, erit
$$6 . 5 \text{ F} = 5 . 11 (1 + aa) \text{ E} + (ii - 25)(1 - aa)^2 \text{ D}.$$

etc. etc.

§. 77. Cum igitur sit

$$A = 1 + i + (1 - i)\,a\,a,$$

pro prima aequatione erit

$$(1 + a\,a)\,A = 1 + i + 2\,a\,a + (1 - i)\,a^4,$$

hujus triplo addi oportet

$$(i\,i - 1)\,(1 - a\,a)^2 = i\,i - 1 - 2\,(i\,i - 1)\,a\,a + (i\,i - 1)\,a^4,$$

unde oritur primo terminus absolutus $= (2 + i)\,(1 + i)$, deinde coefficiens ipsius $a\,a$ erit $8 - 2\,i\,i$, coefficiens vero ipsius a^4 erit $(2 - i)\,(1 - i)$, unde concludimus litteram

$$B = \frac{(2+i)}{1.}\frac{(1+i)}{2} + (2+i)\,(2-i)\,a\,a + \frac{(2-i)}{1.}\frac{(1-i)}{2}\,a^4.$$

§. 78. Ista forma nos manuducit ad coefficientes potestatum binomii, quos ut jam moninus per characteres peculiares repraesentamus, sicque per tales characteres erit

$$A = \left(\frac{1+i}{1}\right) + \left(\frac{1-i}{1}\right)a\,a, \text{ tum vero}$$

$$B = \left(\frac{2+i}{2}\right) + \left(\frac{2+i}{1}\right)\left(\frac{2-i}{1}\right)a\,a + \left(\frac{2-i}{2}\right)a^4$$

Videamus autem, quomodo haec lex in sequentibus valoribus se sit habitura.

§. 79. Evolvamus igitur aequationem secundam, pro qua sequentes duas multiplicationes institui oportet

$$10\left[\frac{2+3\,i+i\,i}{2} + (4 - i\,i)\,a\,a + \frac{2-3\,i+i\,i}{2}\,a^4\right] \text{ per } 1 + a\,a,$$

ultimum autem membrum postulat hanc multiplicationem

$$(i\,i - 4)\,(1 - 2\,a\,a + a^4) \text{ per } 1 + i + (1 - i)\,a\,a;$$

unde primo oritur iste terminus absolutus

$$10 + 15\,i + 5\,i\,i + (i\,i - 4)\,(1 + i),$$

quae reducitur ad hanc formam $(2 + i)\,(1 + i)\,(3 + i)$. Pro termino autem $a\,a$ erit

$$40 - 10\,ii + 5\,(2+i)\,(1+i) + (ii-4)\,[-2\,(1+i) + 1 - i]$$
$$= (4 - ii)\,(11 + 3\,i) + 5\,(2+i)\,(1+i),$$

quae expressio reducitur ad

$$(2+i)\,(27 - 3\,ii) = 3\,(2+i)\,(3+i)\,(3-i).$$

Porro coefficiens ipsius a^4 erit

$$(2-i)\,(27 - 3\,ii) = 3\,(2-i)\,(3+i)\,(3-i).$$

Denique coefficiens ipsius a^6 erit $(2-i)\,(1-i)\,(3-i)$.

§. 80. Calculo ergo hoc peracto habebimus

$$3.2\,C = (3+i)\,(2+i)\,(1+i) + 3\,(3+i)\,(2+i)\,(3-i)\,aa$$
$$+ 3\,(3+i)\,(2-i)\,(3-i)\,a^4 + (3-i)\,(2-i)\,(1-i)\,a^6,$$

quae forma commode redigitur ad istam per characteres coefficientium binomii

$$C = \binom{3+i}{3} + \binom{3+i}{2}\binom{3-i}{1}aa + \binom{3+i}{1}\binom{3-i}{2}a^4 + \binom{3-i}{3}a^6.$$

Hic ordo maxime confirmat conjecturam ex casibus praecedentibus deductam, neque dubium ullum esse potest, quin sequentes litterae istos sortiantur valores

$$D = \binom{4+i}{4} + \binom{4+i}{3}\binom{4-i}{1}aa + \binom{4+i}{2}\binom{4-i}{2}a^4$$
$$+ \binom{4+i}{1}\binom{4-i}{3}a^6 + \binom{4-i}{4}a^8.$$
$$E = \binom{5+i}{5} + \binom{5+i}{4}\binom{5-i}{1}aa + \binom{5+i}{3}\binom{5-i}{2}a^4$$
$$+ \binom{5+i}{2}\binom{5-i}{3}a^6 + \binom{5+i}{1}\binom{5-i}{4}a^8 + \binom{5-i}{5}a^{10}.$$

etc. etc.

Interim tamen fatendum est, hunc ordinem egregium tantum per conjecturam se nobis obtulisse; cujus ergo demonstratio rigorosa adhuc desideratur.

§. 81. Cum igitur supra ingenere posuerimus

$$\int \frac{\partial \Phi \cos. i\Phi}{\Delta^{n+1}} \left[\begin{array}{l} \text{a } \Phi = 0 \\ \text{ad } \Phi = 180 \end{array}\right] = \frac{\pi a^i}{(1-aa)^{2n+1}} V,$$

erit nunc

$$V = \left(\frac{n+i}{n}\right) + \left(\frac{n+i}{n-1}\right)\left(\frac{n-i}{1}\right) aa + \left(\frac{n+i}{n-2}\right)\left(\frac{n-i}{2}\right) a^4$$
$$+ \left(\frac{n+i}{n-3}\right)\left(\frac{n-i}{3}\right) a^6 + \left(\frac{n+i}{n-4}\right)\left(\frac{n-i}{4}\right) a^8 + \text{etc.}$$

unde sponte deducitur forma in articulo praecedenti conclusa, ubi erat $i = 0$. Pro hoc enim casu erit

$$V = \left(\frac{n}{n}\right) + \left(\frac{n}{n-1}\right)\left(\frac{n}{1}\right) aa + \left(\frac{n}{n-2}\right)\left(\frac{n}{2}\right) a^4 + \left(\frac{n}{n-3}\right)\left(\frac{n}{3}\right) a^6 + \text{etc.}$$

Cum autem in hujusmodi characteribus perpetuo sit $\left(\frac{n}{p}\right) = \left(\frac{n}{n-p}\right)$, erit prorsus uti supra conjectavimus

$$V = \left(\frac{n}{0}\right) + \left(\frac{n}{1}\right)^2 aa + \left(\frac{n}{2}\right)^2 a^4 + \left(\frac{n}{3}\right)^2 a^6 + \left(\frac{n}{4}\right)^2 a^8 + \text{etc.}$$

Hinc igitur operae pretium erit sequens theorema constituere.

Theorema generale.

§. 82. Si formula integralis

$$\int \frac{\partial \Phi \cos. i\Phi}{(1 + aa - 2a\cos.\Phi)^{n+1}},$$

a termino $\Phi = 0$ usque ad terminum $\Phi = 180^\circ$ extendatur, valor integralis semper habebit talem formam

$$\frac{\pi a^i}{(1-aa)^{2n+1}} V, \text{ existente}$$

$$V = \left(\frac{n+i}{i}\right) + \left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{1}\right) aa + \left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right) a^4$$
$$+ \left(\frac{n+i}{i+3}\right)\left(\frac{n-i}{3}\right) a^6 + \left(\frac{n+i}{i+4}\right)\left(\frac{n-i}{4}\right) a^8 + \text{etc.}$$

dummodo fuerit i numerus integer, atque adeo tam positivus quam negativus; quandoquidem etiam posteriori casu ista forma veritati

consentanea deprehenditur, ita ut ista expressio latius pateat, quam omnes casus speciales junctim sumti, unde eam per conjecturam conclusimus; namque in omnibus casibus specialibus littera i necessario denotabat numeros integros tantum positivos.

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4) Demonstratio Theorematis insignis per conjecturam eruti, circa integrationem formulae

$$\int \frac{\partial \Phi \cos. i \Phi}{(1 + aa - 2 a \cos. \Phi)^{n+1}}$$

M. S. Academiae exhib. die 10 Septembris 1778.

§. 83. Cum nuper hanc formulam integralem tractassem, ac potissimum in ejus valorem inquisivissem, quem accipit, si integrale a termino $\Phi = 0$ ad terminum $\Phi = 180^\circ$ usque extendatur; ex pluribus casibus, quos evolvere licuit, conclusi ejus integrale in genere ita expressum iri

$$\frac{\pi a^i}{(1 - aa)^{2n+1}} \mathrm{V},$$

ubi V denotat summam hujus seriei

$$\mathrm{V} = \left(\frac{n-i}{0}\right)\left(\frac{n+i}{i}\right) + \left(\frac{n-i}{1}\right)\left(\frac{n+i}{i+1}\right) a^2 + \left(\frac{n-i}{2}\right)\left(\frac{n+i}{i+2}\right) a^4 + \text{etc.}$$

Hic scilicet isti characteres clausulis inclusi designant coefficientes potestatis binomialis, dum statuimus

$$(1 + x)^m = 1 + \left(\frac{m}{1}\right) x + \left(\frac{m}{2}\right) x^2 + \left(\frac{m}{3}\right) x^3 + \left(\frac{m}{4}\right) x^4 + \text{etc.}$$

§. 84. Circa hanc autem formulam integralem ante omnia tenendum est, litteram i perpetuo significare numeros integros, quandoquidem in analysi constanter assumitur, casu $\Phi = 180^\circ$ sem-

per esse sin. $i\phi = 0$; tum vero etiam ejus valores perpetuo ut positivi spectari possunt, propterea quod cos. $(-i\phi) =$ cos. $(+i\phi)$. Interim tamen mox ostendemus nostram formam integralem etiam veritati esse consentaneam, quamvis litterae i valores negativi tribuantur. Ad hoc ostendendum circa characteres in subsidium vocatos sequentia sunt observanda.

1°). Si p et q designent numeros integros, ac primo quidem positivos, quoniam in evolutione potestatis binomialis omnes termini primum antecedentes sunt nulli, quoties fuerit q numerus negativus, semper erit $\left(\frac{p}{q}\right) = 0$.

2°). Quia coefficiens tam primi termini quam ultimi semper est unitas, erit tam $\left(\frac{p}{0}\right) = 1$ quam $\left(\frac{p}{p}\right) = 1$.

3°). Quia termini ultimum sequentes pariter sunt nulli, quoties fuerit $q < p$, valor characteris $\left(\frac{p}{q}\right)$ semper pro nihilo haberi poterit.

4°). Quia in evolutione potestatis binomialis coefficientes ordinem tenent retrogradum, hinc sequitur semper fore $\left(\frac{p}{q}\right) = \left(\frac{p}{p-q}\right)$. Sin autem superior numerus p fuerit negativus, ob rationem praecedentem semper etiam erit $\left(\frac{-p}{-q}\right) = 0$.

5°). At si q denotet numeros positivos, character $\left(\frac{-p}{q}\right)$, perpetuo dabit valores alternatim positivos et negativos; cum sit

$$\left(\frac{-p}{0}\right) = 1;\ \left(\frac{-p}{1}\right) = -p;\ \left(\frac{-p}{2}\right) = +\frac{p(p+1)}{1.2};\ \left(\frac{-p}{3}\right) = -\frac{p(p+1)(p+2)}{1.2.3}\ \text{etc.}$$

Atque hinc

6°). In genere tales characteres, ubi superior numerus est negativus, ad positivos reduci poterunt, cum sit $\left(\frac{-p}{q}\right) = \pm\left(\frac{p+q-1}{q}\right)$, ubi signum $+$ valet si q fuerit numerus par, inferius $-$ vero, si impar.

§. 85 His proprietatibus circa characteres hic adhibitos notatis, in forma nostra integrali loco i scribamus $-i$, eritque

$$\int \frac{\partial \Phi \cos. -i\Phi}{(1+aa-2a\cos.\Phi)^{n+1}} = \frac{\pi a^{-i}}{(1-aa)^{2n+1}} V,$$

existente

$$V = \left(\frac{n+i}{0}\right)\left(\frac{n-i}{-i}\right) + \left(\frac{n+i}{1}\right)\left(\frac{n-i}{-i+1}\right) a^2 + \left(\frac{n+i}{2}\right)\left(\frac{n-i}{-i+2}\right) a^4$$
$$+ \left(\frac{n+i}{3}\right)\left(\frac{n-i}{-i+3}\right) a^6 + \text{etc.}$$

ubi posteriores factores evanescunt, quamdiu denominatores sunt negativi: primum igitur membrum significatum habens erit $\left(\frac{n+i}{i}\right)$ $\left(\frac{n-i}{-i+i}\right) a^{2i}$, cujus valor erit $\left(\frac{n+i}{i}\right) a^{2i}$; sequentia autem membra erunt

$$\left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{-i+i+1}\right) a^{2i+2} = \left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{1}\right) a^{2i+2},$$

tum vero $\left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right) a^{2i+4}$, etc. Hoc igitur modo erit

$$V = a^{2i}\left[\left(\frac{n+i}{i}\right)\left(\frac{n-i}{0}\right) + \left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{1}\right) a^2 + \left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right) a^4 + \text{etc.}\right]$$

qui valor ductus in $\frac{\pi a^{-i}}{(1-aa)^{2n+1}}$ praebet hanc formam

$$\frac{\pi a^i}{(1-aa)^{2n+1}}\left[\left(\frac{n+i}{i}\right)\left(\frac{n-i}{0}\right) + \left(\frac{n+i}{i+1}\right)\left(\frac{n-i}{1}\right) a^2 + \left(\frac{n+i}{i+2}\right)\left(\frac{n-i}{2}\right) a^4 + \text{etc.}\right]$$

quae prorsus congruit cum nostra formula valori positivo ipsius i respondente, qui egregius consensus haud contemnendum firmamentum pro veritate nostrae formae integralis continet.

§. 86. Praeterea vero circa formam nostram integralem imprimi notari debet, seriem pro V supra datam semper alicubi abrumpi quoties n fuerit numerus integer positivus, quippe quod eveniet, quando vel in priore factore, cujus forma est $\left(\frac{n-i}{\lambda}\right)$, pervenitur

ad terminum quo $\lambda > n - i$, vel in posteriore factore, cujus forma est $\left(\frac{n+i}{i+\lambda}\right)$, evadet $\lambda > n$; quae proprietas eo magis est observanda, quod, si series V in infinitum porrigeretur, parum lucrati essemus censendi, id quod praecipue de iis casibus est notandum, quibus n foret numerus fractus, quos ergo casus penitus ab instituto nostro removemus, ita ut pro n tantum numeros integros simus assumturi.

§. 87. Consideremus ergo etiam casus, quibus n est numerus negativus, ac primo quidem jam per se clarum est, quamdiu is minor fuerit quam i, ideoque $n + i$ etiamnum numerus positivus, tum seriem pro V datam adeo citius abruptum iri; tum igitur demum in infinitum excurret, quando etiam $n + i$ fuerit numerus positivus. His autem casibus forma integralis supra data ita transformari potest, ut abruptio pariter locum inveniat.

§. 88. Ad hoc ostendendum statuamus $n = -m - 1$, ut formula nostra integralis evadat

$$\int \partial \phi \cos. i \phi (1 + aa - 2a \cos. \phi)^m,$$

ejusque igitur valor $= \pi a^i (1 - aa)^{2m+1}$ V, existente jam

$$V = \left(\frac{-m-1-i}{0}\right)\left(\frac{-m-1+i}{i}\right) + \left(\frac{-m-1-i}{1}\right)\left(\frac{-m-1+i}{i+1}\right) a^2$$
$$+ \left(\frac{-m-1-i}{2}\right)\left(\frac{-m-1+i}{i+2}\right) a^4 + \left(\frac{-m-1-i}{3}\right)\left(\frac{-m-1+i}{i+3}\right) a^6 + \text{etc.}$$

quae series manifesto in infinitum excurrit, quam autem ope sequentes lemmatis transformare poterimus.

Lemma.

§. 89. Ista series per characteres hic introductos procedens

$$s = \left(\frac{f}{0}\right)\left(\frac{h}{e}\right) + \left(\frac{f}{1}\right)\left(\frac{h}{e+1}\right) x + \left(\frac{f}{2}\right)\left(\frac{h}{e+2}\right) x^2 + \left(\frac{f}{3}\right)\left(\frac{h}{e+3}\right) x^3 + \text{etc.}$$

in hanc sui similem transmutari potest

$$\text{♂} = \left(\frac{-h-1}{0}\right)\left(\frac{-f-1}{e}\right) + \left(\frac{-h-1}{1}\right)\left(\frac{-f-1}{e+1}\right)x + \left(\frac{-h-1}{2}\right)\left(\frac{-f-1}{e+2}\right)x^2 + \text{etc.}$$

quandoquidem inter earum valores ♄ et ♂ ista relatio semper locum habere, non ita pridem a me est demonstrata

$$\left(\frac{e+f}{1}\right)\text{♄} = \left(\frac{e-h-1}{e}\right)(1-x)^{f+h+1}\,\text{♂},$$

cujus demonstratio profundissimae est indaginis, dum adeo per aequationes differentiales secundi gradus procedit.

§. 90. Applicemus jam istud lemma ad casum nostrum propositum, atque ut series ♄ cum nostro V consentiens reddatur, ut fiat ♄ $=$ V, sumi debet $f = -m-1-i$, $h = -m-1+i$, $e = i$ et $x = aa$, unde altera series ♂ hanc accipiet formam

$$\text{♂} = \left(\frac{m-i}{0}\right)\left(\frac{m+i}{i}\right) + \left(\frac{m-i}{1}\right)\left(\frac{m+i}{i+1}\right)aa + \left(\frac{m-i}{2}\right)\left(\frac{m+i}{i+2}\right)a^4 + \text{etc.}$$

quae series jam certe abrumpitur alicubi, propterea quod hic m denotat numerum integrum positivum: at vero relatio inter superiorem V $=$ ♄ et novam hanc seriem ♂ ita se habebit

$$\left(\frac{-m-1}{i}\right)V = \left(\frac{m}{i}\right)(1-aa)^{-2m-1}\,\text{♂}.$$

§. 91. Hinc igitur formulae nostrae integralis hujus

$$\int \partial\Phi \cos. i\Phi (1 + aa - 2a\cos.\Phi)^m = \frac{\left(\frac{m}{i}\right)\pi a^i\,\text{♂}}{\left(\frac{-m-1}{i}\right)},$$

ubi ♂ denotat seriem modo ante §. 89. expositam, qui valor cum factorem habeat $\left(\frac{m}{i}\right)$ semper evanescet, quamdiu fuerit $i > m$, ita ut his casibus valor integralis semper nihilo sit aequalis. Ceterum hic notasse juvabit, facta evolutione esse

$$\left(\frac{m}{i}\right) : \left(\frac{-m-1}{i}\right) = \pm\frac{m(m-1)}{(m+1)(m+2)}\cdots\cdots\frac{(m-i+1)}{(m+i)},$$

ubi signum superius $+$ valet si i fuerit numerus par, inferius $-$

vero si impar. His circa indolem nostri theorematis notatis, ipsam ejus demonstrationem aggrediamur, quam quo clarior evadat in varias partes distribuamus.

Demonstrationis pars prima.

§. 92. Quoniam valorem nostrum integralem ad duas formulas accommodavimus, eas distinctionis gratia signis ⊙ et ☾ designemus, sitque

$$\odot = \int \frac{\partial \Phi \cos. i\Phi}{(1 + aa - 2a \cos. \Phi)^{n+1}} \left[\begin{array}{l} a\ \Phi = 0 \\ ad\ \Phi = 180^\circ \end{array}\right],$$

$$☾ = \int \partial \Phi \cos. i\Phi\, (1 + aa - 2a \cos. \Phi)^m \left[\begin{array}{l} a\ \Phi = 0 \\ ad\ \Phi = 180^\circ \end{array}\right],$$

quarum posterior ☾ in priorem ⊙ convertitur si loco m scribamus $-n-1$; modo autem vidimus, has duas formulas a se invicem pendere, unde a posteriori tanquam simpliciori, siquidem denominatore $(1-aa)^{2n+1}$ caret, incipiamus, quam quo simpliciorem reddamus statuamus $\frac{a}{1+aa} = b$; sic enim habebimus

$$☾ = (1+aa)^m \int \partial \Phi \cos. i\Phi\, (1 - 2b \cos. \Phi)^m,$$

cujus ergo integrale nobis erit investigandum.

§. 93. Ante omnia igitur conveniet potestatem $(1-2b \cos. \Phi)^m$ evolvi, unde fiet

$$(1-2b\cos.\Phi)^m = 1 - \left(\tfrac{m}{1}\right) 2b \cos.\Phi + \left(\tfrac{m}{2}\right) 4b^2 \cos.\Phi^2$$
$$- \left(\tfrac{m}{3}\right) 8b^3 \cos.\Phi^3 + \text{etc.}$$

cujus ergo terminus quicunque erit $\pm \left(\frac{m}{\lambda}\right) 2^\lambda b^\lambda \cos. \Phi^\lambda$; ubi signum $+$ valet si λ fuerit numerus par, alterum vero $-$ si impar. Jam quia hic potestates ipsius cos. Φ occurrunt, eas per praecepta satis cognita in cosinus simplices converti oportet, quibus fit

$$2^2\cos.\Phi^2 = 2\cos.2\Phi + 1\left(\tfrac{2}{1}\right),$$
$$2^3\cos.\Phi^3 = 2\cos.3\Phi + 2\left(\tfrac{3}{1}\right)\cos.\Phi,$$
$$2^4\cos.\Phi^4 = 2\cos.4\Phi + 2\left(\tfrac{4}{1}\right)\cos.2\Phi + 1\left(\tfrac{4}{2}\right).$$
$$2^5\cos.\Phi^5 = 2\cos.5\Phi + 2\left(\tfrac{5}{1}\right)\cos.3\Phi + 2\left(\tfrac{5}{2}\right)\cos.\Phi,$$
$$2^6\cos.\Phi^6 = 2\cos.6\Phi + 2\left(\tfrac{6}{1}\right)\cos.4\Phi + 2\left(\tfrac{6}{2}\right)\cos.2\Phi + 1\left(\tfrac{6}{3}\right),$$
etc. etc.

Ubi notandum, in potestatibus paribus postremum membrum $\cos. 0\Phi = 1$ dimidio tantum coefficiente esse affectum. Hinc igitur in genere erit

$$2^\lambda\cos.\Phi^\lambda = 2\cos.\lambda\Phi + 2\left(\tfrac{\lambda}{1}\right)\cos.(\lambda+2)\Phi + 2\left(\tfrac{\lambda}{2}\right)\cos.(\lambda-4)\Phi + 2\left(\tfrac{\lambda}{3}\right)\cos.(\lambda-6)\Phi + \text{etc.}$$

ubi notetur, quoties fuerit λ numerus par, puta $\lambda = 2i$, ultimum membrum fore tantum $1.\left(\frac{2i}{i}\right)\cos. 0\Phi$.

§. 94 Postquam igitur omnes cosinuum potestates ad cosinus simplices fuerint reductae, integrationes nostrae semper ad talem formam redigentur $\int\partial\Phi\cos. i\Phi\cos.\lambda\Phi$, de qua forma hic imprimis est notandum, ejus integrale a $\Phi = 0$ ad $\Phi = 280^\circ$ extensum semper esse nullum, solo casu $\lambda = i$ excepto, Cum enim sit

$$\cos. i\Phi\cos.\lambda\Phi = \tfrac{1}{2}\cos.(i+\lambda)\Phi + \tfrac{1}{2}\cos.(i-\lambda)\Phi,$$

erit illud integrale indefinitum

$$= \frac{\sin.(i+\lambda)\Phi}{2(i+\lambda)} + \frac{\sin.(i-\lambda)\Phi}{2(i-\lambda)},$$

quod pro termino $\Phi = 0$ manifesto evanescit; pro altero vero termino $\Phi = 180^\circ = \pi$, ob i et λ numeros integros, manifestum est, hoc integrale denuo evanescere, solo casu excepto quo $\lambda = i$. Si enim $i-\lambda$ ut infinite parvum spectetur, puta $=\omega$, pars posterior hujus integralis erit $\frac{\sin.\omega\Phi}{2\omega} = \frac{\pi}{2}$, id quod etiam inde patet, quod sit

$$\cos. i\Phi^2 = \tfrac{1}{2} + \tfrac{1}{2}\cos. 2i\Phi,$$

ideoque

$$\int \partial\Phi \cos. i\Phi^2 = \tfrac{1}{2}\Phi + \tfrac{1}{4}\sin. 2i\Phi = \tfrac{1}{2}\pi.$$

§. 95. Ad integrale igitur quaesitum obtinendum, ex potestate $(1 - 2b\cos.\Phi)^m$ evoluta, eos tantum terminos, qui cos. $i\Phi$ continent, excerpsisse sufficiet, cum reliqui omnes nihil plane producant, qui si junctim sumti praebeant N cos. $i\Phi$, totum nostrum integrale pro ☾ erit

$$☾ = (1 + aa)^m . \tfrac{1}{2}N\pi;$$

quocirca nobis incumbet, in omnes superioris formae partes inquirere, quae formula cos. $i\Phi$ erunt affectae; unde evidens est, quamdiu in illo termino generali $\pm \left(\frac{m}{\lambda}\right) 2^\lambda b^\lambda \cos.\Phi^\lambda$ exponens λ minor fuerit quam i, inde nihil plane in integrale inferri.

§. 96. Primus igitur terminus, qui hic in computum venit, erit $\pm \left(\frac{m}{i}\right) 2^i b^i \cos.\Phi^i$, pro quo signum superius $+$ valebit si i fuerit numerus par, inferius $-$ vero si impar. Hinc autem par superiorem reductionem proveniet

$$2^i \cos.\Phi^i = 2\cos. i\Phi,$$

ita ut hinc pro N oriatur pars prima $\pm \left(\frac{m}{i}\right) 2b^i$. Tum vero ex termino immediate sequente, qui erit

$$\mp \left(\frac{m}{i+1}\right) 2^{i+1} b^{i+1} \cos.\Phi^{i+1},$$

nullus angulus $i\Phi$ oritur, cum sit

$$2^{i+1}\cos.\Phi^{i+1} = 2\cos.(i+1)\Phi + 2\left(\frac{i+1}{1}\right)\cos.(i-1)\Phi + \text{etc.}$$

At vero terminus sequens

$$\pm \left(\frac{m}{i+2}\right) 2^{i+2} b^{i+2} \cos.\Phi^{i+2}, \text{ ob}$$

$$2^{i+2}\cos.\Phi^{i+2} = 2\cos.(i+2)\Phi + 2\left(\frac{i+2}{1}\right)\cos. i\Phi + \text{etc.}$$

partem hinc in litteram N resultantem dat

$$2\left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right) b^{i+2}.$$

Simili modo ex casu $\lambda = i + 3$ nihil nascitur. At ex sequente

$$\pm\left(\frac{m}{i+4}\right) 2^{i+4} b^{i+4} \cos.\Phi^{i+4}, \text{ ob}$$

$$2^{i+4}\cos.\Phi^{i+4} = 2\cos.(i+4)\Phi + 2\left(\frac{i+4}{1}\right)\cos.(i+2)\Phi + 2\left(\frac{i+4}{2}\right)\cos.i\Phi + \text{etc.}$$

pars ad litteram N accedens erit

$$2\left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right) b^{i+4}.$$

Eodem modo ex casu $\lambda = i + 6$ pars ad litteram N accedens erit

$$2\left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right) b^{i+6}, \text{ et ita porro.}$$

§. 97. His igitur omnibus partibus colligendis, nanciscemur valorem completum litterae N, qui erit

$$N = \pm 2 b^i\left[\left(\frac{m}{i}\right)+\left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right)b^2+\left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right)b^4+\left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right)b^6+\text{etc.}\right]$$

ubi notasse juvabit esse, ut sequitur

$$\left(\frac{i+2}{1}\right)\left(\frac{m}{i+2}\right) = \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right),$$
$$\left(\frac{i+4}{2}\right)\left(\frac{m}{i+4}\right) = \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right),$$
$$\left(\frac{i+6}{3}\right)\left(\frac{m}{i+6}\right) = \left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right),$$
$$\text{etc.}$$

Per hos igitur valores erit

$$N = \pm 2 b^i\left[\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)+\left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)b^2+\left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)b^4+\left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right)b^6 \text{ etc.}\right]$$

quo valore invento, erit integrale nostrum quaesitum

$$C = \pm\pi(1+aa)^m b^i\left[\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)+\left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)b^2+\text{etc.}\right]$$

quae series manifesto abrumpitur, quoties fuerit m numerus integer

positivus. Statim enim atque in hoc charactere $\left(\frac{m-\lambda}{i+\lambda}\right)$ denominator $i+\lambda$ superare incipit numeratorem $m-\lambda$, valor ejus in nihilum abit.

Demonstrationis pars secunda.

§. 98. Ut autem hanc integralis expressionem ad solam litteram a revocamus, prouti in nostro theoremate supra est repraesentata, hic loco b restituamus valorem assumtum $\frac{a}{1+aa}$, fietque

$$☾ = \pm\pi\, a^i (1+aa)^{m-i} \left[\left(\frac{m}{0}\right)\left(\frac{m}{i}\right) + \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)\frac{a^2}{(1+aa)^2} + \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)\frac{a^4}{(1+aa)^4} + \text{etc.}\right]$$

ubi, ut formam supra datam eliciamus, potestates ipsius $1+aa$ evolvi oportet. Hunc in finem statuamus $☾ = \pm\pi\, a^i\, A$, ita ut jam sit

$$A = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)(1+aa)^{m-i} + \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)a^2(1+aa)^{m-i-2}$$
$$+ \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)a^4(1+aa)^{m-i-4} + \left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right)a^6(1+aa)^{m-i-6} + \text{etc.}$$

Facta autem harum potestatum evolutione, fiat

$$A = \alpha + \beta a^2 + \gamma a^4 + \delta a^6 + \varepsilon a^8 + \zeta a^{10} + \eta a^{12} + \text{etc.}$$

quarum litterarum α, β, γ, δ, etc. valores investigemus.

§. 99. Primo igitur statim patet esse $\alpha = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)$; deinde vero reperietur

$$\beta = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{1}\right) + \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right).$$

At vero pars posterior per priorem divisa, facta evolutione, praebet $\frac{m-i-1}{i+1}$, quo observato erit

$$\beta = \frac{m}{i+1}\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{1}\right),$$

quod reducitur ad $\beta = \left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right)$. Simili modo littera γ con-

stabit ex tribus partibus: erit enim

$$\gamma = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{2}\right) + \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)\left(\frac{m-i-2}{1}\right) + \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right),$$

ubi pars secunda per primam divisa dat $\frac{2(m-i-2)}{i+1}$. At tertius terminus per primum divisus praebet $\frac{(m-i-2)(m-i-3)}{(i+1)(i+2)}$, unde fit

$$\gamma = 1 + \frac{2(m-i-2)}{i+1} + \frac{(m-i-2)(m-i-3)}{(i+1)(i+2)}.$$

At vero est

$$1 + \frac{m-i-2}{i+1} = \frac{m-1}{i+1}, \text{ et}$$

$$\left(\frac{m-i-2}{i+1}\right)\left(1 + \frac{m-i-3}{i+2}\right) = \frac{m-1}{i+2}\cdot\frac{m-i-2}{i+1}.$$

unde colligitur

$$\gamma = \frac{m-1}{i+1}\cdot\frac{m}{i+2}\left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m-i}{2}\right),$$

quae expressio contrahitur in hanc $\left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right)$.

§. 100. Cum igitur sit

$$\alpha = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right),\ \beta = \left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right),\ \gamma = \left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right),$$

hinc jam satis tuto concludere liceret, fore

$$\delta = \left(\frac{m}{3}\right)\left(\frac{m}{i+3}\right),\ \varepsilon = \left(\frac{m}{4}\right)\left(\frac{m}{i+4}\right), \text{ etc.}$$

Verum ne hic quicquam conjecturae vel inductioni tribuamus, in genere pro valore litterae A investigemus coëfficientem potestatis indefinitae $a^{2\lambda}$, quem vocemus $= \lambda$, eritque

$$\Lambda = \left(\frac{m-i}{\lambda}\right)\left(\frac{m}{0}\right)\left(\frac{m}{i}\right) + \left(\frac{m}{1}\right)\left(\frac{m-1}{i+1}\right)\left(\frac{m-i-2}{\lambda-1}\right) + \left(\frac{m}{2}\right)\left(\frac{m-2}{i+2}\right)\left(\frac{m-i-4}{\lambda-2}\right)$$
$$+ \left(\frac{m}{3}\right)\left(\frac{m-3}{i+3}\right)\left(\frac{m-i-6}{\lambda-3}\right) + \text{etc.}$$

§. 101. Hujus seriei pro Λ inventae singulos terminos sub hac forma generali complecti licet $\left(\frac{m}{\theta}\right)\left(\frac{m-\theta}{i+\theta}\right)\left(\frac{m-i-2\theta}{\lambda-\theta}\right)$, quae secundum factores evoluta transmutatur in hanc formam

$$\frac{m(m-1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (m-i-\lambda-\theta+1)}{1 \quad . \quad . \quad . \quad \theta \times 1 \quad . \quad . \quad . \quad . \quad (i+\theta) \times 1 \quad . \quad . \quad (\lambda-\theta)},$$

ubi numeratoris factores ab m incipientes continuo unitate decrescent usque ad ultimum $(m-i-\lambda-\theta+1)$. Jam ista fractio supra et infra multiplicetur per hoc productum

$$\lambda(\lambda-1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (\lambda-\theta+1),$$

ac prodibit ista fractio

$$\frac{\lambda(\lambda-1) \quad . \quad . \quad . \quad (\lambda-\theta+1) \times m(m-1) \quad . \quad . \quad . \quad (m-i-\lambda-\theta+1)}{1.\,2.\,3. \quad . \quad . \quad . \quad \theta \times 1.\,2.\,3. \quad . \quad . \quad . \quad (i+\theta) \times 1.\,2.\,3 \quad . \quad . \quad . \quad \lambda},$$

in qua primo continetur character $\left(\frac{\lambda}{\theta}\right)$, deinde etiam ibi continetur character $\left(\frac{m}{\lambda}\right)$; quod restat dabit characterem $\left(\frac{m-\lambda}{i+\theta}\right)$, sicque habebitur forma A generalis $= \left(\frac{\lambda}{\theta}\right)\left(\frac{m}{\lambda}\right)\left(\frac{m-\lambda}{i+\theta}\right)$. Unde si loco θ successive scribamus 0, 1, 2, 3, etc., quia in singulis terminis communis inest factor $\left(\frac{m}{\lambda}\right)$, erit valor litterae

$$A = \left(\frac{m}{\lambda}\right)\left[\left(\frac{\lambda}{0}\right)\left(\frac{m-\lambda}{i}\right) + \left(\frac{\lambda}{1}\right)\left(\frac{m-\lambda}{i+1}\right) + \left(\frac{\lambda}{2}\right)\left(\frac{m-\lambda}{i+2}\right) + \text{etc.}\right]$$

Verum ante aliquod tempus demonstravi, hujus similis seriei

$$\left(\frac{p}{0}\right)\left(\frac{q}{r}\right) + \left(\frac{p}{1}\right)\left(\frac{q}{r+1}\right) + \left(\frac{p}{2}\right)\left(\frac{q}{r+2}\right) + \left(\frac{p}{3}\right)\left(\frac{q}{r+3}\right) + \text{etc.}$$

summam semper esse $= \left(\frac{p+q}{p+r}\right) = \left(\frac{p+q}{q-r}\right)$. Facta ergo applicatione, erit $p = \lambda$, $q = m - \lambda$, $r = i$: sicque finito modo habebimus

$$A = \left(\frac{m}{\lambda}\right)\left(\frac{m}{\lambda+i}\right) = \left(\frac{m}{\lambda}\right)\left(\frac{m}{m-\lambda-i}\right),$$

quae est demonstratio conjecturae supra allatae et ex valoribus α, β, γ, conclusae.

§. 102. Quod si jam hic loco λ successive scribamus numeros 0, 1, 2, 3, etc., nanciscemur verum valorem seriei, quam sub littera A complexi; erit scilicet

$$\begin{aligned} A = \left(\frac{m}{0}\right)\left(\frac{m}{i}\right) + \left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right) a^2 + \left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right) a^4 \\ + \left(\frac{m}{3}\right)\left(\frac{m}{i+3}\right) a^6 + \text{etc.} \end{aligned}$$

atque hinc valor integralis sub signo ☾ indicatae formulae erit

$$☾ = \pm \pi a^i \left[\left(\frac{m}{0}\right)\left(\frac{m}{i}\right) + \left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right) a^2 + \left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right) a^4 + \text{etc.}\right]$$

quae expressio manifesto semper abrumpitur, quoties m est numerus integer positivus. Hic autem meminisse oportet, signi ambigui $\pm$ superius locum habere quando i fuerit numerus par, inferius vero si impar.

Demonstrationis pars tertia.

§. 103. Ista forma, quam pro valore integrali ☾ hic sumus adepti multo adeo est simplicior ea, quam theorema nostrum nobis suppeditaverat, quippe quae, si loco ♂ seriem quam designat scribamus, erit

$$☾ = \frac{\pi a^i \left(\frac{m}{1}\right)}{\left(\frac{-m-1}{i}\right)} \left[\left(\frac{m-i}{0}\right)\left(\frac{m+i}{i}\right) + \left(\frac{m-i}{1}\right)\left(\frac{m+i}{i+1}\right) a^2 + \left(\frac{m-i}{2}\right)\left(\frac{m+i}{i+2}\right) a^4 + \text{etc.}\right]$$

Superest igitur, ut perfectum consensum inter has duas expressiones specie multum a se invicem discrepantes ostendamus. Hic autem plurimum notasse juvabit, esse $\left(\frac{-m-1}{i}\right) = \pm\left(\frac{m+i}{i}\right)$, propterea quod supra §. 88. jam observavimus, esse in genere $\left(\frac{-p}{q}\right) = \pm \left(\frac{p+q-1}{q}\right)$, ubi signum superius valet si fuerit q numerus par, inferius vero si impar; quo notato posterior forma pro ☾ inventa erit

$$☾ = \pm \frac{\pi a^i \left(\frac{m}{i}\right)}{\left(\frac{m+i}{i}\right)} \left[\left(\frac{m-i}{0}\right)\left(\frac{m+i}{i}\right) + \left(\frac{m-i}{1}\right)\left(\frac{m+i}{i+1}\right) a^2 + \text{etc.}\right].$$

§. 104. Quoniam nunc ambae formae affectae sunt signo ambiguo $\pm$, demonstrandum nobis incumbit, si utramque expressionem per $\left(\frac{m+i}{i}\right)$ multiplicemus, duas sequentes series inter se pror-

sus esse aequales

$$\text{I.}\quad \left(\frac{m}{0}\right)\left(\frac{m}{i}\right)\left(\frac{m+i}{i}\right)+\left(\frac{m}{1}\right)\left(\frac{m}{i+1}\right)\left(\frac{m+i}{i}\right)a^2$$
$$+\left(\frac{m}{2}\right)\left(\frac{m}{i+2}\right)\left(\frac{m+i}{i}\right)a^4+\text{etc.}$$
$$\text{II.}\quad \left(\frac{m-i}{0}\right)\left(\frac{m+i}{i}\right)\left(\frac{m}{i}\right)+\left(\frac{m-i}{1}\right)\left(\frac{m+i}{i+1}\right)\left(\frac{m}{i}\right)a^2$$
$$+\left(\frac{m-i}{2}\right)\left(\frac{m+i}{i+2}\right)\left(\frac{m}{i}\right)a^4+\text{etc.}$$

ubi aequalitas primorum terminorum ob $\left(\frac{m}{0}\right)$ et $\left(\frac{m-i}{0}\right)=1$ sponte se prodit: deinde vero non difficulter aequalitas inter terminos secundos ipso $a\,a$ affectos ostendi poterit, similique modo etiam de sequentibus hoc idem est tenendum.

§. 105. Verum ne etiam hic inductione uti cogamur, convenientiam binorum terminorum eadem potestate $a^{2\lambda}$ demonstremus. In priore vero serie ista potestas $a^{2\lambda}$ hunc habet coëfficientem $\left(\frac{m}{\lambda}\right)\left(\frac{m}{i+\lambda}\right)\left(\frac{m+i}{i}\right)$; in altera vero ejusdem coëfficiens est $\left(\frac{m-i}{\lambda}\right)\left(\frac{m+i}{i+\lambda}\right)\left(\frac{m}{i}\right)$. Evolvatur igitur uterque in factores simplices, ac prior deducit ad hanc fractionem

$$\frac{m\;\ldots\;(m-\lambda+1)\times m\;\ldots\;(m-i-\lambda+1)\times(m+i)\;\ldots\;(m+1)}{1\;\ldots\;\lambda\times 1\;\ldots\;(i+\lambda)\times 1\;\ldots\;i};$$

posterior vero praebet istam

$$\frac{(m-i)\;\ldots\;(m-i-\lambda+1)\times(m+i)\;\ldots\;(m-\lambda+1)\times m\;\ldots\;(m-i+1)}{1\;\ldots\;\lambda\times 1\;\ldots\;(i+\lambda)\times 1\;\ldots\;i};$$

ubi denominatores utrinque manifesto sunt iidem, ita ut tantum aequalitas inter numeratores sit demonstranda.

§. 106. Primo autem in priore numeratore tertius factor generalis cum primo conjunctus praebet hoc productum

$$(m+i)\;\ldots\;\ldots\;(m-\lambda+1),$$

quod etiam in forma posteriori occurrit: his igitur sublatis aequalitatem monstrari oportet inter partes residuas quae sunt,

in priori forma $m \quad . \quad . \quad . \quad . \quad . \quad (m-i-\lambda+1)$

in altera $m \ldots (m-i+1) \times (m-i) \ldots (m-i-l+1)$

quae nunc iterum est manifesta. Sic igitur veritas nostri theorematis, quod demonstrandum suscepimus, jam rigide est ob oculos posita pro formula integrali

$$☾ = \int \partial \Phi \cos. i \Phi (1 + aa - 2a \cos. \Phi)^n \begin{bmatrix} a\Phi = 0 \\ ad\Phi = \pi \end{bmatrix}.$$

Demonstrationis pars quarta.

§. 107. Invento valore formulae ☾, tota demonstratio jam confecta est censenda, quandoquidem jam initio ex valore formulae ⊙ ille rite est derivatus. Interim tamen hic quoque vicissim ex valore ☾ alterum valorem ⊙ derivari conveniet. Utamur autem forma simpliciori ipsius ☾, ad quem nos ipsa demonstratio immediate perduxit, qui erat

$$☾ = \pm \pi a^i [(\tfrac{m}{0})(\tfrac{m}{i}) + (\tfrac{m}{1})(\tfrac{m}{i+1}) a^2 + (\tfrac{m}{2})(\tfrac{m}{i+2}) a^4 + \text{etc.}]$$

ubi signum superius valet si i fuerit numerus par, inferius si impar.

§. 108. Ex hoc jam valore formulae ☾ alterius formulae ⊙ valor deducitur, si modo loco m scribamus $-n-1$, qui ergo valor hinc erit

$$⊙ = \pm \pi a^i [(\tfrac{-n-1}{0})(\tfrac{-n-1}{i}) + (\tfrac{-n-1}{1})(\tfrac{-n-1}{i+1}) a^2 + (\tfrac{-n-1}{2})(\tfrac{-n-1}{i+2}) a^4 + \text{etc.}]$$

quae autem series nunc in infinitum progreditur, siquidem n fuerit numerus integer positivus; quamobrem hanc seriem in aliam converti oportet, quae abrumpatur, quoties n fuerit numerus integer positivus, id quod ope lemmatis supra initio allati praestari poterit.

§. 109. Seriem igitur hic inventam cum serie ♄ in lemmate comparemus. id quod fit statuendo

$$f = -n-1,\ h = -n-1 \text{ et } e = i,$$

ita ut jam sit $\odot = \pm \pi a^i$ ♄. Ex his autem valoribus altera series signo ♂ notata fiet, ob

$$-h-1 = n,\ -f-1 = n, \text{ et } x = a^2,$$

$$♂ = \tbinom{n}{0}\tbinom{n}{i} + \tbinom{n}{1}\tbinom{n}{i+1} a^2 + \tbinom{n}{2}\tbinom{n}{i+2} a^4 + \text{etc.}$$

At vero relatio inter has duas series erit

$$\binom{i-n-1}{i} ♄ = \frac{\binom{n+i}{i} ♂}{(1-aa)^{2n+1}};$$

ubi notetur, cum supra jam observaverimus esse

$$\binom{-p}{q} = \pm \binom{p+q-1}{q}, \text{ hic fore } \binom{-n-1+i}{i} = \pm \binom{n}{i};$$

ubi iterum signum superius valet, si i fuerit numerus par. Hinc igitur erit

$$♄ = \pm \frac{\binom{n+i}{i} ♂}{\binom{n}{i}(1-aa)^{2n+1}}.$$

§. 110. Substituatur igitur iste valor loco ♄, quo ipso duplex signorum ambiguitas e medio tolletur, loco ♂ autem series modo data scribatur, atque pro $\odot$ sequentem nanciscemur expressionem

$$\odot = \frac{\pi a^i \binom{n+i}{i}}{\binom{n}{i}(1-aa)^{2n+1}} \left[\tbinom{n}{0}\tbinom{n}{i} + \tbinom{n}{1}\tbinom{n}{i+1} a^2 + \tbinom{n}{2}\tbinom{n}{i+2} a^4 + \text{etc.}\right]$$

quae series manifesto semper abrumpitur, quoties n fuerit numerus integer positivus. Verumtamen hoc laborat defectu, quod casibus quibus $n < i$, ob $\binom{n}{i} = 0$, infinita evadere videtur. Verum notan-

dum est, his casibus etiam omnes terminos seriei $\circlearrowleft$ in nihilum abire; ex quo necesse est, ut in ejus verum valorem totiusque expressionis inquiramus. At vero reliquis casibus, quibus $n > i$ haec expressio adeo illi quam in theoremate dedimus praeferenda videtur.

§. 111. Ostendi ergo hic debet, omnes terminos nostrae seriei ita transformari posse, ut per denominatorem $(\frac{n}{i})$ divisionem admittant. At vero quilibet nostrae seriei terminus sub hac forma continetur $(\frac{n}{\lambda})\,(\frac{n}{i+\lambda})$, quae per factorem comunem $(\frac{n+i}{i})$ multiplicata fit $(\frac{n+i}{i})\,(\frac{n}{\lambda})\,(\frac{n}{i+\lambda})$, quae in factores evoluta ad hanc fractionem reducitur

$$\frac{(n+i)\,\ldots\ldots\,(n+1)\times n\,\ldots\ldots\,(n-\lambda+1)\times n\,\ldots\ldots\,(n-i-\lambda+1)}{1\,\ldots\ldots\,i\times 1\,\ldots\ldots\,\lambda\times\,.\,1\,\ldots\ldots\,(i+\lambda)};$$

ubi tam numerator quam denominator tres habet factores principales; factores autem singulares in numeratore continuo unitate decrescunt, in denominatore unitate increscunt. Cum igitur sit $(\frac{n}{i}) = \frac{n\,\ldots\ldots\,(n-i+1)}{1\,\ldots\ldots\,i}$, superior fractio per hanc divisa, ob

$$\frac{n\,\ldots\ldots\,(n-i-\lambda+1)}{n\,\ldots\ldots\,(n-i+1}=(n-i)\,\ldots\ldots\,(n-i-\lambda+1),$$

proveniet

$$\frac{(n+i)\,\ldots\ldots\,(n+i)\times n\,\ldots\ldots\,(n-\lambda+1)\times(n-i)\,\ldots\ldots\,(n-i-\lambda+1)}{1.\,2.\,3\,\ldots\ldots\,\lambda\times 1.\,2.\,3\,\ldots\ldots\,(i+\lambda)},$$

quae manifesto in hanc transit (ob duo priores factores cohaerentes)

$$\frac{n+i)\,\ldots\ldots\ldots\,(n-\lambda+1)\times(n-i)\,\ldots\ldots\,(n-i-\lambda+1)}{1.\,2\,\ldots\ldots\,\lambda\times 1\,.\,2\,\ldots\ldots\,(i+\lambda)},$$

ita ut omnibus ad characteres reductis, sit forma generalis cujusque termini $= (\frac{n+i}{i+\lambda})\,(\frac{n-i}{\lambda})$.

§. 112. Nunc igitur loco λ successive scribantur valores 0, 1, 2, 3, etc. atque valor integralis formulae $\odot$ prodibit, pror-

sus uti in theoremate est enunciatus, scilicet

$$\odot = \frac{\pi a^i}{(1-aa)^{2n+1}} \left[\left(\frac{n-i}{0}\right)\left(\frac{n+i}{i}\right) + \left(\frac{n-i}{1}\right)\left(\frac{n+i}{i+1}\right)a^2 + \left(\frac{n-i}{2}\right)\left(\frac{n+i}{i+2}\right)a^4 + \text{etc.}\right]$$

quae expressio jam non solum semper abrumpitur, quoties n fuerit numerus integer positivus, nec ullo amplius laborat defectu, cum omnibus casibus valorem ipsius $\odot$ determinatum exhibeat, sicque adeo nostrum theorema, quod antea sola conjectura innitebatur, solidissima demonstratione est confirmatum.

SUPPLEMENTUM V.

AD TOM. I. CAP. VIII.

DE

VALORIBUS INTEGRALIUM QUOS CERTIS TANTUM CASIBUS RECIPIUNT.

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1) **Nova Methodus quantitates integrales determinandi.** *Novi Commentarii Academiae Scient. Petropolitanae Tom. XIX. Pag.* 66 — 102.

§. 1. Cum mihi saepius occurrissent formulae differentiales, quae per logarithmum quantitatis variabilis erant divisae, veluti $\frac{P\,\partial z}{lz}$, nunquam perspicere potui, ad quodnam genus quantitatum earum integralia sint referenda, quin etiam maxime difficile videbatur eorum valores saltem vero proxime assignare. Quod quidem ad formulam integralem simplicissimam hujus generis $\int\frac{\partial z}{lz}$ attinet, facile patet, si eam ita integrari concipiam, ut evanescat posito $z = 0$, tum vero statuatur $z = 1$, quantitatem infinite magnam esse prodituram; quod si enim variabilis z jam proxime ad unitatem accesserit, ut sit $z = 1 - u$, existente u quantitate infinite parva, tum ob

$$\partial z = -\partial u \text{ et } lz = l(1-u) = -u,$$

haec formula erit $\int\frac{\partial u}{u}$, cujus valor utique fit infinitus. At vero dantur omnino hujusmodi formulae integrales $\int\frac{P\,\partial z}{lz}$, quae, etiamsi po-

natur $z = 1$, tamen valores finitae magnitudinis sortiuntur: quod determinasse eo magis operae pretium videtur, quod nulla adhuc cognita est via istos valores investigandi.

§. 2. Consideremus exempli gratia hanc formulam satis simplicem $\int \frac{(z-1)\partial z}{l z}$, quae memorata lege integrata valorem finitum habere facile ostendi potest. Posito enim $\frac{z-1}{l z} = y$, ut formula nostra fiat $\int y \partial z$, ideoque exprimat aream curvae, pro abscissa z applicatam habentis $= y$, ista area a termino $z = 0$ usque ad terminum $z = 1$ extensa utique valorem finitum non multo majorem quam $\frac{1}{2}$ repraesentabit; posita enim abscissa $z = 0$, fiet etiam applicata $y = 0$, at sumta $z = 1$, pro applicata $y = \frac{z-1}{l z}$ tam numerator quam denominator evanescit, ergo eorum loco substitutis suis differentialibus, fiet $y = z = 1$. Pro abscissis autem mediis ponamus $z = e^{-n}$, existente e numero, cujus logarithmus hyperbolicus est unitas, erit

$$y = \frac{e^{-n} - 1}{-n} = \frac{e^n - 1}{n e^n},$$

quae, si n fuerit numerus valde magnus, ut abscissa z fiat minima, applicata erit proxime $y = \frac{1}{n}$; qui ergo valor multo major erit quam abscissa z; forma scilicet hujus curvae similis erit figurae adjectae, ubi A P denotat abscissam z et P M applicatam y, abscissae vero A B $= 1$ respondet applicata B C $= 1$, qua curva descripta, Fig. I. ejus area A M C B non multum superabit aream trianguli A B C quae est $= \frac{1}{2}$.

§. 3. Nuper autem, in aliis investigationibus occupatus, praeter exspectationem inveni, hanc aream aequalem esse logarithmo hyperbolico binarii, ita ut ea per fractiones decimales sit l

$2 = 0{,}6931471805$; sequenti autem ratiocinio huc sum perductus. Cum revera sit $l\,z = \frac{z^0 - 1}{0}$, quia differentiando utrinque prodit $\frac{\partial z}{z} = \frac{\partial z}{z}$, et sumto $z = 1$ utraque expressio evanescit, loco 0 scribo $\frac{1}{i}$, denotante i numerum infinitum, eritque $l\,z = i\,(z^{\frac{1}{i}} - 1)$, hincque applicata

$$y = \frac{z - 1}{i(z^{\frac{1}{i}} - 1)} = \frac{1 - z}{i(1 - z^{\frac{1}{i}})},$$

et formula integralis

$$\int \frac{(1 - z)\partial z}{i(1 - z^{\frac{1}{i}})}.$$

Nunc igitur statuo $z^{\frac{1}{i}} = x$, ut fiat $z = x^i$, ubi notetur, pro utroque integrationis termino $z = 0$ et $z = 1$ etiam fore $x = 0$ et $x = 1$; quia igitur hinc fit $\partial z = i x^{i-1} \partial x$, formula integralis evadit

$$\int \frac{x^{i-1}\partial x\,(1 - x^i)}{(1 - x)},$$

quam ergo integrari oportet a termino $x = 0$ usque ad terminum $x = 1$.

§. 4. Spectemus nunc i ut numerum valde magnum, et fractio $\frac{1 - x^i}{1 - x}$ resolvitur in hanc progressionem geometricam

$$1 + x + x^2 + x^3 + x^4 + x^5 + x^6 + \ldots . x^{i-1},$$

cujus singuli termini in $x^{i-1}\partial x$ ducti et integrati praebent hanc seriem

$$\frac{x^i}{i} + \frac{x^{i+1}}{i+1} + \frac{x^{i+2}}{i+2} + \frac{x^{i+3}}{i+3} + \ldots\ldots + \frac{x^{2i-1}}{2i-1},$$

quae utique evanescit facto $x = 0$. Nunc igitur sumatur $x = 1$, et valor quaesitus nostrae formulae integralis erit

$$\frac{1}{i} + \frac{1}{i+1} + \frac{1}{i+2} + \frac{1}{i+3} + \ldots\ldots + \frac{1}{2i-1},$$

ubi quidem littera i denotat numerum infinite magnum, ita ut numerus horum terminorum sit revera infinitus. Nihilo vero minus, quia singuli termini sunt infinite parvi, haec series summam habebit finitam, quam sequenti modo ad seriem ordinariam reducere licet.

§. 5. Series inventa spectari potest tanquam differentia inter binas sequentes progressiones harmonicas

$$A = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \ldots\ldots + \frac{1}{2i-1}$$

$$B = 1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \tfrac{1}{5} + \tfrac{1}{6} + \ldots\ldots + \frac{1}{i-1}$$

quandoquidem differentia $A - B$ ipsam seriem inventam exhibet; quia autem numerus terminorum seriei A est $2i - 1$, seriei vero $B = i - 1$, ille duplo major est quam hic, quocirca, ut seriem regularem obtineamus, singulos terminos seriei B per saltum a seriei A termino secundo, quarto, sexto, octavo etc. auferamus, quo pacto simul ad finem utriusque pervenietur, eritque

$$A - B = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \tfrac{1}{5} - \tfrac{1}{6} + \text{etc.}$$

in infinitum, cujus ergo valor est $l\,2$, ita ut nunc quidem solide sit demonstratum, formulae integralis propositae $\int \frac{(z-1)\partial z}{lz}$, casu $z = 1$, valorem revera esse $= l\;2$.

§. 6. Simile ratiocinium etiam ad formulam integralem generaliorem $\int \frac{(z^m - 1)\partial z}{lz}$ accommodari potest, ac tandem reperietur, casu $z = 1$ ejus valorem fore $l\,(m+1)$; quia igitur pari modo erit

$$\int \frac{(z^n - 1)\partial z}{lz} = l(n+1),$$

si hanc ab illa subtrahamus, prodit sequens integratio

$$\int \frac{(z^m - z^n)\partial z}{lz} = l\frac{m+1}{n+1},$$

si scilicet integratio a termino $z = 0$ usque ad terminum $z = 1$ extendatur.

§. 7. Quia autem haec demonstratio per quantitates infinitas et infinite parvas procedit, merito aliam methodum planam et consuetam desideramus, quae ad easdem summas perducere valeat; quae quidem investigatio maxime ardua videbitur. Interim tamen, cum nuper consideratio functionum duas variabiles involventium me ad integrationem formularum differentialium prorsus singularium perduxisset, quae aliis methodis frustra tentantur, ex eodem principio quoque integrationes hic exhibitas derivandas esse intellexi. Hanc igitur methodum tanquam fontem prorsus novum, ex quo integrationes, aliis methodis inaccessas, haurire liceat, clare et perspicue explicabo, cui negotio istam disquisitionem praecipue destinavi.

Lemma I.

§. 8. Si P fuerit functio quaecunque duarum variabilium z et u, ac ponatur $\int P\,\partial z = S$, ut etiam S sit functio binarum variabilium z et u, tum erit

$$\int \partial z \left(\frac{\partial P}{\partial u}\right) = \left(\frac{\partial S}{\partial u}\right).$$

Demonstratio.

Cum in integratione formulae $\int P\,\partial z$ sola z ut variabilis spectetur, erit $\left(\frac{\partial S}{\partial z}\right) = P$, quae formula denuo differentiata, sola u

pro variabili habita, praebet $\left(\frac{\partial\partial S}{\partial u \partial z}\right) = \left(\frac{\partial P}{\partial u}\right)$, quae in ∂z ducta et integrata producit $\left(\frac{\partial S}{\partial u}\right) = \int \partial z \left(\frac{\partial P}{\partial u}\right)$, quandoquidem ex principiis calculi integralis est

$$\int \partial z \left(\frac{\partial\partial S}{\partial z \partial u}\right) = \left(\frac{\partial S}{\partial u}\right) \text{ q. e. d.}$$

Corollarium I.

§. 9. Eodem modo per hujusmodi differentialia, ubi tantum u pro variabili spectatur, ulterius progredi licet, unde sequentes oriuntur integrationes

$$\left(\frac{\partial\partial S}{\partial u^2}\right) = \int \partial z \left(\frac{\partial\partial P}{\partial u^2}\right) \text{ et}$$
$$\left(\frac{\partial^3 S}{\partial u^3}\right) = \int \partial z \left(\frac{\partial^3 P}{\partial u^3}\right)$$
$$\text{etc.} \qquad\qquad \text{etc.}$$

Corollarium 2.

§. 10. Quod si ergo formula $\int P \partial z$ fuerit integrabilis, ita ut ejus integrale S exhiberi possit, tum etiam omnes istae formulae integrales

$$\int \partial z \left(\frac{\partial P}{\partial u}\right), \int \partial z \left(\frac{\partial\partial P}{\partial u^2}\right), \int \partial z \left(\frac{\partial^3 P}{\partial u^3}\right) \text{ etc.}$$

integrationem admittent, atque adeo ipsa integralia exhiberi poterunt.

Scholion.

§. 11. Ex his quidem formulis si in genere tractentur, parum utilitatis in calculum integralem redundat. At si functio P ita fuerit comparata, ut integrale $\int P \partial z$, casu saltem particulari, quo post integrationem variabili z certus quidam valor puta $z = a$ tribuitur, commode exhiberi potest, ut hoc casu quantitas S abeat in functionem solius variabilis u satis simplicem, tum integrationes memoratae perinde locum habebunt, si quidem post singulas integratio-

nes ponatur $z = a$, atque hinc ad ejusmodi integrationes plerumque pervenitur, quas aliis methodis vix, ac ne vix quidem perficere liceat: atque hinc oritur

Primum principium integrationum.

§. 12. Si P ejusmodi fuerit functio binarum variabilium z et u, ut valor integralis $\int P \partial z$ saltem casu certo $z = a$ commode exprimi queat, qui valor sit $= S$, functio scilicet ipsius u tantum; tum etiam sequentia integralia, si quidem post integrationem pariter statuatur $z = a$, commode exhiberi poterunt, scilicet

$$\int P \partial z = S$$
$$\int \partial z \left(\frac{\partial P}{\partial u}\right) = \left(\frac{\partial S}{\partial u}\right)$$
$$\int \partial z \left(\frac{\partial \partial P}{\partial u^2}\right) = \left(\frac{\partial \partial S}{\partial u^2}\right)$$
$$\int \partial z \left(\frac{\partial^3 P}{\partial u^3}\right) = \left(\frac{\partial^3 S}{\partial u^3}\right)$$
$$\int \partial z \left(\frac{\partial^4 P}{\partial u^4}\right) = \left(\frac{\partial^4 S}{\partial u^4}\right)$$
$$\text{etc.} \qquad \text{etc.}$$

Exemplum I.

§. 13. Si fuerit $P = z^u$, erit quidem in genere

$$\int P \partial z = \frac{z^{u+1}}{u+1};$$

unde casu $z = 1$ hic valor satis simplex nascitur $\frac{1}{u+1}$, ita ut sit $S = \frac{1}{u+1}$; cum deinde per differentiationes continuas, dum sola u pro variabili habetur, prodeat $\left(\frac{\partial P}{\partial u}\right) = z^u \, l\, z$, tum vero $\left(\frac{\partial \partial P}{\partial u^2}\right) = z^u \, (l\, z)^2$, porro

$$\left(\frac{\partial^3 P}{\partial u^3}\right) = z^u \, (l\, z)^3, \quad \left(\frac{\partial^4 P}{\partial u^4}\right) = z^u \, (l\, z)^4, \text{ etc.}$$

hinc sequentes obtinentur valores integrales, si quidem post singulas integrationes statuatur $z = 1$

$$\int z^u \partial z = + \frac{1}{u+1}$$
$$\int z^u \partial z\, l z = - \frac{1}{(u+1)^2}$$
$$\int z^u \partial z\, (l z)^2 = + \frac{1.\,2}{(u+1)^3}$$
$$\int z^u \partial z\, (l z)^3 = - \frac{1.\,2.\,3}{(u+1)^4}$$
$$\int z^u \partial z\, (l z)^4 = + \frac{1.\,2.\,3.\,4}{(u+1)^5}$$
$$\int z^u \partial z\, (l z)^5 = - \frac{1.2.3.4.5}{(u+1)^6}$$
$$\int z^u \partial z\, (l z)^6 = + \frac{1 \ldots\ldots 6}{(u+1)^7}$$
$$\int z^u \partial z\, (l z)^7 = - \frac{1 \ldots\ldots 7}{(u+1)^8}$$

unde concludimus generaliter fore

$$\int z^u \partial z\, (l z)^n = \pm \frac{1.\,2.\,3.\,4 \ldots n}{(u+1)^{n+1}},$$

ubi signum $+$ valet si n sit numerus par, alterum vero $-$ si n sit numerus impar. Hae quidem integrationes jam aliunde satis sunt notae, id quod mirum non est, quoniam tam simplicem formulam pro P assumsimus: breviter igitur repetamus eos casus, quos jam nuper expedivi.

Exemplum 2.

§. 14. Si fuerit

$$P = \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2n}},$$

jam dudum demonstravi, formulae $\int P \partial z$ valorem integralem casu quo post integrationem ponitur $z = 1$, esse

$$S = \frac{\pi}{2 n \cos. \frac{\pi u}{2 n}}.$$

Hinc ergo cum sit

$$\left(\frac{\partial P}{\partial u}\right) = - \frac{z^{n-u-1} + z^{n+u-1}}{1 + z^{2-n}}\, l z,$$

tum vero

$$\left(\frac{\partial\partial P}{\partial u^2}\right) = + \frac{z^{n-u-1} + z^{n+u-1}}{1+z^{2n}} (lz)^2 \text{ et}$$

$$\left(\frac{\partial^3 P}{\partial u^3}\right) = - \frac{z^{n-u-1} + z^{n+u-1}}{1+z^{2n}} (lz)^3$$

etc. etc.

ex cognito valore S sequentes nacti sumus integrationes

$$\text{I. } \int \frac{z^{n-u-1} + z^{n+u-1}}{1+z^{2n}} \partial z = S = \frac{\pi}{2n \cos. \frac{\pi u}{2u}}$$

$$\text{II. } \int - \frac{z^{n-u-1} + z^{n+u-1}}{1+z^{2n}} \partial z\, lz = \left(\frac{\partial S}{\partial u}\right)$$

$$\text{III. } \int \frac{z^{n-u-1} + z^{n+u-1}}{1+z^{2n}} \partial z\, (lz)^2 = \left(\frac{\partial\partial S}{\partial u^2}\right)$$

$$\text{IV. } \int - \frac{z^{n-u-1} + z^{n+u-1}}{1+z^{2n}} \partial z\, (lz)^3 = \left(\frac{\partial^3 S}{\partial u^3}\right)$$

$$\text{V. } \int \frac{z^{n-u-1} + z^{n+u-1}}{1+z^{2n}} \partial z\, (lz)^4 = \left(\frac{\partial^4 S}{\partial u^4}\right)$$

etc. etc.

Exemplum 3.

§. 15. Si fuerit

$$P = \frac{z^{n-u-1} - z^{n+u-1}}{1-z^{2n}},$$

simili modo demonstravi, valorem formulae integralis $\int P\, \partial z$, casu quo post integrationem ponitur $z = 1$, fore

$$S = \frac{\pi}{2n} \text{ tang. } \frac{\pi u}{2n};$$

atque hinc sequentes integrationes pro eodem casu $z = 1$ fuerunt deductae.

$$\text{I. } \int \frac{z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}} \partial z = S = \frac{\pi}{2n} \text{ tang. } \frac{\pi u}{2n}$$

$$\text{II. } \int \frac{-z^{n-u-1} + z^{n+u-1}}{1 - z^{2n}} \partial z \, l z = \left(\frac{\partial S}{\partial u}\right)$$

$$\text{III. } \int \frac{z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}} \partial z \, (l z)^2 = \left(\frac{\partial \partial S}{\partial u^2}\right)$$

$$\text{IV. } \int \frac{-z^{n-u-1} + z^{n+u-1}}{1 - z^{2n}} \partial z \, (l z)^3 = \left(\frac{\partial^3 S}{\partial u^3}\right)$$

$$\text{V. } \int \frac{-z^{n} \quad u-1 - z^{n-u-1}}{1 - z^{2n}} \partial z \, (l z)^4 = \left(\frac{\partial^4 S}{\partial u^4}\right)$$

etc. etc.

Scholion.

§. 16. Quo igitur uberiores fructus ex hoc principio exspectare queamus, praecipuum negotium huc redit, ut ejusmodi functiones binarum variabilium z et u pro P investigemus, ita ut valor formulae integralis saltem certo quodam casu puta $z = 1$ succincte assignari possit, quemadmodum in allatis exemplis fieri licuit. Quemadmodum autem hoc principium ex continua differentiatione est deductum, ita eodem modo continua integratio ad usum nostrum accommodari poterit.

Lemma II.

§. 17. Si P fuerit functio duarum variabilium z et u, ac ponatur $\int P \partial z = S$, ut etiam S sit functio duarum variabilium z et u, tum erit $\int S \partial u = \int \partial z \int P \partial u$, ubi in integralibus formulis $\int P \partial u$ et $\int S \partial u$ sola u pro variabili habetur, in formula autem $\int \partial z \int P \partial u$ sola z.

Demonstratio.

Ponatur $\int S \partial u = V$, ut sit $S = (\frac{\partial V}{\partial u})$, ideoque

$$(\frac{\partial V}{\partial u}) = \int P \partial z, \text{ eritque } (\frac{\partial \partial V}{\partial z \partial u}) = P;$$

unde, per ∂u multiplicando et integrando, erit $(\frac{\partial V}{\partial z}) = \int P \partial u$, ex quo sequitur

$$V = \int \partial z \int P \partial u = \int S \partial u. \text{ q. e. d.}$$

Corollarium 1.

§. 18. Hoc modo etiam integratio repeti potest, unde orietur talis aequatio

$$\int \partial u \int S \partial u = \int \partial z \int \partial u \int P \partial u;$$

hinc autem plerumque parum utilitatis exspectari potest, nisi forte istae integrationes commode succedant.

Corollarium 2.

§. 19. Quod si ergo formula $\int P \partial z$ fuerit integrabilis, scilicet $= S$, altera hinc deducta $\int \partial z \int P \partial u$ eatenus tantum integrari poterit, quatenus integrale $\int S \partial u$ integrare licet.

Secundum principium integrationum.

§. 20. Si P ejusmodi fuerit functio duarum variabilium z et u, ut formulae integralis $\int P \partial z$ valor certo saltem casu, puta $z = a$, commode exhiberi queat, ita ut hoc casu quantitas S fiat functio solius variabilis u; tum etiam pro eodem casu $z = a$ hujus formulae integralis $\int \partial z \int P \partial u$ valor assignari poterit, si modo formulam $\int S \partial u$ integrare licuerit.

Exemplum I.

§. 21. Sumamus $P = z^u$, eritque $\int P \partial z = \frac{z^{u+1}}{u+1}$; quae formula casu $z = 1$ abit in $\frac{1}{u+1}$, quod ergo loco S scribatur. Tum vero quia est

$$\int P \partial u = \int z^u \partial u = \frac{z^u}{l z},$$

et quia

$$\int S \partial u = l\,(u+1), \text{ erit}$$

$$\int \frac{z^u \partial z}{l z} = l\,(u+1);$$

si quidem post illam integrationem ponatur $z = 1$. Quia autem omnis integratio additionem constantis postulat, hic potius statui oportebit

$$\int \frac{z^u \partial z}{l z} = l\,(u+1) + C;$$

atque hic quidem facile intelligitur, hanc constantem C esse debere infinitam, quoniam in formula integrali fractio $\frac{z^u}{l z}$ posito $z = 1$ fit infinita, ita ut hinc parum pro instituto nostro sequi videatur.

Corollarium 1.

§. 22. Quoniam autem haec constans C non a variabili u pendet, ea retinebit eundem valorem, quicunque numeri determinati pro u accipiantur. Sumamus igitur primo $u = m$, tum vero etiam $u = n$, ut habeamus istos valores

$$\text{I. } \int \frac{z^m \partial z}{l z} = l\,(m+1) + C \text{ et}$$

$$\text{II. } \int \frac{z^n \partial z}{l z} = l(n+1) + C,$$

quarum altera ab altera subtracta relinquet istam integrationem notatu dignissimam

$$\int \frac{(z^m - z^n) \partial z}{l z} = l \frac{m+1}{n+1},$$

quemadmodum jam supra ex longe aliis principiis demonstravimus.

Corollarium 2.

§. 23. Si ad alteram integrationem ascendamus, quia est $\int P \partial u = \frac{z^u}{l z}$, erit $\int \partial u \int P \partial u = \frac{z^u}{(l z)^2}$; tum vero ob

$$\int S \partial u = l(u+1) + C, \text{ erit}$$
$$\int \partial u \int S \partial u = (u+1)[l(u+1) - 1] + Cu + D,$$

sicque habebimus

$$\int \frac{z^u \partial z}{(l z)^2} = (u+1)[l(u+1) - 1] + Cu + D,$$

ubi constantes C et D non ab u pendent: quare ut eas eliminemus tres casus determinatos evolvamus

$$\text{I. } \int \frac{z^m \partial z}{(l z)^2} = (m+1) l(m+1) - m - 1 + Cm + D,$$

$$\text{II. } \int \frac{z^n \partial z}{(l z)^2} = (n+1) l(n+1) - n - 1 + Cn + D,$$

$$\text{III. } \int \frac{z^k \partial z}{(l z)^2} = (k+1) l(k+1) - k - 1 + Ck + D,$$

eritque

$$\text{I} - \text{III} = (m+1) l(m+1) - (k+1) l(k+1) + k - m + C(m-k) \text{ et}$$
$$\text{II} - \text{III} = (n+1) l(n+1) - (k+1) l(k+1) + k - n + C(n-k)$$

hincque deducimus

$$(\mathrm{I-III})(n-k)-(\mathrm{II-III})(m-k)=\begin{cases}+(m+1)(n-k)l(m+1)\\-(k+1)(n-k)l(k+1)+(k-m)(n-k)\\-(n+1)(m-k)l(n+1)-(k-n)(m-k)\\+(k+1)(m-k)l(k+1)\end{cases}$$

atque hinc pervenimus ad sequentem integrationem

$$\int \frac{\partial z\,[(n-k)z^m-(m-k)z^n+(m-n)z^k]}{(lz)^2}=$$

$$\begin{aligned}&+(m+1)\ (n-k)\ l(m+1)\\&-(n+1)\ (m-k)\ l(n+1)\\&+(k+1)\ (m-n)\ l(k+1).\end{aligned}$$

Corollarium 3.

§. 24. Operae pretium erit aliquot casus evolvere, ubi quidem numeros m, n et k inter se inaequales accipi convenit, quia aliter omnes termini se destruerent.

I. Sit igitur $m=2$, $n=1$ et $k=0$, erit

$$\int \frac{(z-1)^2\partial z}{(lz)^2}=3\,l\,3-4\,l\,2=l\tfrac{27}{16},$$

II. Sit $m=3$, $n=1$ et $k=0$, eritque

$$\int \frac{(z^3-3z+2)\partial z}{(lz)^2}=\int \frac{\partial z\,(z-1)^2(z+2)}{(lz)^2}=4\,l\,4-6\,l\,2=2\,l\,2=l\,4,$$

III. Sit $m=3$, $n=2$ et $k=0$, et erit

$$\int \frac{(2z^3-3zz+1)\partial z}{(lz)^2}=\int \frac{\partial z\,(z-1)^2(2z+1)}{(lz)^2}=8\,l\,4-9\,l\,3=l\tfrac{4^8}{3^9},$$

IV. Sit $m=3$, $n=2$ et $k=1$, et prodit

$$\int \frac{(z^3-2zz+z)\partial z}{(lz)^2}=\int \frac{z\,\partial z\,(z-1)^2}{(lz)^2}=4\,l\,4-6\,l\,3+2\,l\,2=l\tfrac{2^{10}}{3^6}.$$

Corollarium 4.

§. 25. In his casibus notatu dignum occurrit, quod numerator in formulis integralibus factorem habet $(z-1)^2$, quod

ideo necessario usu venit, ne valores integralium evadant infiniti. Quia enim denominator $(lz)^2$ evanescit casu $z = 1$, si ponamus $z = 1 - \omega$, existente ω infinite parvo, erit

$$lz = -\omega \text{ et } (lz)^2 = +\omega\omega.$$

Necesse ergo est ut in numeratore adsit factor, qui casu $z = 1 - \omega$ itidem praebeat $\omega\omega$, quod evenit si ibi factor fuerit $(z-1)^2$.

Scholion.

§. 26. Integratio, quam in corollario primo sumus nacti, ideo omni digna videtur attentione, quod valores integrales inde nati casu $z = 1$ nullo adhuc modo assignare potuerim, etiamsi tam simpliciter per logarithmos exprimantur. At vero integrationes in corollario secundo inventae, etiamsi multo magis arduae, videantur, tamen ex prioribus ope reductionum cognitarum non difficulter derivari possunt; id quod pro unico casu ostendisse sufficiet. Ponamus

$$\int \frac{\partial z (z-1)^2}{(lz)^2} = \frac{p}{lz} + \int \frac{q\,\partial z}{lz},$$

eritque differentiando

$$\frac{\partial z (z-1)^2}{(lz)^2} = \frac{\partial p}{lz} - \frac{p\,\partial z}{z(lz)^2} + \frac{q\,\partial z}{lz},$$

unde aequatis terminis seorsim vel per $(lz)^2$ vel per lz divisis, habebimus has duas aequalitates

$$(z-1)^2 = -\frac{p}{z} \text{ et } \partial p = -q\,\partial z,$$

ex quarum priore oritur $p = -z(z-1)^2$, hincque

$$\frac{\partial p}{\partial z} = -3zz + 4z - 1,$$

ideoque

$$q = 3zz - 4z + 1,$$

ita ut sit

$$\int \frac{\partial z (z-1)^2}{(lz)^2} = \frac{-z(z-1)^2}{lz} + \int \frac{(3zz - 4z + 1)\,\partial z}{lz},$$

hic autem prius membrum posito $z = 1$ sponte evanescit; posito enim $z = 1 - \omega$, ut sit $lz = -\omega$, erit

$$p = -\omega\omega(1-\omega), \text{ ideoque}$$

$$\frac{p}{lz} = \omega(1-\omega) = 0, \text{ ob } \omega = 0:$$

posterius vero membrum in has partes discerpi potest

$$3\int\frac{(zz-z)\partial z}{lz} - \int\frac{(z-1)\partial z}{lz},$$

cujus prioris partis integrale est $3\,l\frac{3}{2}$, posterioris vero $-1\,l\,2$; sicque totum hoc integrale erit

$$3\,l\tfrac{3}{2} - l\,2 = 3\,l\,3 - 4\,l\,2 = l\tfrac{27}{16},$$

prorsus uti invenimus. Hoc igitur modo si in genere statuamus

$$\int\frac{V\partial z}{(lz)^2} = \frac{p}{lz} + \int\frac{q\partial z}{lz},$$

erit differentiando

$$\frac{V\partial z}{(lz)^2} = \frac{\partial p}{lz} - \frac{p\partial z}{z(lz)^2} + \frac{q\partial z}{lz},$$

unde istae duae fluunt aequalitates

$$p = -Vz \text{ et } q = -\frac{\partial p}{\partial z}.$$

Jam ut terminus $\frac{p}{lz}$ evanescat posito $z = 1$, numerator p factorem habere debet $(z-1)^2$; qui ergo etiam factor esse debet quantitatis V. Sit igitur

$$V = \frac{U(z-1)^2}{z}, \text{ eritque } p = -U(z-1)^2,$$

unde fit

$$\partial p = -\partial U(z-1)^2 - 2U\partial z(z-1) = (z-1)[\partial U(z-1) - 2U\partial z],$$

hincque

$$q\partial z = (z-1)[2U\partial z - \partial U(z-1)];$$

quia ergo q factorem habet $z-1$, formula $\int\frac{q\partial z}{lz}$ semper in partes resolvi potest, quarum integralia per corollarium primum assig-

nare licet, si modo U fuerit aggregatum ex quotcunque potestatibus ipsius z; unde sequens deducitur theorema.

Theorema.

§. 27. Si fuerit

$$P = A z^{\alpha} + B z^{\beta} + C z^{\gamma} + D z^{\delta} + \text{etc.}$$

ita ut summa coëfficientium

$$A + B + C + D + \text{etc.} = 0,$$

tum erit

$$\int \frac{P \partial z}{l z} = A\, l(\alpha + 1) + B\, l(\beta + 1) + C\, l(\gamma + 1) + D\, l(\delta + 1) + \text{etc.}$$

si quidem post integrationem statuatur $z = 1$.

Demonstratio.

Cum hoc ipso casu, quo post integrationem ponitur $z = 1$, sit

$$\int \frac{z^n \partial z}{l z} = l(n + 1) + \Delta,$$

denotante Δ illam constantem infinitam integratione ingressam, erit

$$A \int \frac{z^{\alpha} \partial z}{l z} = A\, l(\alpha + 1) + A\, \Delta,$$

eodemque modo

$$B \int \frac{z^{\beta} \partial z}{l z} = B\, l(\beta + 1) + B\, \Delta,$$

etc. etc.

si nunc haec integralia omnia in unam summam colligantur, erit ob

$$(A + B + C + D + \text{etc.})\, \Delta = 0$$

integrale quaesitum

$$\int\frac{P\,\partial z}{lz}=A\,l(\alpha+1)+B\,l(\beta+1)+C\,l(\gamma+1)+D\,l(\delta+1)\text{ etc.}$$

q. e. d.

Corollarium 1.

§. 28. Quia supponimus

$$A+B+C+D+\text{ etc.}=0,$$

evidens est, formulam

$$P=A\,z^{\alpha}+B\,z^{\beta}+C\,z^{\gamma}+D\,z^{\delta}+\text{etc.}$$

factorem habere $z-1$, quemadmodum jam ante notavimus.

Corollarium 2.

§. 29. Quia est

$$(z-1)^n=z^n-\tfrac{n}{1}z^{n-1}+\frac{n(n-1)}{1\cdot 2}z^{n-2}-\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}z^{n-3},$$

hoc valore loco P posito, erit $A=1$ et $\alpha=n$, deinde

$$B=-\tfrac{n}{1}\text{ et }\beta=n-1,$$

porro

$$C=\frac{n(n-1)}{1\cdot 2}\text{ et }\gamma=n-2,\text{ etc.}$$

hinc igitur erit

$$\int\frac{(z-1)^n\partial z}{lz}=l(n+1)-\tfrac{n}{1}ln+\frac{n\cdot(n-1)}{1\cdot 2}l(n-1)-\frac{n\cdot(n-1)(n-2)}{1\cdot 2\cdot 3}l(n-2)$$
$$+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}l(n-3)+\text{ etc.}$$

si modo exponens n fuerit nihilo major, vel saltem unitate non minor, quia alioquin casu $z=1$ fractio $\frac{(z-1)^n}{lz}$ fieret infinita; hoc autem non obstante area supra considerata fiet finita, ita ut sufficiat, dummodo sit $n>0$.

Exemplum 2.

§. 30. Sit

$$=\frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}, \text{ erit } \int P\,\partial z=\frac{\pi}{2\,n\cos.\frac{\pi u}{2n}};$$

si quidem post integrationem ponatur $z=1$, quem ergo valorem litterae S tribuimus. Nunc spectata z ut constante, erit

$$\int P\,\partial u=\frac{1}{1+z^{2n}}\left(\int z^{n-u-1}\,\partial u+\int z^{n+u-1}\,\partial u\right),$$

ideoque

$$\int P\,\partial u=-\frac{z^{n-u-1}+z^{n+u-1}}{(1+z^{2n})lz},$$

unde fiet

$$\int S\,\partial u=\int\frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\cdot\frac{\partial z}{lz};$$

cum igitur sit $\cos.\frac{\pi u}{2n}=\sin.\frac{\pi(n-u)}{2n}$, erit

$$\int S\,\partial u=\int\frac{\pi\,\partial u}{2\,n\sin.\frac{\pi(n-u)}{2n}},$$

hinc si ponamus

$$\frac{\pi(n-u)}{2n}=\Phi,\quad \text{erit } \partial\Phi=-\frac{\pi\,\partial u}{2n},$$

ideoque

$$\int S\,\partial u=-\int\frac{\partial\Phi}{\sin.\Phi}=-l\,\text{tang.}\tfrac{1}{2}\Phi,$$

quocirca habebimus

$$\int S\,\partial u=-l\,\text{tang.}\frac{n(n-u)}{4n},$$

ita ut posito post integrationem $z=1$, assecuti sumus hanc integrationem

$$\int\frac{z^{n-u-1}+z^{n+u-1}}{1+z^{2n}}\cdot\frac{\partial z}{lz}=-l\,\text{tang.}\frac{\pi(n-u)}{4n}=$$
$$+l\,\text{tang.}\frac{\pi(n+u)}{4n}.$$

Exemplum 3.

§. 31. Sit

$$P = \frac{z^{n-u-1} - z^{n+u-1}}{1 - z^{2n}}, \text{ erit}$$

$$\int P \partial z = \frac{\pi}{2n} \text{ tang. } \frac{\pi u}{2n} = S$$

unde fit

$$\int S \partial u = -l \cos. \frac{\pi u}{2n},$$

hinc cum sit

$$\int P \partial u = -\frac{z^{n-u-1} - z^{n+u-1}}{(1 - z^{2n}) l z},$$

nanciscimur sequentem integrationem, si quidem integrale a termino $z = 0$ usque ad terminum $z = 1$ extendatur,

$$\int \frac{z^{n-u-1} + z^{n+u-1}}{1 - z^{2n}} \cdot \frac{\partial z}{l z} = + l \cos. \frac{\pi u}{2n}.$$

Haec quidem duo posteriora exempla jam ante fusius expedivi; unde iis magis evolvendis non immoror, sed ad sequens problema progredior.

Problema.

§. 32. *Si proponantur hae duae series infinitae*

$P = z \cos. u + z^2 \cos. 2u + z^3 \cos. 3u + z^4 \cos. 4u + z^5 \cos. 5u + \text{etc. et}$
$Q = z \sin. u + z^2 \sin. 2u + z^3 \sin. 3u + z^4 \sin. 4u + z^5 \sin. 5u + \text{etc.}$
quae binas variabiles z et u involvunt, invenire relationes inter formulas integrales $\int \frac{P \partial z}{z}$, $\int P \partial u$ *et* $\int \frac{Q \partial z}{z}$, $\int Q \partial u$, *aliasque formulas integrales per continuam integrationem inde natas.*

Solutio.

Cum utraque series sit recurrens, reperitur per formulas finitas

$$P = \frac{z \cos. u - zz}{1 - 2z \cos. u + zz} \text{ et } Q = \frac{z \sin. u}{1 - 2z \cos. u + zz},$$

unde fit

$$\int \frac{P \partial z}{z} = \int \frac{\partial z \cos. u - z \partial z}{1 - 2z \cos. u + zz} = -l\sqrt{(1 - 2z \cos. u + zz)} \text{ et}$$

$$\int Q \partial u = \int \frac{z \partial u \sin. u}{1 - 2z \cos. u + zz} = +l\sqrt{(1 - 2z \cos. u + zz)},$$

ita ut sit

$$\int \frac{P \partial z}{z} = -\int Q \partial u;$$

tum vero etiam erit

$$\int \frac{Q \partial z}{z} = \int \frac{\partial z \sin. u}{1 - 2z \cos. u + zz} = \text{arc. tang. } \frac{z \sin. u}{1 - z \cos. u};$$

at si iste arcus differentietur sumto solo angulo u variabili, erit

$$\frac{1}{\partial u} \partial \,.\, \text{arc. tang. } \frac{z \sin. u}{1 - z \cos. u} = \frac{z \cos. u - zz}{1 - 2z \cos. u + zz},$$

ita ut sit

$$\int \frac{Q \partial z}{z} = \int P \partial u.$$

§. 33. Verum eaedem relationes facilius ex ipsis seriebus derivantur: cum enim sit

$$P = z \cos. u + z^2 \cos. 2u + z^3 \cos. 3u + z^4 \cos. 4u + \text{etc.}$$

erit

$$\int \frac{P \partial z}{z} = \frac{z \cos. u}{1} + \frac{zz \cos. 2u}{2} + \frac{z^3 \cos. 3u}{3} + \text{etc. et}$$

$$\int P \partial u = \frac{z \sin. u}{1} + \frac{zz \sin. 2u}{2} + \frac{z^3 \sin. 3u}{3} + \text{etc.}$$

et quia est

$$Q = z \sin. u + zz \sin. 2u + z^3 \sin. 3u + \text{etc. erit}$$

$$\int \frac{Q \partial z}{z} = \frac{z \sin. u}{1} + \frac{zz \sin. 2u}{2} + \frac{z^3 \sin. 3u}{3} + \text{etc. et}$$

$$\int Q \partial u = -\frac{z \cos. u}{1} - \frac{zz \cos. 2u}{2} - \frac{z^3 \cos. 3u}{3} - \text{etc.}$$

unde manifestum est fore

$$\int \frac{P\partial z}{z} = -\int Q\partial u \text{ et } \int \frac{Q\partial z}{z} = \int P\partial u.$$

§. 34. Quo hoc modo ulterius progredi liceat, statuamus brevitatis gratia

$$P' = \frac{z\cos.u}{1} + \frac{zz\cos.2u}{2} + \frac{z^3\cos.3u}{3} + \text{etc. et } Q' = \frac{z\sin.u}{1} + \frac{zz\sin.2u}{2} + \frac{z^3\sin.3u}{3} + \text{etc.}$$
$$P'' = \frac{z\cos.u}{1^2} + \frac{zz\cos.2u}{2^2} + \frac{z^3\cos.3u}{3^2} + \text{etc. et } Q'' = \frac{z\sin.u}{1^2} + \frac{zz\sin.2u}{2^2} + \frac{z^3\sin.3u}{3^2} + \text{etc.}$$
$$P''' = \frac{z\cos.u}{1^3} + \frac{zz\cos.2u}{2^3} + \frac{z^3\cos.3u}{3^3} + \text{etc. et } Q''' = \frac{z\sin.u}{1^3} + \frac{zz\sin.2u}{2^3} + \frac{z^3\sin.3u}{3^3} + \text{etc.}$$
$$P'''' = \frac{z\cos.u}{1^4} + \frac{zz\cos.2u}{2^4} + \frac{z^3\cos.3u}{3^4} + \text{etc. et } Q'''' = \frac{z\sin.u}{1^4} + \frac{zz\sin.2u}{2^4} + \frac{z^3\sin.3u}{3^4} + \text{etc.}$$

etc. etc. etc. etc.

et hinc comparationes ante inventae continuabuntur

$$P' = \int \frac{P\partial z}{z} = -\int Q\partial u, \quad Q' = \int \frac{Q\partial z}{z} = \int P\partial u,$$
$$P'' = \int \frac{P'\partial z}{z} = -\int Q'\partial u, \quad Q'' = \int \frac{Q'\partial z}{z} = \int P'\partial u,$$
$$P''' = \int \frac{P''\partial z}{z} = -\int Q''\partial u, \quad Q''' = \int \frac{Q''\partial z}{z} = \int P''\partial u,$$
$$P'''' = \int \frac{P'''\partial z}{z} = -\int Q'''\partial u, \quad Q'''' = \int \frac{Q'''\partial z}{z} = \int P'''\partial u,$$

etc. etc. etc. etc.

unde plures insignes relationes deduci possunt.

§. 35. Maxime autem notatu dignae et ad nostrum institutum accommodatae sunt eae relationes, ubi formulae integrales, in quibus sola z est variabilis, reducuntur ad alias formulas integrales, in quibus sola u est variabilis; cujusmodi sunt, quae sequuntur

$$P' = \int \frac{P\partial z}{z} = -\int Q\partial u,$$
$$P'' = \int \frac{\partial z}{z} \int \frac{P\partial z}{z} = -\int \partial u \int P\partial u,$$
$$P''' = \int \frac{\partial z}{z} \int \frac{\partial z}{z} \int \frac{P\partial z}{z} = +\int \partial u \int \partial u \int Q\partial u,$$

$$P'''' = \int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{P\partial z}{z} = +\int\partial u\int\partial u\int\partial u\int P\partial u,$$
$$P^{V} = \int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{P\partial z}{z} = -\int\partial u\int u\int\partial u\int\partial u\int Q\partial u,$$
etc. etc. etc.

Similique modo pro altero genere

$$Q' = \int\frac{Q\partial z}{z} = +\int P\partial u,$$
$$Q'' = \int\frac{\partial z}{z}\int\frac{Q\partial z}{z} = -\int\partial u\int Q\partial u,$$
$$Q''' = \int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{Q\partial z}{z} = -\int\partial u\int\partial u\int\partial P u,$$
$$Q'''' = \int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{Q\partial z}{z} = +\int\partial u\int\partial u\int\partial u\int P\partial u,$$
$$Q^{V} = \int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{Q\partial z}{z} = +\int\partial u\int\partial u\int\partial u\int\partial u\int P\partial u,$$
etc. etc. etc.

§. 36. Quod si jam nostrarum serierum, sive quod eodem redit, quantitatum

P, P', P'', P''', P'''', etc. et Q, Q', Q'', Q''', Q'''', etc. eos tantum valores desideremus, quos adipiscuntur posito $z = 1$, hoc commodi assequimur, ut in formulis integralibus, ubi solus angulus u pro variabili habetur, statim ante integrationes ponere liceat $z = 1$, hoc autem facto erit

$$P = \frac{\cos. u - 1}{2 - 2\cos. u} = -\tfrac{1}{2} \text{ et } Q = \frac{\sin. u}{2 - 2\cos. u} = \tfrac{1}{2}\cot.\tfrac{1}{2}u,$$

tum vero porro

$$\int P\partial u = A - \tfrac{1}{2}u,$$
$$\int\partial u\int P\partial u = B + Au - \tfrac{1}{4}uu,$$
$$\int\partial u\int\partial u\int P\partial u = C + Bu + \tfrac{1}{2}Auu - \tfrac{1}{12}u^3,$$
$$\int\partial u\int\partial u\int\partial u\int P\partial u = D + Cu + \tfrac{1}{2}Buu + \tfrac{1}{6}Au^3 - \tfrac{1}{48}u^4,$$

at pro formulis, ubi est Q, calculus non tam concinne succedit; erit enim

$$Q = \tfrac{1}{2}\cot.\tfrac{1}{2}u,$$
$$\int Q\partial u = l\sin.\tfrac{1}{2}u,$$
$$\int\partial u\int Q\partial u = \int\partial u\, l\sin.\tfrac{1}{2}u,$$

quae formula cum omnem integrationem respuat, vix ulterius progredi licet; interim tamen erit

$$\int\partial u\int\partial u\int Q\partial u = \int\partial u\int\partial u\, l\sin.\tfrac{1}{2}u,$$
$$\int\partial u\int\partial u\int\partial u\int\partial Q\, u = \int\partial u\int\partial u\int\partial u\, l\sin.\tfrac{1}{2}u.$$

§. 37. Quod ad priores formulas variabilem z involventes attinet, per notas reductiones elicitur

$$\int\frac{P'\partial z}{z} = \int\frac{\partial z}{z}\int\frac{P\partial z}{z} = lz\int\frac{P\partial z}{z} - \int\frac{P\partial z}{z}lz,$$

ubi prius membrum $lz\int P\partial z$ evanescit posito $z = 1$, tum vero

$$\int\frac{\partial z}{z}\int\frac{P'\partial z}{z} = \int\frac{\partial z}{z}\int\frac{\partial z}{z}\int\frac{P\partial z}{z} = +\int\frac{P\partial z}{z}\cdot\frac{(lz)^2}{2},$$

quibus expressionibus ulterius exhibitis colligimus fore

$$P' = \int\frac{P\partial z}{z}, \qquad Q' = \int\frac{Q\partial z}{z},$$
$$P'' = -\int\frac{P\partial z}{z}lz, \qquad Q'' = -\int\frac{Q\partial z}{z}lz,$$
$$P''' = +\int\frac{P\partial z}{z}\cdot\frac{(lz)^2}{1\cdot 2}, \qquad Q''' = +\int\frac{Q\partial z}{z}\cdot\frac{(lz)^2}{1\cdot 2},$$
$$P^{IV} = -\int\frac{P\partial z}{z}\cdot\frac{(lz)^3}{1\cdot 2\cdot 3} \qquad Q^{IV} = -\int\frac{Q\partial z}{z}\cdot\frac{(lz)^3}{1\cdot 2\cdot 3}.$$

§. 38. Ex his igitur sequentium formularum integralium valores assignare possumus, casu quo $z = 1$,

$$P = -\tfrac{1}{2},$$
$$P' = \int\frac{P\partial z}{z} = -l\sin.\tfrac{1}{2}u,$$
$$P'' = -\int\frac{P\partial z}{z}lz = -B - Au + \tfrac{1}{4}uu,$$
$$P''' = +\int\frac{P\partial z}{z}\cdot\frac{(lz)^2}{1\cdot 2} = \int\partial u\int\partial u\, l\sin.\tfrac{1}{2}u,$$

$$P'''' = -\int \frac{P\partial z}{z} \cdot \frac{(lz)^3}{1\cdot2\cdot3} = D + Cu + \tfrac{1}{2}Buu + \tfrac{1}{6}Au^3 - \tfrac{1}{48}u^4,$$
$$P^{V} = +\int \frac{P\partial z}{z} \cdot \frac{(lz^4)}{1\cdot2\cdot3\cdot4} = \int\partial u\int\partial u\int\partial u\int\partial u\, l\sin.\tfrac{1}{2}u,$$
etc. etc.

Eodem modo

$$Q = \tfrac{1}{2}\cot.\tfrac{1}{2}u,$$
$$Q' = \int\frac{Q\partial z}{z} = A - \tfrac{1}{2}u,$$
$$Q'' = -\int\frac{Q\partial z}{z}\cdot\frac{lz}{1} = -\int\partial u l\sin.\tfrac{1}{2}u,$$
$$Q''' = +\int\frac{Q\partial z}{z}\cdot\frac{(lz)^2}{2} = -C - Bu - \tfrac{1}{2}Auu + \tfrac{1}{12}u^3,$$
$$Q'''' = -\int\frac{Q\partial z}{z}\cdot\frac{(lz)^3}{6} = \int\partial u\int\partial u\int\partial u\, l\sin.\tfrac{1}{2}u,$$
$$Q^{V} = +\int\frac{Q\partial z}{z}\cdot\frac{(lz)^4}{24} = E + Du + \tfrac{1}{2}Cuu + \tfrac{1}{6}Bu^3 + \tfrac{1}{24}Au^4 - \tfrac{1}{240}u^5,$$
etc. etc.

§. 39. Cum igitur sit

$$P = \frac{z\cos.u - zz}{1 - 2z\cos.u + zz} \quad\text{et}\quad Q = \frac{z\sin.u}{1 - 2z\cos.u + zz},$$

hactenus id sumus assecuti, ut harum duarum formularum integralium

$$\int\frac{\partial z(\cos.u - z)}{1 - 2z\cos.u + zz}(lz)^n \quad\text{et}\quad \int\frac{\partial z\sin.u}{1 - 2z\cos.u + zz}(lz)^n$$

valores casu $z = 1$ commode per angulum u assignare valeamus, si modo constaret, quo facto quantitates A, B, C, D, etc. determinari oporteat, id quod vix alio modo nisi per ipsas series, unde hae quantitates sunt natae, fieri posse videtur.

§. 40. Omissis igitur formulis integralibus, quae quantitatem Q involvunt, quippe quarum integratio minus succedit, alteras tantum consideremus, et posito statim $z = 1$ ubi sit $P = -\frac{1}{2}$, ita ut sit

$$\cos.u + \cos.2u + \cos.3u + \cos.4u + \text{etc.} = -\tfrac{1}{2},$$

si per ∂u multiplicemus et integremus, habebimus

$$Q' = \frac{\sin. u}{1} + \frac{\sin. 2u}{2} + \frac{\sin. 3u}{3} + \frac{\sin. 4u}{4} + \frac{\sin. 5u}{5} + \text{etc.} = A - \tfrac{1}{2}u,$$

quae constans nihilo aequalis videri potest, quia posito $u = 0$ summa seriei evanescere videtur; at sumto angulo u infinite parvo series praebebit

$$u + u + u + u + u + u + \text{etc. et infinitum;}$$

notum autem est, talem seriem summam finitam habere posse, unde hoc casu omisso statuamus $u = \pi$, seu potius $u = \pi + \omega$, prodibitque haec series existente ω angulo infinite parvo,

$$-\omega + \omega - \omega + \omega - \omega + \omega - \omega + \text{etc.}$$

ubi, quia signa alternantur, nullum est dubium, quin summa seriei evanescat, quae cum esse debeat $A - \frac{\pi}{2}$, evidens est, fieri constantem $A = \frac{1}{2}\pi$, ita, ut jam habeamus

$$Q' = \frac{\sin. u}{1} + \frac{\sin. 2u}{2} + \frac{\sin. 3u}{3} + \frac{\sin. 4u}{4} + \frac{\sin. 5u}{5} + \text{etc.} = \frac{\pi - u}{2}.$$

Hoc modo constantem determinandi Illustr. *Daniel Bernoulli* primus est usus, qui praeterea multa praeclara circa indolem harum serierum annotavit.

§. 41. Multiplicemus porro hanc ultimam seriem per $-\partial u$, et integratio dabit

$$P'' = \frac{\cos. u}{1^2} + \frac{\cos. 2u}{2^2} + \frac{\cos. 3u}{3^2} + \frac{\cos. 4u}{4^2} + \text{etc.} = B - \frac{\pi u}{2} + \frac{uu}{4},$$

ad quam constantem inveniendam ponamus primo $u = 0$, fietque

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \text{etc.} = B.$$

Cujus seriei summam jam pridem primus demonstravi esse $= \frac{\pi\pi}{6}$; verum si haec veritas nobis esset ignota, egregia illa methodo a magno *Bernoullio* adhibita utamur, ac ponamus $u = \pi$ eritque

$$-\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \frac{1}{5^2} + \frac{1}{6^2} - \text{etc.} = B - \frac{\pi\pi}{2} + \frac{\pi\pi}{4} = B - \frac{\pi\pi}{4};$$

ambae hae series additae dabunt

$$\frac{2}{2^2}+\frac{2}{4^2}+\frac{2}{6^2}+\frac{2}{8^2}+\text{etc.}=2B-\frac{\pi\pi}{4},$$

cujus duplum praebet

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}=4B-\frac{\pi\pi}{2}=B;$$

unde colligitur $B=\frac{\pi\pi}{6}$, ita ut sit

$$P''=\frac{\cos.u}{1^2}+\frac{\cos.2u}{2^2}+\frac{\cos.3u}{3^2}+\frac{\cos.4u}{4^2}+\text{etc.}=\frac{\pi\pi}{6}-\frac{\pi u}{2}+\frac{uu}{4}.$$

§. 42. Eodem modo ulterius progrediamur, et denuo per ∂u multiplicando et integrando adipiscimur

$$Q'''=\frac{\sin.u}{1^3}+\frac{\sin.2u}{2^3}+\frac{\sin.3u}{3^3}+\frac{\sin.4u}{4^3}+\text{etc.}$$
$$=C+\frac{\pi\pi u}{6}-\frac{\pi uu}{4}+\frac{u^3}{12},$$

ubi si statuatur $u=0$, summa seriei manifesto evanescit, prodiret enim posito $u=\omega$

$$\frac{\omega}{1^2}+\frac{\omega}{2^2}+\frac{\omega}{3^2}+\frac{\omega}{4^2}+\text{etc.}=\frac{\omega\pi\pi}{6},$$

quae ob $\omega=0$ fit $=0$, sicque erit $C=0$, ideoque

$$Q'''=\frac{\sin.u}{1^3}+\frac{\sin.2u}{2^3}+\frac{\sin.3u}{3^3}+\frac{\sin.4u}{4^3}+\text{etc.}=\frac{\pi\pi u}{6}-\frac{\pi uu}{4}+\frac{u^3}{12}$$

§. 43. Ducatur haec series in $-\partial u$, et integratio praebebit

$$P^{IV}=\frac{\cos.u}{1^4}+\frac{\cos.2u}{2^4}+\frac{\cos.3u}{3^4}+\frac{\cos.4u}{4^4}+\text{etc.}$$
$$=D-\frac{\pi\pi uu}{12}+\frac{\pi u^3}{12}-\frac{u^4}{48},$$

hinc sumto $u=0$ fiet

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\text{etc.}=D,$$

nunc vero fiat etiam $u=\pi$, fietque

$$-\frac{1}{1^4}+\frac{1}{2^4}-\frac{1}{3^4}+\frac{1}{4^4}-\frac{1}{5^4}+\text{etc.}=D-\frac{\pi^4}{48},$$

hae autem ambae series additae dant

$$\frac{2}{2^4}+\frac{2}{4^4}+\frac{2}{6^4}+\frac{2}{8^4}+\text{etc.} = 2D-\frac{\pi^4}{48},$$

quae octies sumta ut numeratores fiant $= 2^4$, praebebit

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.} = 16D-\frac{\pi^4}{6},$$

unde oritur $D=\frac{\pi^4}{90}$, quae est eadem summa seriei

$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\text{etc.}$$

quam jam dudum inveneram, habebimus jam

$$P'''' = \frac{\cos. u}{1^4}+\frac{\cos. 2u}{2^4}+\frac{\cos. 3u}{3^4}+\frac{\cos. 4u}{4^4}+\text{etc.}$$
$$= \frac{\pi^4}{90}-\frac{\pi^2 u^2}{12}+\frac{\pi u^3}{12}-\frac{u^4}{48}.$$

§. 44. Multiplicando iterum per ∂u et integrando consequimur

$$Q^{V} = \frac{\sin. u}{1^5}+\frac{\sin. 2u}{2^5}+\frac{\sin. 3u}{3^5}+\frac{\sin. 4u}{4^5}+\text{etc.}$$
$$= E+\frac{\pi^4 u}{90}-\frac{\pi^2 u^3}{36}+\frac{\pi u^4}{48}-\frac{u^5}{240};$$

ubi uti in casu penultimo constans E iterum fit $= 0$, ita ut habeamus

$$Q^{V} = \frac{\sin. u}{1^5}+\frac{\sin. 2u}{2^5}+\frac{\sin. 3u}{3^5}+\frac{\sin. 4u}{4^5}+\text{etc.}$$
$$= \frac{\pi^4 u}{90}-\frac{\pi^2 u^3}{36}+\frac{\pi u^4}{48}-\frac{u^5}{240}.$$

§. 45. Multiplicemus denuo per $-\partial u$, prodibitque integrando

$$P^{VI} = \frac{\cos. u}{1^6}+\frac{\cos. 2u}{2^6}+\frac{\cos. 3u}{3^6}+\frac{\cos. 4u}{4^6}+\text{etc.}$$
$$= F-\frac{\pi^4}{90}\cdot\frac{uu}{2}+\frac{\pi\pi}{6}\cdot\frac{u^4}{24}-\frac{\pi}{2}\cdot\frac{u^5}{120}+\frac{1}{2}\cdot\frac{u^6}{720};$$

ubi ad constantem determinandam ponatur $u = 0$, eritque

$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.} = F,$$

tum vero sumatur $u=\pi$, et fiet

$$-\frac{1}{1^6}+\frac{1}{2^6}-\frac{1}{3^6}+\frac{1}{4^6}-\text{etc.} = F-\frac{\pi^6}{480},$$

quae additae dant

$$\frac{2}{2^6}+\frac{2}{4^6}+\frac{2}{6^6}+\frac{2}{8^6}+\text{etc.}=2F-\frac{\pi^6}{480},$$

quae multiplicetur per 32, ut omnes numeratores fiant $64=2^6$, et orietur

$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\text{etc.}=64F-\frac{\pi^6}{15}=F;$$

unde colligitur $F=\frac{\pi^6}{945}$, ita ut sit

$$P^{VI}=\frac{\cos. u}{1^6}+\frac{\cos. 2u}{2^6}+\frac{\cos. 3u}{3^6}+\frac{\cos. 4u}{4^6}+\text{etc.}$$
$$=\frac{\pi^6}{945}-\frac{\pi^4}{90}\cdot\frac{u^2}{2}+\frac{\pi^2}{6}\cdot\frac{u^4}{24}-\frac{\pi}{2}\cdot\frac{u^5}{120}+\frac{1}{2}\cdot\frac{u^6}{720}.$$

§. 46. Has series ulterius continuare superfluum foret, cum lex progressionis jam satis sit manifesta, praecipue si in subsidium vocentur summationes potestatum reciprocarum parium, quas olim usque ad potestatem trigesimam supputatas dedi. Quod quo clarius perspiciatur, istas summas sequenti modo repraesentemus

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\text{etc.}=\alpha\pi\pi,\quad \text{ut sit } \alpha=\frac{1}{6}$$
$$\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\frac{1}{5^4}+\frac{1}{6^4}+\text{etc.}=\beta\pi^4,\quad \text{ut sit } \beta=\frac{1}{90}$$
$$\frac{1}{1^6}+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\text{etc.}=\gamma\pi^6,\quad \text{ut sit } \gamma=\frac{1}{945}$$
$$\frac{1}{1^8}+\frac{1}{2^8}+\frac{1}{3^8}+\frac{1}{4^8}+\frac{1}{5^8}+\frac{1}{6^8}+\text{etc.}=\delta\pi^8,\quad \text{ut sit } \delta=\frac{1}{9450}$$

etc. etc. etc.

atque his positis, sequentes habebimus integrationes, pro casu scilicet $z=1$,

$$Q'=+\int\frac{\partial z\,\sin. u}{1-2z\cos. u+zz}=\tfrac{1}{2}\pi-\tfrac{1}{2}u=\text{Arc. tang. }\frac{\sin. u}{1-\cos. u}$$
$$P''=-\int\frac{\partial z(\cos. u-z)}{1-2z\cos. u+zz}\cdot\frac{lz}{1}=\alpha\pi\pi-\tfrac{1}{2}\pi u+\tfrac{1}{2}\cdot\frac{uu}{2}$$
$$Q'''=+\int\frac{\partial z\,\sin. u}{1-2z\cos. u+zz}\cdot\frac{(lz)^2}{2}=\alpha\pi\pi\frac{u}{1}-\tfrac{1}{2}\pi\cdot\frac{uu}{2}+\tfrac{1}{2}\cdot\frac{u^3}{6}$$
$$P^{IV}=-\int\frac{\partial z(\cos. u-z)}{1-2z\cos. u+zz}\cdot\frac{(lz)^3}{6}=\beta\pi^4-\alpha\pi\pi\cdot\frac{uu}{2}+\tfrac{1}{2}\pi\cdot\frac{u^3}{6}-\tfrac{1}{2}\cdot\frac{u^4}{24}$$

$$Q^{V} = +\int \frac{\partial z \sin. u}{1 - 2z \cos. u + zz} \cdot \frac{(lz)^4}{24} = \beta\pi^4 . \frac{u}{1} - \alpha\pi\pi \cdot \frac{u^3}{6} + \frac{1}{2}\pi \cdot \frac{u^4}{24} - \frac{1}{2} \cdot \frac{u^5}{120}$$

$$P^{VI} = -\int \frac{\partial z (\cos. u - z)}{1 - 2z \cos. u + zz} \cdot \frac{(lz)^5}{120} = \gamma\pi^6 - \beta\pi^4 \cdot \frac{uu}{2} + \alpha\pi\pi \cdot \frac{u^4}{24} - \frac{1}{2}\pi \cdot \frac{u^5}{120} + \frac{1}{2} \cdot \frac{u^6}{720}$$

$$Q^{VII} = +\int \frac{\partial z \sin. u}{1 - 2z \cos. u + zz} \cdot \frac{(lz)^6}{720} = \gamma\pi^6 . \frac{u}{1} - \beta\pi^4 . \frac{u^3}{6} + \alpha\pi\pi \cdot \frac{u^5}{120} - \frac{1}{2}\pi \cdot \frac{u^6}{720} + \frac{1}{2} \cdot \frac{u^7}{5040}$$

etc. etc. etc.

§. 47. Operae pretium erit, aliquos casus, quibus angulo u datus valor tribuitur, ob oculos exponere. Ponamus igitur $u = 0$, quo casu formulae nostrae alternatim evanescunt, reliquae vero praebebunt

$$-\int \frac{\partial z}{1-z} lz = \alpha\pi\pi = \frac{\pi\pi}{6}$$

$$-\int \frac{\partial z}{1-z} \cdot \frac{(lz)^3}{6} = \beta\pi^4 = \frac{\pi^4}{90}$$

$$-\int \frac{\partial z}{1-z} \cdot \frac{(lz)^5}{120} = \gamma\pi^6 = \frac{\pi^6}{945}$$

his affines sunt formulae, quae oriuntur ex positione $u = \pi$, ubi iterum abeunt alternae sinum u involventes, et remanebunt sequentes

$$\int \frac{\partial z}{1+z} lz = -\frac{\pi\pi}{12} = -\frac{1}{2}\alpha\pi\pi$$

$$\int \frac{\partial z}{1+z} \cdot \frac{(lz)^3}{6} = -\frac{7\pi^4}{720} = -\frac{7}{8}\beta\pi^6$$

$$\int \frac{\partial z}{1+z} \cdot \frac{(lz)^5}{120} = -\frac{31}{32}\gamma\pi^6$$

$$\int \frac{\partial z}{1+z} \cdot \frac{(lz)^7}{720} = -\frac{127}{128}\delta\pi^8.$$

§. 48. Hic notatu dignum occurrit, quod valores alterni, quos hic omisimus, etiam evanescant posito $u = \pi$; deinde non minus notatu dignum est, easdem formulas quoque evanescere posito $u = 2\pi$, sola prima excepta, quippe quae etiam non evanescit posito $u = 0$; reliquae vero, scilicet tertia, quinta, septima etc. certe evanescunt casibus $u = 0$ et $u = \pi$, quin etiam $u = 2\pi$. Quod quo clarius appareat, has formulas per factores repraesentemus, eritque tertiae valor

$$= \tfrac{1}{12}u\,(\pi-u)\,(2\pi-u),$$

quintae vero valor reperitur

$$\tfrac{u}{120}\,(\pi-u)\,(2\pi-u)\,(4\pi\pi+6\pi u-3uu),$$

quod etiam in sequentibus usu venit. In genere autem observari meretur, omnes nostras formulas sola prima excepta eosdem sortiri valores, sive ponatur $u=0$ sine $u=2\pi$, quippe quibus tam idem sinus quam cosinus respondet. Videtur quidem eundem consensum locum habere debere, si ponatur $u=4\pi$ et $u=6\pi$, verum Illustr. *Bernoullius* jam luculenter ostendit, angulum u in his valoribus non ultra quatuor rectos augeri posse. Hujusmodi autem anomalia etiam in omnibus vulgaribus seriebus quibus arcus exprimuntur occurrit; atque adeo in *Leibniziana*, inqua est

$$u=\frac{tang.\,u}{1}-\frac{(tang.\,u)^3}{3}+\frac{(tang.\,u)^5}{5}-\frac{(tang.\,u)^7}{7}+\frac{(tang.\,u)^9}{9}-\text{etc.}$$

angulum u non ultra 180 gr. augere licet. Si enim poneremus $u=180^\circ+u$, foret utique tang. $u=$ tang. u, neque tamen series illa exprimeret arcum $\pi+u$ sed tantum arcum u, cujusmodi phaenomena etiam in aliis similibus seriebus locum habent. Quod autem prima series hinc plerumque excipi debeat, ratio in eo est sita, quod in formula integrali posito $u=0$ denominator fiat $(1-z)$, qui casu $z=1$ evanescit, ideoque formula in infinitum excrescit, id quod in sequentibus, quae per lz sunt multiplicatae, non amplius evenit, quia $\frac{lz}{1-z}$ casu $z=1$ non amplius fit infinitus sed tantum $=-1$, et si major potestas logarithmi adsit, fit adeo $=0$.

§. 49. Ponamus nunc etiam $u=90^\circ$, seu $u=\frac{\pi}{2}$, ut sit cos. $u=0$ et sin. $u=1$, hocque casu omnes formulae generales sequentes obtinebunt valores

$$\int\frac{\partial z}{1+zz}=\frac{\pi}{4}$$

$$\int \frac{z\partial z}{1+zz}\, lz = -\frac{\pi\pi}{48}$$
$$\int \frac{\partial z}{1+zz} \cdot \frac{(lz)^2}{2} = \frac{\pi^3}{32}$$
$$\int \frac{z\partial z}{1+zz} \cdot \frac{(lz)^3}{6} = -\frac{7\pi^4}{90\cdot 128}.$$
etc.

§. 50. Consideremus etiam casum $u = 60^\circ$, sive $u = \frac{\pi}{3}$, ut sit cos. $u = \frac{1}{2}$ et sin. $u = \frac{\sqrt{3}}{2}$, et formulae generales perducent ad sequentia integralia

$$\frac{\sqrt{3}}{2}\int \frac{\partial z}{1-z+zz} = \frac{\pi}{3}$$
$$\frac{1}{2}\int \frac{\partial z\,(1-2z)}{1-z+zz}\, lz = -\frac{\pi\pi}{36}$$
$$\frac{\sqrt{3}}{2}\int \frac{\partial z}{1-z+zz} \cdot \frac{(lz)^2}{2} = \frac{5\pi^3}{162}.$$

Simili modo si ponamus $u = 120^\circ = \frac{2\pi}{3}$, ut sit cos. $u = -\frac{1}{2}$ et sin. $u = \frac{\sqrt{3}}{2}$, sequentes integrationes istis affines prodibunt

$$\frac{\sqrt{3}}{2}\int \frac{\partial z}{1+z+zz} = \frac{\pi}{6}$$
$$\frac{1}{2}\int \frac{\partial z\,(1+2z)}{1+z+zz}\, lz = -\frac{\pi\pi}{18}$$
$$\frac{\sqrt{3}}{2}\int \frac{\partial z}{1+z+zz} \cdot \frac{(lz)^2}{2} = \frac{2\pi^3}{81},$$

sicque pro lubitu numerus hujusmodi integrationum specialium augeri poterit.

§. 51. Quemadmodum istae integrationes memorabiles ex priore serie nostra P posito $z = 1$ sunt deductae, ita eodem modo alteram seriem Q pertractemus. Cum igitur sit

$$Q = \sin. u + \sin. 2u + \sin. 3u + \sin. 4u + \text{etc.} = \tfrac{1}{2}\cot. \tfrac{1}{2}u,$$

si per $-\partial u$ multiplicemus et integremus, reperitur series

$$P' = \frac{\cos. u}{1} + \frac{\cos. 2u}{2} + \frac{\cos. 3u}{3} + \frac{\cos. 4u}{4} + \text{etc.} = \tfrac{1}{2} - l\sin. \tfrac{1}{2}u + A,$$

pro qua constante determinanda ponatur $u = \pi$, ut sit

$$-1 + \tfrac{1}{2} - \tfrac{1}{3} + \tfrac{1}{4} - \tfrac{1}{5} + \tfrac{1}{6} - \text{etc.} = A,$$

quocirca fit $A = -l2$, ita ut habeamus

$$P' = \frac{\cos. u}{1} + \frac{\cos. 2u}{2} + \frac{\cos. 3u}{3} + \frac{\cos. 4u}{4} + \text{etc.} = -l2 \sin. \tfrac{1}{2}u,$$

pro quo valore scribamus brevitatis gratia $\Delta : u$, si quidem eum spectamus tanquam certam ipsius u functionem, ita ut sit $P' = \Delta : u$.

§. 52. Multiplicando porro per ∂u et integrando, nanciscimur hanc seriem

$$Q'' = \frac{\sin. u}{1^2} + \frac{\sin. 2u}{2^2} + \frac{\sin. 3u}{3^2} + \frac{\sin. 4u}{4^2} + \text{etc.} = \int \partial u \Delta : u = \Delta' : u;$$

ubi haec formula integralis involvet certam constantem, quam facile definire licet ex casu $u = 0$, quia enim series evanescit, fieri debet $\Delta' : 0 = 0$, sicque integratio plene determinatur.

§. 53. Si eodem modo ulterius progrediamur, multiplicando per $-\partial u$, prodibit haec series

$$P''' = \frac{\cos. u}{1^3} + \frac{\cos. 2u}{2^3} + \frac{\cos. 3u}{3^3} + \frac{\cos. 4u}{4^3} + \text{etc.} = -\int \partial u \Delta' : u = \Delta'' : u.$$

Jam ad constantem, quae in hac expressione continetur, definiendam, sit $1^0\ u = 0$, eritque

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.} = \Delta'' : 0.$$

Sit $2^0.\ u = \pi$, et fiet

$$-\frac{1}{1^3} + \frac{1}{2^3} - \frac{1}{3^3} + \frac{1}{4^3} - \frac{1}{5^3} + \text{etc.} = \Delta'' : \pi,$$

quibus additis prodit

$$\frac{2}{2^3} + \frac{2}{4^3} + \frac{2}{6^3} + \frac{2}{8^3} + \text{etc.} = \Delta'' 0 + \Delta'' : \pi,$$

hacque quatuor sumta erit

$$\frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \text{etc.} = 4\Delta'' : 0 + 4\Delta'' : \pi = \Delta'' : 0,$$

unde oritur

$$3\,\Delta'' : 0 + 4\Delta'' : \pi = 0;$$

ex qua constans in formulam nostram integralem

$$\Delta'' : u = -\int \partial u \Delta' u$$

ingressa determinari debet.

§. 54. Multiplicemus denuo per ∂u, et integremus, prodibitque

$$Q^{IV} = \frac{\sin. u}{1^4} + \frac{\sin. 2u}{2^4} + \frac{\sin. 3u}{3^4} + \frac{\sin. 4u}{4^4} + \text{etc.} = \int \partial u \Delta'' : u = \Delta''' : u,$$

atque haec functio $\Delta''' : u$ ita debet determinari, ut evanescat sumto $u = 0$, sive ut fiat $\Delta''' : 0 = 0$. Eodem modo ulterius progrediendo fiet

$$P^V = \frac{\cos. u}{1^5} + \frac{\cos. 2u}{2^5} + \frac{\cos. 3u}{3^5} + \frac{\cos. 4u}{4^5} + \text{etc.} = -\int \partial u \Delta''' : u = \Delta'''' : u,$$

hujusque functionis indoles sequenti modo determinabitur: ponatur scilicet ut hactenus $u = 0$, et $u = \pi$, eritque

$$\frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \text{etc.} = \Delta^{IV} : 0, \text{ et}$$

$$-\frac{1}{1^5} + \frac{1}{2^5} - \frac{1}{3^5} + \frac{1}{4^5} - \frac{1}{5^5} + \text{etc.} = \Delta^{IV} : \pi,$$

hinc addendo

$$\frac{2}{2^5} + \frac{2}{4^5} + \frac{2}{6^5} + \frac{2}{8^5} + \text{etc.} = \Delta^{IV} : 0 + \Delta^{IV} : \pi,$$

et multiplicando per 16

$$\frac{1}{1^5} + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \text{etc.} = 16\Delta^{IV} : 0 + 16\Delta^{IV} : \pi = \Delta^{IV} : 0,$$

sicque fieri debebit

$$15\Delta^{IV} : 0 + 16\Delta^{IV} \pi = 0 \text{ etc.}$$

§. 55. Hinc igitur sequentes adipiscemur integrationes pro casu $z = 1$

$$\text{I. } -\int \frac{\partial z (\cos. u - z)}{1 - 2z\cos. u + zz} = -l2 \sin. \tfrac{1}{2} u = \Delta : u$$

$$\text{II. } \int \frac{\partial z \sin. u}{1 - 2z \cos. u + zz} lz = \int \partial u \Delta u = \Delta' : u$$

$$\text{III. } -\int \frac{\partial z (\cos. u - z)}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^2}{2} = -\int \partial u \Delta' u = \Delta'' : u$$

$$\text{IV.}\quad \int \frac{\partial z \sin. u}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^3}{6} = \int \partial u \Delta'' u = \Delta''' u$$

$$\text{V.} - \int \frac{\partial z (\cos. u - z)}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^4}{24} = - \int \partial u \Delta''' u = \Delta^{\mathrm{IV}} u$$

$$\text{VI.}\quad \int \frac{\partial z \sin. u}{1 - 2z\cos. u + zz} \cdot \frac{(lz)^5}{120} = \int \partial u \Delta^{\mathrm{IV}} u = \Delta^{\mathrm{V}} u$$

etc. etc. etc. etc.

Has autem expressiones facile quousque libuerit continuare licet, si modo integratio cujusque integralis rite instituatur; conditiones autem, quas impleri oportet, sequenti modo referri possunt

$\Delta' : 0 = 0$	$3\Delta'' : 0 + 4\Delta'' : \pi = 0$
$\Delta''' : 0 = 0$	$15\Delta^{\mathrm{IV}} : 0 + 16\Delta^{\mathrm{IV}} : \pi = 0$
$\Delta^{\mathrm{V}} : 0 = 0$	$63\Delta^{\mathrm{VI}} : 0 + 64\Delta^{\mathrm{VI}} : \pi = 0$
$\Delta^{\mathrm{VII}} : 0 = 0$	$255\Delta^{\mathrm{VIII}} : 0 + 256\Delta^{\mathrm{VIII}} : \pi = 0$
etc. etc.	etc. etc.

caeterum quia posteriores integrationes absolvere non licet, hinc parum utilitatis exspectare possumus.

§. 56. Caeterum methodus, qua hic sumus usi, ad constantes per quamque integrationem ingressas determinandas, a celeberrimo *Bernoullio* primum est adhibita, atque eo majori attentione digna est aestimanda, quod ejus ope summationes meae serierum reciprocarum potestatum obtineri possunt; quandoquidem credideram, eas non aliter nisi ex consideratione infinitorum arcuum, qui vel eodem sinu vel cosinu gaudent, demonstrari posse.

2). Comparatio valorum formulae integralis

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}},$$

a termino $x = 0$ usque ad $x = 1$ extensae. *Nova Acta Acad. Imp. Scient. Petropolitanae. Tom. V. Pag.* 86 — 117.

§. 57. In hac formula litterae n, p et q perpetuo designant numeros integros positivos, et pro quolibet numero n binis litteris p et q omnes valores tribui concipiuntur, ita ut hinc pro quovis numero n innumerae nascantur hujusmodi formulae integrales, quarum valores plurimas egregias relationes inter se servant; unde si eorum aliquot fuerint cogniti, reliquae omnes ex iis definiri queant. Jam dudum equidem plures hujusmodi relationes demonstravi; cum autem hoc argumentum tum temporis neutiquam exhausissem, nunc accuratius in istas relationes inquirere constitui, et ejusmodi methodum adhibebo, quae omnes plane hujus generis relationes sit exhibitura; his enim inventis innumerabilia theoremata condi poterunt, quibus universa analysis non mediocriter locupletari erit censenda.

§. 58. Quoniam igitur hoc modo pro quolibet numero n ambae litterae p et q infinitos valores recipere possunt, ante omnia hic observari convenit, omnes hos innumerabiles casus semper ad numerum finitum revocari posse. Quantumvis enim magni numeri pro litteris p et q accipiantur, eos casus semper ad alios reducere licet, in quibus numeri p et q quantitate n futuri sint diminuti. Hoc igitur modo omnes hujusmodi casus tandem eo redigi poterunt, ut ambo numeri p et q infra exponentem n deprimantur; unde pro quolibet numero n eos tantum casus con-

siderasse sufficiet, quibus litterae p et q minores valores recipiant quam n, vel saltem hunc limitem non superent. Hoc igitur modo pro quovis numero n multitudo casuum, qui in computum veniunt, et quos inter se comparari oportet, prorsus erit determinata.

§. 59. Quemadmodum autem ista reductio litterarum p et q ad numeros continuo minores institui debeat, quamquam id satis in vulgus est notum, tamen ad formulam praesentem accommodasse juvabit. Statuatur scilicet haec formula algebraica

$x^p (1-x^n)^{\frac{q}{n}} = V$, eritque

$$lV = plx + \tfrac{q}{n}\, l(1-x^n),$$

hinc differentiando

$$\frac{\partial V}{V} = \frac{p\partial x}{x} - \frac{qx^{n-1}\partial x}{1-x^n} = \frac{p\partial x - (p+q)x^n\partial x}{x(1-x^n)},$$

ubi si per V multiplicemus, ac per partes integremus, orietur ista aequatio

$$V = p\int x^{p-1}\partial x(1-x^n)^{\frac{q-n}{n}}$$
$$-(p+q)\int x^{p+n-1}\partial x(1-x^n)^{\frac{q-n}{n}}.$$

Quoniam igitur quantitas V pro utroque integrationis termino evanescit, hinc adipiscimur istam reductionem

$$\int x^{p+n-1}\partial x(1-x^n)^{\frac{q-n}{n}} = \frac{p}{p+q}\int x^{p-1}\partial x(1-x^n)^{\frac{q-n}{n}},$$

cujus ergo reductionis ope exponens ipsius x continuo quantitate n diminui poterit, donec tandem infra n deprimatur.

§. 60. Deinde formula pro

$$\frac{\partial V}{V} = \frac{p\partial x - (p+q)x^n\partial x}{x(1-x^n)}$$

inventa hoc modo referri poterit

$$\frac{\partial V}{V} = \frac{(p+q)\,\partial x\,(1-x^n) - q\,\partial x}{x\,(1-x^n)},$$

quae forma per V multiplicata ac denuo per partes integrata dabit

$$V = (p+q)\int x^{p-1}\,\partial x\,(1-x^n)^{\frac{q}{n}} - q\int x^{p-1}\,\partial x\,(1-x^n)^{\frac{q-n}{n}},$$

unde quia posito $x = 1$ fit $V = 0$, oritur haec reductio

$$\int x^{p-1}\,\partial x\,(1-x^n)^{\frac{q}{n}} = \frac{q}{p+q}\int x^{p-1}\,\partial x\,(1-x^n)^{\frac{q-n}{n}},$$

cujus reductionis ope exponens Binomii $1 - x^n$ unitate minuitur, sive quod eodem redit, numerus q numero n imminuitur. Tali igitur reductione, quoties opus fuerit, repetita, exponens q tandem infra n deprimi poterit.

§. 61. Quoniam igitur pro quovis numero n ambos exponentes p et q tanquam minores quam n spectare licet, formulam propositam hoc modo expresssam repraesentemus

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}}.$$

Hic scilicet pro quovis numero n sufficiet litteris p et q omnes valores ipso n minores tribuisse, quo pacto multitudo omnium casuum ad quemlibet exponentem n pertinentium ad numerum satis modicum reducetur, qui tamen eo major evadit, quo major fuerit exponens n.

§. 62. Multo magis autem numerus casuum diversorum diminuetur, si perpendamus, ambas litteras p et q inter se permutari posse, ita ut hujus formulae

$$\frac{x^{q-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$$

valor ab illo prorsus non discrepet. Ad quod ostendendum ponamus

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} = S,$$

si scilicet ista formula integralis ab $x = 0$ usque ad $x = 1$ extendatur. Jam faciamus $1 - x^n = y^n$, ut formula sit

$$S = \int \frac{x^{p-1}\,\partial x}{y^{n-q}};$$

tum vero quia $x^n = 1 - y^n$, erit $x = (1-y^n)^{\frac{1}{n}}$, hincque $x^p = (1-y^n)^{\frac{p}{n}}$, unde differentiando fit

$$px^{p-1}\,\partial x = -py^{n-1}\,\partial y\,(1-y^n)^{\frac{p-n}{n}},$$

quo valore substituto erit

$$S = -\int y^{q-1}\,\partial y\,(1-y^n)^{\frac{p-n}{n}},$$

quam formulam ab $x = 0$ usque ad $x = 1$, hoc est ab $y = 1$ usque ad $y = 0$, extendi oportet; permutatis igitur his terminis erit

$$S = \int \frac{y^{q-1}\,\partial y}{\sqrt[n]{(1-y^n)^{n-p}}} \begin{bmatrix} \text{ab } y = 0 \\ \text{ad } y = 1 \end{bmatrix}.$$

Sicque demonstratum est ambas litteras p et q semper inter se esse permutabiles.

§. 63. His praemissis, quo calculos sequentes magis in compendium redigere liceat, loco formulae hujus integralis

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} = \int \frac{x^{q-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$$

scribamus hunc characterem (p, q), ubi perinde est, sive p ante q, sive q ante p collocetur; semper autem hic certus exponens n subintelligi debet. Hic autem duo casus prae reliquis maxime memorabiles occurrunt. Prior casus est, quo numerorum p et q alteruter ipsi exponenti n est aequalis; si enim fuerit $q = n$, erit ex priore formula $(p, n) = \int x^{p-1} \partial x = \frac{1}{p}$, sicque perpetuo habebimus $(p, n) = \frac{1}{p}$, hincque etiam $(n, q) = \frac{1}{q}$. Alter casus notatu dignissimus locum habet, quando $p + q = n$, quo casu semper est

$$(p, q) = \frac{\pi}{n \sin. \frac{p\pi}{n}} = \frac{\pi}{n \sin. \frac{q\pi}{n}}.$$

Ad hoc ostendendum sit $q = n - p$, hincque formula proposita $\int \frac{x^{p-1} \partial x}{\sqrt[n]{(1 - x^n)^p}}$, tum ponatur $\frac{x}{\sqrt[n]{(1 - x^n)}} = z$, et quia $\frac{x^p}{\sqrt[n]{(1 - x^n)^p}} = z^p$, erit $S = \int \frac{z^p \partial x}{x}$. Ex facta autem positione sequitur $x^n = \frac{z^n}{1 + z^n}$, hincque

$$n\,l\,x = n\,l\,z - l\,(1 + z^n),$$

ergo differentiando

$$\frac{\partial x}{x} = \frac{\partial z}{z} - \frac{z^{n-1} \partial z}{1 + z^n} = \frac{\partial z}{z\,(1 + z^n)},$$

ita ut jam sit

$$S = \int \frac{z^{p-1} \partial z}{1 + z^n}.$$

Quia autem sumto $x = 0$ fit etiam $z = 0$, at vero sumto $x = 1$ prodit $z = \infty$, hoc integrale a termino $z = 0$ usque

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ad $z = \infty$ extendi debet. Notum autem est valorem hoc modo resultantem esse $\frac{\pi}{n \sin. \frac{\pi p}{n}}$.

§. 64. Progrediamur nunc ad ipsum fundamentum, unde omnes relationes, quas quaerimus, derivari convenit, et quod reductioni priori innititur; unde fit

$$\int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} = \frac{p+q}{p} . \int \frac{x^{n+p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-q}}},$$

ubi loco $\sqrt[n]{(1-x^n)^{n-q}}$ scribamus X, ut sit

$$\int \frac{x^{p-1}\partial x}{X} = \frac{p+q}{p} . \int \frac{x^{n+p-1}\partial x}{X}:$$

hinc jam simili modo, si loco p scribamus $n+p$, erit

$$\int \frac{x^{n+p-1}\partial x}{X} = \frac{n+p+q}{n+p} . \int \frac{x^{2n+p-1}\partial x}{X},$$

hincque sequitur fore

$$\int \frac{x^{p-1}\partial x}{X} = \frac{p+q}{p} . \frac{n+p+q}{n+p} \int \frac{x^{2n+p-1}\partial x}{X}.$$

Quodsi simili modo ulterius progrediamur, perveniemus ad hanc aequationem

$$\int \frac{x^{p-1}\partial x}{X} = \frac{p+q}{p} . \frac{n+p+q}{n+p} . \frac{2n+p+q}{2n+p} \int \frac{x^{3n+p-1}\partial x}{X}.$$

Quare si hoc modo in infinitum progrediamur, habebimus

$$\int \frac{x^{p-1}\partial x}{X} = \frac{p+q}{p} . \frac{n+p+q}{n+p} . \frac{2n+p+q}{2n+p} \times$$

$$\times \; . \; . \; . \; . \; \frac{in+p+q}{in+p} \int \frac{x^{(i+1)n+p-1}\partial x}{X},$$

ubi i denotat numerum infinite magnum.

§. 66. Quodsi jam loco p alium quemcunque numerum r, pariter ipso n minorem, assumamus, erit simili modo

$$\int \frac{x^{r-1}\partial x}{X} = \frac{r+q}{r} \cdot \frac{n+r+q}{n+r} \cdot \frac{2n+r+q}{2n+r} \times$$
$$\times \quad \cdot \; \cdot \; \cdot \; \cdot \; \cdot \quad \frac{in+r+q}{in+r} \int \frac{x^{(i+1)n+r-1}\partial x}{X},$$

ubi littera i eundem numerum infinitum designat, ita ut utrinque idem factorum numerus adsit. Dividamus jam priorem expressionem per istam, et quoniam extremae formulae integrales, ob litteras p et r prae $(i+1)n$ evanescentes, pro aequalibus inter se sunt habendae, facta divisione per singulos factores reperiemus hanc aequationem

$$\frac{\int x^{p-1}\partial x : X}{\int x^{r-1}\partial x : X} = \frac{r(p+q)}{p(r+q)} \cdot \frac{(n+r)(n+p+q)}{(n+p)(n+r+q)} \times$$
$$\times \frac{(2n+r)(2n+p+q)}{(2n+p)(2n+r+q)} \cdot \frac{(3n+r)(3n+p+q)}{(3n+p)(3n+r+q)} \times \text{etc.}$$

Restituamus jam loco harum formularum integralium characteres ante stabilitos, atque adipiscemur istam relationem notatu dignissimam

$$\frac{(p,q)}{(r,q)} = \frac{r(p+q)}{p(r+q)} \cdot \frac{(n+r)(n+p+q)}{(n+p)(n+r+q)} \cdot \frac{(2n+r)(2n+p+q)}{(2n+p)(2n+r+q)} \cdot \text{etc.}$$

quod productum ex infinitis membris componitur, quorum singula sunt fractiones, quarum tam numeratores quam denominatores ex binis factoribus constant. Hos factores singulos eodem numero n augeri oportet, dum a quovis membro ad sequens progredimur, unde sufficiet solum primum productum nosse, quod ergo ita repraesentabimus

$$\frac{(p,q)}{(r,q)} = \frac{r(p+q)}{p(r+q)} \text{ etc.}$$

§. 66. Quoniam litterae p et q nobis numeros quasi indefinitos significant, utamur litteris alphabeti initialibus ad numeros determinatos designandos, eritque eodem modo

$$\frac{(\alpha, b)}{(\alpha, b)} = \frac{\alpha(\alpha+b)}{\alpha(\alpha+b)} \cdot \frac{(n+\alpha)(n+\alpha+b)}{(n+\alpha)(n+\alpha+b)} \text{ etc.}$$

Hic jam loco α scribamus $a + c$, et productum infinitum hanc induet formam

$$\frac{(a, b)}{(a+c, b)} = \frac{(a+c)(a+b)}{a(a+c+b)} \cdot \frac{(n+a+c)(n+a+b)}{(n+a)(n+a+c+b)} \cdot \text{etc.}$$

in quo producto ambae litterae b et c manifesto permutari possunt, unde idem productum infinitum etiam exprimet valorem hujus formae $\frac{(a, c)}{(a+b, c)}$, unde sequitur ista aequalitas maxime memorabilis $\frac{(a, b)}{(a+c, b)} = \frac{(a, c)}{(a+b, c)}$; fractionibus igitur sublatis habebimus istud insigne theorema

$$(a, b)\ (a + b, c) = (a, c)\ (a + c, b),$$

huicque theoremati universa analysis, qua utemur, erit superstructa.

§. 67. Cum ob rationes supra allegatas numeri p et q exponentem n superare non debeant, etiam in forma theorematis modo allati singuli termini ibi occurrentes, qui sunt a, b, c, $a + b$ et $a + c$, quovis casu exponentem n superare non debent, sicque nec $a + b$, neque $a + c$ major capi poterit quam n. Hic autem primo observo litteras b et c, inter se inaequales statui debere: si enim esset $c = b$, aequalitas in theoremate expressa foret identica; hanc ob rem perpetuo assumemus $b > c$, ita ut maximus terminus in theoremate sit $a + b$, quem ergo exponentem n quovis casu excedere non oportet, quamobrem evolutionem formae generalis in theoremate contentae ita in classes distribuamus, quae inter se per maximum valorem termini $a + b$ distinguantur. Cum igitur nulla litterarum a, b, c nihilo aequalis sumi queat, ac esse debeat $b > c$, minimus valor, quem

terminus $a + b$ recipere potest, erit 3, in quo ergo primam classem constituemus; sequentes vero classes constituentur, dum termino $a + b$ valores 4, 5, 6, 7, etc. tribuantur.

I. Evolutio classis qua $a+b = 3$.

§. 68. Hic ergo necessario erit $a = 1$, $b = 2$ et $c = 1$, ita ut hic nulla varietas locum inveniat, unde theorema nostrum suppeditat hanc unicam relationem (1, 2) (3, 1) = (1, 1) (2, 2). Dummodo igitur exponens n non fuerit minor quam 3, semper haec insignis relatio locum habet

$$\int \frac{\partial x}{\sqrt[n]{(1-x^n)^{n-2}}} \cdot \int \frac{xx\partial x}{\sqrt[n]{(1-x^n)^{n-1}}} = \int \frac{\partial x}{\sqrt[n]{(1-x^n)^{n-1}}} \cdot \int \frac{x\partial x}{\sqrt[n]{(1-x^n)^{n-2}}},$$

quae forma, quia in quolibet charactere terminos inter se permutare licet, etiam hoc modo repraesentari poterit

$$\int \frac{x\partial x}{\sqrt[n]{(1-x^n)^{n-1}}} \cdot \int \frac{\partial x}{\sqrt[n]{(1-x^n)^{n-3}}} = \int \frac{\partial x}{\sqrt[n]{(1-x^n)^{n-1}}} \cdot \int \frac{x\partial x}{\sqrt[n]{(1-x^n)^{n-2}}}.$$

II. Evolutio classis qua $a+b = 4$.

§. 69. Quoniam b binario minor esse nequit, hic erit vel $b = 2$, vel $b = 3$. Sit igitur primo $b = 2$, eritque $a = 2$ et $c = 1$; unde ex nostro theoremate sequitur haec relatio (2, 2) (4, 1) = (2, 1) (3, 2), quae forma manifesto oritur ex classe prima, si ibi termini priores cujusque characteris unitate augeantur; id quod etiam inde intelligere licet, quod omnes termini priores litteram a continent, qua unitate aucta processus semper fit ad classem sequentem.

§. 70. Deinde vero hic quoque statui potest $b = 3$, unde fit $a = 1$; at vero littera c jam duos valores, vel 1, vel 2 sortiri poterit; priore casu, quo $c = 1$, prodibit ista aequatio $(1, 3)(4, 1) = (1, 1)(2, 3)$; alter vero casus, quo $c = 2$, praebet hanc aequationem $(1, 3)(4, 2) = (1, 2)(3, 3)$. Sicque haec classis omnino sequentes tres relationes continebit

$$
\begin{aligned}
1^\circ.\ & (2, 2)(4, 1) = (2, 1)(3, 2),\\
2^\circ.\ & (1, 3)(4, 1) = (1, 1)(2, 3),\\
3^\circ.\ & (1, 3)(4, 2) = (1, 2)(3, 3).
\end{aligned}
$$

III. Evolutio classis qua $a + b = 5$.

§. 71. In hac igitur classe primo occurrent tres relationes praecedentes, si modo termini priores cujusque characteris unitate augeantur: hinc enim casus exsurgent, quibus est vel $b = 2$, vel $b = 3$. De novo igitur hic accedent casus, quibus $b = 4$ et $a = 1$, ubi ergo erit vel $c = 1$, vel $c = 2$, vel $c = 3$, quibus ergo tribus casibus evolutis omnino in hac classe sex continebuntur relationes, quae erunt

$$
\begin{aligned}
1^\circ.\ & (3, 2)(5, 1) = (3, 1)(4, 2),\\
2^\circ.\ & (2, 3)(5, 1) = (2, 1)(3, 3),\\
3^\circ.\ & (2, 3)(5, 2) = (2, 2)(4, 3),\\
4^\circ.\ & (1, 4)(5, 1) = (1, 1)(2, 4),\\
5^\circ.\ & (1, 4)(5, 3) = (1, 2)(3, 4),\\
6^\circ.\ & (1, 4)(5, 3) = (1, 3)(4, 4).
\end{aligned}
$$

IV. Evolutio classis quo $a + b = 6$.

§. 72. Hic igitur primum occurrent omnes relationes proxime praecedentes, si modo termini priores cujusque cha-

racteris unitate augeantur: hi scilicet nascuntur, si fuerit vel $b = 2$, vel $b = 3$, vel $b = 4$. Praeterea vero insuper accedent casus $b = 5$ et $a = 1$, ubi littera c recipere poterit valores 1, 2, 3, 4, sicque, omnino in hac classe occurrent decem relationes sequentes

$$
\begin{array}{ll}
1^{\circ}. & (4, 2)(6, 1) = (4, 1)(5, 2), \\
2^{\circ}. & (3, 3)(6, 1) = (3, 1)(4, 3), \\
3^{\circ}. & (3, 3)(6, 2) = (3, 2)(5, 2), \\
4^{\circ}. & (2, 4)(6, 1) = (2, 1)(3, 4), \\
5^{\circ}. & (2, 4)(6, 2) = (2, 2)(4, 4), \\
6^{\circ}. & (2, 4)(6, 3) = (2, 3)(5, 4), \\
7^{\circ}. & (1, 5)(6, 1) = (1, 1)(2, 5), \\
8^{\circ}. & (1, 5)(6, 2) = (1, 2)(3, 5), \\
9^{\circ}. & (1, 5)(6, 3) = (1, 3)(4, 5), \\
10^{\circ}. & (1, 5)(6, 4) = (1, 4)(5, 5),
\end{array}
$$

V. Evolutio classis

qua $a + b = 7$.

§. 73. Hic igitur primo occurrent omnes relationes classis IV. postquam scilicet omnes terminos priores singulorum characterum unitate auxerimus, quos igitur hic apposuisse non erit necesse, ac sufficiet eas tantum relationes hic exponere, quae de novo accedunt et ex valore $b = 6$ oriuntur, existente $a = 1$; ubi pro c sumi poterunt numeri 1, 2, 3, 4, 5, ita ut harum numerus sit quinque. Haec ergo relationes sunt

$$
\begin{array}{l}
(1, 6)(7, 1) = (1, 1)(2, 6) \\
(1, 6)(7, 2) = (1, 2)(3, 6) \\
(1, 6)(7, 3) = (1, 3)(4, 6) \\
(1, 6)(7, 4) = (1, 4)(5, 6) \\
(1, 6)(7, 5) = (1, 5)(6, 6).
\end{array}
$$

VI. Evolutio classis qua $a+b=8$.

§. 74. In hac jam classe primo occurrent omnes decem relationes classis IV., dum scilicet omnes termini priores binario augentur; praeterea quoque accedent quinque relationes in classe V allatae, dum partes priores unitate augebuntur; praeter has vero de novo accedent 6 sequentes relationes ex valoribus $a=1$ et $b=7$ oriundae, dum litterae c valores 1, 2, 3, 4, 5, 6 ordine tribuuntur, quae ergo erunt

$$(1, 7)\ (8, 1) = (1, 1)\ (2, 7)$$
$$(1, 7)\ (8, 2) = (1, 2)\ (3, 7)$$
$$(1, 7)\ (8, 3) = (1, 3)\ (4, 7)$$
$$(1, 7)\ (8, 4) = (1, 4)\ (5, 7)$$
$$(1, 7)\ (8, 5) = (1, 5)\ (6, 7)$$
$$(1, 7)\ (8, 6) = (1, 6)\ (7, 7).$$

VII. Evolutio classis qua $a+b=9$.

§. 75. Ut omnes relationes ad hanc classem pertinentes adipiscamur, notandum est primo hic occurrere decem relationes classis IV, dum partes priores ternario augentur. Secundo adjici oportet quinque relationes in classe V exhibitas, ubi partes priores binario augeri debent. Tertio huc referri debent sex relationes classis VI, partes priores unitate augendo. Insuper vero de novo accedent septem relationes ex valoribus $a=1$ et $b=8$ natae, dum litterae c tribuuntur ordine valores 1, 2, 3, 4, 5, 6, 7. Hae relationes sunt

$$(1, 8)\ (9, 1) = (1, 1)\ (2, 8)$$
$$(1, 8)\ (9, 2) = (1, 2)\ (3, 8)$$
$$(1, 8)\ (9, 3) = (1, 3)\ (4, 8)$$

$(1, 8)\ (9, 4) = (1, 4)\ (5, 8)$
$(1, 8)\ (9, 5) = (1, 5)\ (6, 8)$
$(1, 8)\ (9, 6) = (1, 6)\ (7, 8)$
$(1, 8)\ (9, 7) = (1, 7)\ (8, 8).$

§. 76. Hinc jam ordo progressionis tam clare perspicitur, ut superfluum foret has evolutiones ulterius prosequi; quandoquidem ob ingentem multitudinem relationum, quae in sequentibus classibus occurrerent, nimis molestum foret omnes percurrere. Quin etiam nostrum institutum vix permittere videtur, ut in nostra formula generali exponentem n ultra sex vel septem augeamus, si quidem omnes relationes ad eum pertinentes enumerare voluerimus. Sin autem animus sit aliquas tantum expendere, classes allatae abunde sufficiunt, dum termini priores cujusque classis quovis numero augebuntur.

§. 77. His jam classibus expeditis, formulam integralem propositam $\int \frac{x^{p-1} \partial x}{\sqrt[n]{(1-x^n)^{n-q}}}$ secundum diversos valores exponentis n pertractemus, dum scilicet successive assumemus $n = 3$, $n = 4$, $n = 5$, etc. et pro quolibet ordine omnes relationes, quae in eo occurrere possunt, expendamus. Evidens autem est, quicunque numerus exponenti n tributaur, formulas omnium classium inferiorum, in quibus scilicet terminus $a + b$ non superet n, in usum vocari posse. Ex quo intelligitur, si fuerit $n = 3$ unicam relationem locum invenire; statim autem ac n magis augetur, numerus omnium relationum mox ita increscit, ut nimis molestum foret omnes recensere. Hos igitur diversos ordines, ex exponente n constituendos, a primo incipiendo, ordine involvamus.

Ordo I.

quo $n = 3$ et formula

$$(p, q) = \int \frac{x^{p-1} \partial x}{\sqrt[3]{(1-x^3)^{3-q}}} = \int \frac{x^{q-1} \partial x}{\sqrt[3]{(1-x^3)^{3-p}}}.$$

§. 78. Cum hic sit $n = 3$, erit $(3, 1) = 1$; formulae autem integrales hujus ordinis erunt tres; scilicet 1^0. $(1, 1)$, 2^0. $(1, 2)$, 3^0. $(2, 2)$, quarum media, ob $1 + 2 = 3$, a circulo pendet, quae ergo, quia est cognita, ponatur

$$(1, 2) = \frac{\pi}{3 \sin. \frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}} = A.$$

Hic igitur tantum classis prima locum habet, quae nobis hanc unicam aequationem suppeditat $A = (1, 1)(2, 2)$.

§. 79. Hinc ergo patet, productum ex binis formulis transcendentibus $(1, 1)$ et $(2, 2)$ aequari quantitati circulari $A = \frac{2\pi}{3\sqrt{3}}$, ita ut pro ipsis formulis integralibus habeamus hanc relationem

$$\int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} \cdot \int \frac{x \partial x}{\sqrt[3]{(1-x^3)}} = \frac{2\pi}{3\sqrt{3}};$$

unde si altera harum duarum formularum fuerit cognita, etiam valor alterius assignari potest. Spectemus ergo priorem quasi nobis esset cognita, etiamsi sit transcendens, eamque ponamus

$$(1, 1) = \int \frac{\partial x}{\sqrt[3]{(1-x^3)^2}} = P,$$

eritque $(2, 2) = \frac{A}{P}$. Sicque nihil praeterea in hoc ordine notandum relinquitur.

Ordo II.

quo $n = 4$ et formula

$$(p, q) = \int \frac{x^{p-1} \partial x}{\sqrt[4]{(1 - x^4)^{4-q}}} = \frac{x^{q-1} \partial x}{\sqrt[4]{(1 - x^4)^{4-p}}}.$$

§. 80. Cum igitur hic sit $n = 4$, erit $(4, 1) = 1$ et $(4, 2) = \frac{1}{2}$; formulae autem integrales ad hunc ordinem pertinentes erunt sex sequentes: 1°. $(1, 1)$, 2°. $(1, 2)$, 3°. $(1, 3)$, 4°. $(2, 2)$, 5°. $(2, 3)$, 6°. $(3, 3)$, inter quas ergo reperiuntur duae formulae circulares $(1, 3)$ et $(2, 2)$, quas propterea litteris A et B designemus, ponendo

$$(1, 3) = \frac{\pi}{4 \sin. \frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}} = A, \text{ et}$$

$$(2, 2) = \frac{\pi}{4 \sin. \frac{2\pi}{4}} = \frac{\pi}{4} = B,$$

ita ut sit $\frac{A}{B} = \sqrt{2}$.

§. 81. In hoc ergo ordine aequationes tam primae quam secundae classis locum habere possunt; secunda autem classis nobis has tres praebet aequationes

1°. $B = (2,1)(3,2)$, 2°. $A = (1,1)(2,3)$, 3°. $A = 2(1,2)(3,3)$,

classis vero prima insuper dat hanc aequationem $A(1,2) = (1,1)B$, sive $\frac{A}{B} = \frac{(1,1)}{(1,2)}$ quae autem aequatio jam ex duabus prioribus deducitur; namque ob $(3, 2) = (2, 3)$, secunda per primam divisa dabit $\frac{A}{B} = \frac{(1,1)}{(1,2)} = \sqrt{2}$, ita ut ratio inter has duas formulas sit algebraica, quae ergo imprimis notari meretur

$$\int \frac{\partial x}{\sqrt[4]{(1 - x^4)^3}} : \int \frac{\partial x}{\sqrt{(1 - x^4)}} = \sqrt{2}.$$

§. 82. Jam in hoc ordine, praeter binas formulas circulares, $(1, 3) = A$ et $(2, 2) = B$, tanquam cognitam etiam introducamus formulam $(1, 2)$, quae in ordine praecedente erat circularis, nunc autem est transcendens, eamque ponamus $(1, 2) = \int \frac{\partial x}{\sqrt{(1-x^4)}} = P$; ubi caveatur, ne litterae A et P cum iis confundantur, quibus in formulis praecedentibus sumus usi, id quod etiam de ordinibus sequentibus est tenendum. His igitur litteris introductis aequationes nostrae erunt sequentes tres

$$1^{\circ}.\ B = P(3, 2),\quad 2^{\circ}.\ A = (1, 1)(2, 3),\quad 3^{\circ}.\ A = 2P(3, 3),$$

quandoquidem vidimus, quartam in praecedentibus jam contineri.

§. 83. Ope harum trium aequationum ergo ternas formulas integrales etiamnunc incognitas per ternas A, B et P, quas ut datas spectamus, determinare licebit. Ex prima enim fit $(3,2) = \frac{B}{P}$; ex tertia autem fit $(3, 3) = \frac{A}{2P}$; tum vero ex secunda colligitur $(1, 1) = \frac{A}{(3, 2)} = \frac{AP}{B}$. Cum igitur in hoc ordine omnino sint sex formulae integrales, earum ternae per tres reliquas definiri possunt, quas determinationes igitur ob oculos posuisse juvabit

$$(1, 3) = A = \frac{\pi}{2\sqrt{2}};$$
$$(2, 2) = B = \frac{\pi}{4};$$
$$(1, 2) = P = \int \frac{\partial x}{\sqrt{(1-x^4)}};$$
$$(1, 1) = \frac{AP}{B};$$
$$(2, 3) = \frac{B}{P};$$
$$(3, 3) = \frac{A}{2P}.$$

Ex postremis ergo erit

$$(2, 3) : (3, 3) = 2B : A = \sqrt{2} : 1,$$

ita ut etiam hae duae formulae inter se habeant rationem algebraicam, qua est

$$\int \frac{x\,x\,\partial x}{\sqrt{(1-x^4)}} = \sqrt{2} \int \frac{x\,x\,\partial x}{\sqrt[4]{(1-x^4)}}.$$

Aliis insignibus relationibus, utpote satis cognitis, hic non immoramur.

Ordo III.

quo $n = 5$ et formula

$$(p, q) = \int \frac{x^{p-1}\,\partial x}{\sqrt[5]{(1-x^5)^{n-q}}} = \int \frac{x^{q-1}\,\partial x}{\sqrt[5]{(1-x^5)^{n-p}}}.$$

§. 84. Hic igitur ob $n = 5$ ante omnia erit

$$(5, 1) = 1,\ (5, 2) = \tfrac{1}{2},\ (5, 3) = \tfrac{1}{3},$$

formulae autem integrales hujus ordinis erunt hae decem

$1^\circ.\ (1, 1),\ 2^\circ.\ (1, 2),\ 3^\circ.\ (1, 3),\ 4^\circ.\ (1, 4),\ 5^\circ.\ (2, 2),$
$6^\circ.\ (2, 3),\ 7^\circ.\ (2, 4),\ 8^\circ.\ (3, 3),\ 9^\circ.\ (3, 4),\ 10^\circ.\ (4, 4),$

inter quas quarta et sexta sunt circulares, quas ergo ita designemus

$$(1, 4) = \frac{\pi}{5 \sin. \frac{1}{5}\pi} = A \text{ et}$$

$$(2, 3) = \frac{\pi}{5 \sin. \frac{2\pi}{5}} = B.$$

Praeterea vero binas formulas, quae in ordine praecedenti erant circulares, nunc autem sunt transcendentes, etiam peculiaribus litteris notemus, scilicet $(1, 3) = P$ et $(2, 2) = Q$. Mox enim patebit, dummodo etiam istae formulae tanquam cognitae spectentur reliquas sex omnes per has quatuor determinari posse.

§. 85. Quoniam hic tres classes priores locum habere possunt, consideremus primo aequationes, quas tertia classis suppeditat, et quae introductis his valoribus erunt

$$1^\circ.\ B = P(4, 2),$$
$$2^\circ.\ B = (2, 1)(3, 3),$$
$$3^\circ.\ B = 2Q(4, 3),$$
$$4^\circ.\ A = (1, 1)(2, 4),$$
$$5^\circ.\ A = 2(1, 2)(3, 4),$$
$$6^\circ.\ A = 3P(4, 4).$$

Quas hoc modo succinctius repraesentare licet

$$A = (1, 1)(2, 4) = 2(1, 2)(3, 4) = 3P(4, 4),$$
$$B = P(4, 2) = (2, 1)(3, 3) = 2Q(4, 3);$$

ubi sex occurrunt producta ex binis formulis integralibus, quae singula quantitati circulari aequantur, unde totidem egregia theoremata formari possent, nisi hinc jam clare in oculos incurrerent.

§. 86. Jam videamus, quot formulas integrales incognitas ex quatuor cognitis A, B, P et Q definire queamus, at vero prima dat $(4, 2) = \frac{B}{P}$, tertia praebet $(4, 3) = \frac{B}{2Q}$, sexta dat $(4, 4) = \frac{A}{3P}$; hinc autem porro ex quarta deducimus

$$(1, 1) = \frac{A}{(2, 4)} = \frac{AP}{B},$$

ex quinta vero deducimus

$$(1, 2) = \frac{A}{2(3, 4)} = \frac{AQ}{B}.$$

Denique ex secunda elicimus.

$$(3, 3) = \frac{B}{(2, 1)} = \frac{BB}{AQ},$$

sicque ex his sex aequationibus sex determinationes sumus adepti; atque adeo per litteras A, B, P et Q valores omnium reliquarum litterarum assignavimus.

§. 87. Quoniam igitur hactenus tantum classe tertia sumus usi, consideremus etiam aequationes secundae classis, quae sunt

$$1^\circ.\ AQ = B(2, 1),$$
$$2^\circ.\ AP = B(1, 1),\ \text{et}$$
$$3^\circ.\ P(4, 2) = (1, 2)(3, 3);$$

verum si hic valores modo inventos substituamus, aequationes mere identicae resultant, ita ut hinc nulla nova determinatio sequatur. Idem usu venit ex aequatione primae classis, quae erat $(2, 1)(3, 1) = (1, 1).(2, 2)$, quae facta substitutione quoque fit identica, ita ut duae priores classes nihil novi involvant. Neque tamen hinc concludere licet, etiam in sequentibus ordinibus classes praecedentes praetermitti posse, siquidem in ordine sequente statim contrarium se manifestabit.

§. 88. Cum igitur hic ordo complectatur decem formulas integrales, earum valores per quatuor litteras A, B, P et Q ordine ita aspectui exponamus

$$1^\circ.\ (1, 1) = \frac{AP}{B}$$
$$2^\circ.\ (1, 2) = \frac{AQ}{B}$$
$$3^\circ.\ (1, 3) = P$$
$$4^\circ.\ (1, 4) = A$$
$$5^\circ.\ (2, 2) = Q$$
$$6^\circ.\ (2, 3) = B$$
$$7^\circ.\ (2, 4) = \frac{B}{P}$$
$$8^\circ.\ (3, 3) = \frac{BB}{AQ}$$
$$9^\circ.\ (3, 4) = \frac{B}{2Q}$$
$$10^\circ.\ (4, 4) = \frac{A}{3P}.$$

§. 89. Cum sit

$$\frac{A}{B} = \frac{\sin.\frac{2}{5}\pi}{\sin.\frac{1}{5}\pi} = 2\cos.\tfrac{1}{5}\pi,$$

tum vero

$$\cos.\tfrac{1}{5}\pi = \frac{1+\sqrt{5}}{4}, \text{ erit } \frac{A}{B} = \frac{1+\sqrt{5}}{2},$$

ideoque quantitas algebraica. Hinc igitur aliquot paria formularum integralium exhiberi poterunt, quae inter se teneant rationem algebraicam; erit enim

$$\frac{(1,1)}{(1,3)} = \frac{1+\sqrt{5}}{2},\ \frac{(1,2)}{(2,2)} = \frac{1+\sqrt{5}}{2},\ \frac{(3,4)}{(3,3)} = \frac{1+\sqrt{5}}{4},\ \frac{(4,4)}{(2,4)} = \frac{1+\sqrt{5}}{6};$$

unde totidem egregia theoremata condi possent, nisi ex his formulis manifesto elucerent.

Ordo IV.

quo $n = 6$ et formula

$$(p, q) = \int \frac{x^{p-1}\partial x}{\sqrt[6]{(1-x^6)^{6-q}}} = \int \frac{x^{q-1}\partial x}{\sqrt[6]{(1-x^6)^{6-p}}}.$$

§. 90. Quoniam hic est $n = 6$, habebimus ante omnia

$$(6,1) = 1,\ (6,2) = \tfrac{1}{2},\ (6,3) = \tfrac{1}{3},\ (6,4) = \tfrac{1}{4};$$

formularum autem integralium in hoc ordine occurrentium numerus est 15, quae sunt

1°. (1, 1), 2°. (1, 2), 3°. (1, 3), 4°. (1, 4), 5°. (1, 5),
6°. (2, 2), 7°. (2, 3), 8°. (2, 4), 9°. (2, 5), 10°. (3, 3),
11°. (3, 4), 12°. (3, 5), 13°. (4, 4); 14°. (4, 5), 15°. (5, 5);

inter quas reperiuntur tres circulares, quas singulari modo designemus, scilicet

$$1^\circ.\ (1,5) = \frac{\pi}{6\sin.\frac{1}{6}\pi} = \frac{\pi}{3} = A,$$

$$2^{\circ}.\ (2, 4) = \frac{\pi}{6 \sin. \frac{2\pi}{6}} = \frac{\pi}{3\sqrt{3}} = B, \text{ et}$$

$$3^{\circ}.\ (3, 3) = \frac{\pi}{6 \sin. \frac{3\pi}{6}} = \frac{\pi}{6} = C;$$

ita ut sit $A = 2\,C$. Praeterea vero ambas formulas, quae in ordine praecedente erant circulares, nunc vero sunt transcendentes, statuamus $(1, 4) = P$ et $(2, 3) = Q$. His factis denominationibus evolvamus decem aequationes classis quartae, quae sunt

$1^{\circ}.\ B = P\,(5, 2),$
$2^{\circ}.\ C = (3, 1)\,(4, 3),$
$3^{\circ}.\ C = 2\,Q\,(5, 3),$
$4^{\circ}.\ B = (2, 1)\,(3, 4),$
$5^{\circ}.\ B = 2\,(2, 2)\,(4, 4),$
$6^{\circ}.\ B = 3\,Q\,(5, 4),$
$7^{\circ}.\ A = (1, 1)\,(5, 2),$
$8^{\circ}.\ A = 2\,(1, 2)\,(3, 5),$
$9^{\circ}.\ A = 3\,(1, 3)\,(4, 5),$
$10^{\circ}.\ A = 4\,P\,(5, 5),$

quas ita succinctius referre licet

$$A = (1,1)(5,2) = 2(1,2)(3,5) = 3(1,3)(4,5) = 4P(5,5),$$
$$B = P(5,2) = (2,1)(3,4) = 2(2,2)(4,4) = 3Q(4,5),$$
$$C = (3,1)(5,2) = 2Q(5,3).$$

Ecce ergo decem producta ex binis formulis integralibus, quorum singula quantitati circulari aequantur.

§. 91. Cum deinde sit $\frac{A}{B} = \sqrt{3}$ et $\frac{A}{C} = 2$, tum vero etiam $\frac{B}{C} = \frac{2}{\sqrt{3}}$, plura paria binarum formularum integralium exhi-

beri possunt, quae inter se teneant rationem algebraicam; erit enim

$$\frac{A}{B} = \sqrt{3} = \frac{(1,1)}{(1,4)} = \frac{2(3,5)}{(3,4)} = \frac{(1,3)}{(2,3)} = \frac{4(5,5)}{(5,2)},$$

$$\frac{A}{C} = 2 = \frac{(1,1)}{(1,3)} = \frac{(1,2)}{(2,3)} = \frac{3(4,5)}{(2,5)},$$

$$\frac{B}{C} = \frac{2}{\sqrt{3}} = \frac{(1,4)}{(1,3)} = \frac{3(4,5)}{2(3,5)}.$$

§. 92. Quodsi jam quinque formulas litteris A, B, C, P et Q designatas tanquam cognitas spectemus, videamus, quomodo reliquae formulae per eas definire queant. Ac primo quidem percurramus decem aequationes classis quartae supra allatas, quarum prima dabit $(5,2) = \frac{B}{P}$, tertia dat $(5,3) = \frac{C}{2Q}$, sexta praebet $(5,4) = \frac{B}{3Q}$, decima dat $(5,5) = \frac{A}{4P}$. Quodsi jam hos valores in reliquis surrogemus, secunda dabit $(3,1) = \frac{C}{(4,3)} = \frac{AQ}{B}$, septima praebet $(1,1) = \frac{A}{(5,2)} = \frac{AP}{B}$, octava dat $(1,2) = \frac{A}{2(3,5)} = \frac{AQ}{C}$, nona dat $(3,1) = \frac{A}{3(4,5)} = \frac{AQ}{B}$, quem valorem etiam secunda praebuit. Porro vero quarta dat $(3,4) = \frac{B}{(2,1)} = \frac{BC}{AQ}$. At vero ex aequatione quinta nullum valorem elicere possumus, quia neque formula $(2,2)$ nec $(4,4)$ etiamnunc constat. Causa est quia duae reliquarum aequationum eandem determinationem produxerunt.

§. 93. Coacti igitur sumus, ad aequationes praecedentium classium confugere, atque adeo ex prima classe

$$(1,2)(3,1) = (1,1)(2,2)$$

statim colligimus

$$(2,2) = \frac{(1,2)(3,1)}{(1,1)} = \frac{AQQ}{C \cdot P},$$

qui valor in quinta aequatione substitutus suppeditat postremam aequationem, nempe

$$(4,4) = \frac{B}{2(2,2)} = \frac{BCP}{2AQQ}.$$

Omnes igitur hos valores hic ordine referemus

$1^{\circ}.\ (1, 1) = \frac{AP}{B}$.
$2^{\circ}.\ (1, 2) = \frac{AQ}{C}$.
$3^{\circ}.\ (1, 3) = \frac{AQ}{B}$.
$4^{\circ}.\ (1, 4) = P$.
$5^{\circ}.\ (1, 5) = A$.
$6^{\circ}.\ (2, 2) = \frac{AQQ}{CP}$.
$7^{\circ}.\ (2, 3) = Q$.
$8^{\circ}.\ (2, 4) = B$.
$9^{\circ}.\ (2, 5) = \frac{B}{P}$.
$10^{\circ}.\ (3, 3) = C$.
$11^{\circ}.\ (3, 4) = \frac{BC}{AQ}$.
$12^{\circ}.\ (3, 5) = \frac{C}{2Q}$.
$13^{\circ}.\ (4, 4) = \frac{BCP}{2AQQ}$.
$14^{\circ}.\ (4, 5) = \frac{B}{3Q}$.
$15^{\circ}.\ (5, 5) = \frac{A}{4P}$.

§. 94. Cum autem in hoc ordine etiam aequationes tam classis secundae quam tertiae valere debeant, videamus utrum valores inventi his classibus conveniant, an vero forte novam determinationem suppeditent? Facta autem substitutione in tribus aequationibus secundae classis, ad identitatem pervenitur, quod idem quoque in aequationibus tertiae classis contingere debet, id quod evolventi mox patebit. Unde memorabile est omnes aequationes in quatuor primis classibus contentas, quarum numerus est 20, tantum decem determinationes in se complecti.

Ordo V.

quo $n = 7$ et formula

$$(p, q) = \int \frac{x^{p-1}\,\partial x}{\sqrt[7]{(1-x^7)^{7-q}}} = \int \frac{x^{q-1}\,\partial x}{\sqrt[7]{(1-x^7)^{7-p}}}.$$

§. 95. Quia hic $n = 7$, ante omnia habebimus valores absolutos $(7, 1) = 1$, $(7, 2) = \frac{1}{2}$, $(7, 3) = \frac{1}{3}$, $(7, 4) = \frac{1}{4}$, et $(7, 5) = \frac{1}{5}$; deinde inter formulas integrales hujus ordinis imprimis

notari debent circulares, quas hoc modo designemus

$$(1, 6) = \frac{\pi}{7 \sin. \frac{\pi}{7}} = A,$$

$$(2, 5) = \frac{\pi}{7 \sin. \frac{2\pi}{7}} = B,$$

$$(3, 4) = \frac{\pi}{7 \sin. \frac{3\pi}{7}} = C.$$

Praeterea vero peculiaribus litteris notentur eae formulae, quae in ordine praecedenti erant circulares, hic autem valores transcendentes sortiuntur, qui sint $(1, 5) = P$, $(2, 4) = Q$, et $(3, 3) = R$: per has enim sex litteras videbimus omnes reliquas formulas hujus ordinis determinari posse.

§. 96. Quoniam supra non omnes aequationes quintae classis expressimus, eas hic conjunctim exhibeamus, et ad nostrum casum accommodemus

I°.	$(1,6)(7,1) = (1,1)(2,6)$	$A = (1,1)(2,6)$,
II°.	$(1,6)(7,2) = (1,2)(3,6)$	$A = 2(1,2)(3,6)$,
III°.	$(1,6)(7,3) = (1,3)(4,6)$	$A = 3(1,3)(4,6)$,
IV°.	$(1,6)(7,4) = (1,4)(5,6)$	$A = 4(1,4)(5,6)$,
V°.	$(1,6)(7,5) = (1,5)(6,6)$	$A = 5\ P\ (6,6)$,
VI°.	$(2,5)(7,1) = (2,1)(3,5)$	$B = (2,1)(3,5)$,
VII°.	$(2,5)(7,2) = (2,2)(4,5)$	$B = 2(2,2)(4,5)$,
VIII°.	$(2,5)(7,3) = (2,3)(5,5)$	$B = 3(2,3)(5,5)$,
IX°.	$(2,5)(7,4) = (2,4)(6,5)$	$B = 4\ Q\ (6,5)$,
X°.	$(3,4)(7,1) = (3,1)(4,4)$	$C = (3,1)(4,4)$,
XI°.	$(3,4)(7,2) = (3,2)(5,4)$	$C = 2(3,2)(5,4)$,
XII°.	$(3,4)(7,3) = (3,3)(6,4)$	$C = 3\ R\ (6,4)$,
XIII°.	$(4,3)(7,1) = (4,1)(5,3)$	$C = (4,1)(5,3)$,
XIV°.	$(4,3)(7,2) = (4,2)(6,3)$	$C = 2\ Q\ (6,3)$,
XV°.	$(5,2)(7,1) = (5,1)(6,2)$	$B = P\ )(6,2)$.

Hic igitur habemus quina producta formulae A aequalia, totidemque formulis B et C aequalia.

§. 97. Omnino autem in hoc ordine occurrunt 21 formulae integrales, ex quibus sex litteris A, B, C, P, Q et R designavimus, per quas igitur reliquas quindecim formulas integrales definiri oportet, quae sunt: 1°. (1, 1), 2°. (1, 2), 3°. (1, 3), 4°. (2, 2), 5°. (1, 4), 6°. (2, 3), 7°. (2, 6), 8°. (3, 5), 9°. (4, 4), 10°. (3, 6), 11°. (4, 5), 12°. (4, 6), 13°. (5, 5), 14°. (5, 6), 15°. (6, 6).

§. 98. Videamus igitur, quot harum formularum ex superioribus quindecim aequationibus determinare liceat, ac primo quidem ex aequationibus V, IX, XII, XIV et XV, immediate deducuntur sequentes formulae $(6, 6) = \frac{A}{5P}$, $(6, 5) = \frac{B}{4Q}$, $(6, 4) = \frac{C}{3R}$, $(6, 3) = \frac{C}{2Q}$, $(6, 2) = \frac{B}{P}$. His jam inventis ex aequationibus I, II, III et IV, derivamus has formulas $(1, 1) = \frac{AP}{B}$, $(1, 2) = \frac{AQ}{C}$, $(1, 3) = \frac{AR}{C}$, $(1, 4) = \frac{AQ}{B}$. Ex his vero valoribus per aequationes VI, X et XIII, colligimus $(3, 5) = \frac{BC}{AQ}$, $(4, 4) = \frac{CC}{AR}$, et $(5, 3) = \frac{BC}{AQ}$; ubi notasse juvabit eundem valorem pro (5, 3) prodiisse ex aequationibus VI, et XIII. Ex reliquis autem aequationibus VII, VIII et IX, nihil concludere licet, unde istae quatuor formulae (2, 2), (2, 3), (5, 4), et (5, 5), nobis etiamnunc manent incognitae.

§. 99. Recurrere ergo coacti sumus ad aequationes praecedentium classium, quippe quae aeque ad nostrum ordinem pertinent atque aequationes classis quintae; hanc ob rem simili modo aequationes classis quartae hic opponamus et ad nostrum casum applicemus:

I°.	$(1, 5)(6, 1) = (1, 1)(2, 5)$	$PA = (1, 1)B$
II°.	$(1, 5)(6, 2) = (1, 2)(3, 5)$	$P(6, 2) = (1, 2)(3, 5)$
III°.	$(1, 5)(6, 3) = (1, 3)(4, 5)$	$P(6, 3) = (1, 3)(4, 5)$
IV°.	$(1, 5)(6, 4) = (1, 4)(5, 5)$	$P(6, 4) = (1, 4)(5, 5)$
V°.	$(2, 4)(6, 1) = (2, 1)(3, 4)$	$QA = (2, 1)C$
VI°.	$(2, 4)(6, 2) = (2, 2)(4, 4)$	$Q(6, 2) = (2, 2)(4, 4)$
VII°.	$(2, 4)(6, 3) = (2, 3)(5, 4)$	$Q(6, 3) = (2, 3)(5, 4)$
VIII°.	$(3, 3)(6, 1) = (3, 1)(4, 3)$	$RA = (3, 1)C$
IX°.	$(3, 3)(6, 2) = (3, 2)(5, 3)$	$R(6, 2) = (3, 2)(5, 3)$
X°.	$(4, 2)(6, 1) = (4, 1)(5, 2)$	$QA = (4, 1)B.$

§. 100. Ex aequationibus I, V, VIII, et X immediate concludimus has formulas $(1, 1) = \frac{PA}{B}$, $(2, 1) = \frac{QA}{C}$, $(3, 1) = \frac{AR}{C}$, $(4, 1) = \frac{AQ}{B}$, quos autem valores jam ante adepti sumus. Secunda aequatio, si formulae jam inventae substituantur, praebet aequationem identicam. Ex tertia autem poterimus definire formulam $(4, 5)$, cujus valor hinc colligitur $(4, 5) = \frac{CCP}{2AQA}$. Simili modo ex IV elicimus $(5, 5) = \frac{BCP}{3AQR}$. Porro ex aequatione VI concludimus fore $(2, 2) = \frac{ABQR}{CCP}$. Deinde septima aequatio dat $(2, 3) = \frac{AQR}{CP}$. Nona vero aequatio etiam praebet $(3, 2) = \frac{AQR}{CP}$. Sicque omnes quindecim formulas incognitas determinavimus per sex litteras cognitas A, B, C, P, Q et R.

§. 101. Valores igitur omnium formularum hujus ordinis hic aspectui conjunctim exponamus

$(1, 6)$

$(1, 6) = A$	$(6, 2) = \frac{B}{P}$	$(1, 1) = \frac{AP}{B}$	$(3, 5) = \frac{BC}{AQ}$	$(2, 3) = \frac{AQR}{CP}$
$(2, 5) = B$	$(6, 3) = \frac{C}{2Q}$	$(1, 2) = \frac{AQ}{C}$	$(4, 4) = \frac{CC}{AR}$	$(4, 5) = \frac{CCP}{2AQR}$
$(3, 4) = C$	$(6, 4) = \frac{C}{3R}$	$(1, 3) = \frac{AR}{C}$		$(5, 5) = \frac{BCP}{3AQR}$
$(1, 5) = P$	$(6, 5) = \frac{B}{4Q}$	$(1, 4) = \frac{AQ}{B}$		$(2, 2) = \frac{ABQR}{CCP}$
$(2, 4) = Q$	$(6, 6) = \frac{A}{5P}$			
$(3, 3) = R$				

§. 102. Quoniam autem aequationes primae, secundae ac tertiae classis etiam in hoc ordine valent, si in iis valores hic inventos substituamus, perpetuo in aequationes identicas incidemus. Ita cum aequatio primae classis sit $(1, 2)\,(3, 1) = (1, 1)\,(2, 2)$, facta substitutione reperictur $(1, 2)\,(3, 1) = \frac{AAQR}{CC}$; at vero $(1, 1)\,(2, 2)$ fit $= \frac{AAQR}{CC}$, haecque identitas etiam deprehendetur, in tribus aequationibus secundae classis, atque etiam in sex aequationibus tertiae classis, quemadmodum calculum instituenti mox patebit.

§. 103. Simili modo haud difficile erit hanc investigationem ad ordines superiores extendere, neque tamen legem observare licet, secundum quam determinationes singularum formularum cujusque ordinis progrediuntur. Interim tamen observasse juvabit, in ordine sequente sexto, ubi $n = 8$ et formulae occurrunt 28, eas omnes primo per quatuor formulas circulares $(1, 7) = A$, $(2, 6) = B$, $(3, 5) = C$, $(4, 4) = D$, praeterea vero per has tres transcendentes $(1, 6) = P$, $(2, 5) = Q$, et $(3, 4) = R$, determinari posse. Cum igitur quovis ordine determinatio singularum formularum, praeter formulas circulares, quae utique pro cognitis haberi possunt, etiam aliquot formulas transcendentes postulat, si saltem valores harum formularum vero proxime cognoscere voluerimus, methodus adhuc

desideratur, istos valores proxime, veluti in fractionibus decimalibus, definiendi. Talem igitur methodum hic coronidis loco subjungemus.

Problema.

Proposita formula integrali cujusque ordinis

$$S = \int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}},$$

a termino $x = 0$ usque ad $x = 1$ extendenda, investigare seriem convergentem, quae istum valorem S exprimat.

Solutio.

§. 104. Cum sit

$$\frac{1}{\sqrt[n]{(1-x^n)^{n-q}}} = (1-x^n)^{-\frac{(n-q)}{n}},$$

facta evolutione hujus potestatis binomii more solito, reperietur

$$\frac{1}{\sqrt[n]{(1-x^n)^{n-q}}} = 1 + \frac{n-q}{n}x^n + \frac{n-q}{n}\cdot\frac{2n-q}{2n}x^{2n} + \frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{3n-q}{3n}x^{3n} + \text{etc.}$$

Si haec series ducatur in $x^{p-1}\,\partial x$ et integretur, prodibit

$$S = \frac{x^p}{p} + \frac{n-q}{n}\cdot\frac{x^{n+p}}{n+p} + \frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{x^{2n+p}}{2n+p} + \frac{n-q}{n}\cdot\frac{2n-q}{2n}\cdot\frac{3n-q}{3n}\cdot\frac{x^{3n+p}}{3n+p} + \text{etc.}$$

quae series jam evanescit posito $x = 0$; unde si ponamus $x = 1$, valor quaesitus nostrae formulae fiet

$$S = \frac{1}{p} + \frac{n-q}{n} \cdot \frac{1}{n+p} + \frac{n-q}{n} \cdot \frac{2n-q}{2n} \cdot \frac{1}{2n+p}$$
$$+ \frac{n-q}{n} \cdot \frac{2n-q}{2n} \cdot \frac{3n-q}{3n} \cdot \frac{1}{3n+p} + \text{etc.}$$

§. 105. Verum ista series, quicunque numeri pro litteris n, p et q accipiantur, nimis lente convergit, quam ut ex ea valores ipsius S saltem ad tres quatuorve figuras decimales satis exacte definiri queant; quamobrem aliam evolutionem institui conveniet, dum scilicet valorem quaesitum in duas partes resolvemus. Statuamus igitur

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x^n = \frac{1}{2} \end{bmatrix} = \text{P} \text{ et}$$

$$\int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \begin{bmatrix} \text{ab } x^n = \frac{1}{2} \\ \text{ad } x = 1 \end{bmatrix} = \text{Q},$$

atque evidens est fore $S = P + Q$. Nunc autem tam pro P quam pro Q haud difficulter series satis convergentes exhiberi poterunt.

§. 106. Quod primum ad valorem P attinet, eum ex valore generali, quem supra pro S invenimus, facile derivabimus ponendo $x^n = \frac{1}{2}$, ita ut sit $x = \sqrt[n]{\frac{1}{2}}$ et $x^p = \frac{1}{\sqrt[n]{2^p}}$, quo facto pro P obtinebimus hanc seriem

$$P = \frac{1}{\sqrt[n]{2^p}}\left(\frac{1}{p} + \frac{n-q}{2n} \cdot \frac{1}{n+p} + \frac{n-q}{2n} \cdot \frac{2n-q}{4n} \cdot \frac{1}{2n+q}\right.$$
$$\left. + \frac{n-q}{2n} \cdot \frac{2n-q}{4n} \cdot \frac{3n-q}{6n} \cdot \frac{1}{3n+p} + \text{etc.}\right).$$

In qua serie singuli termini plus quam in ratione dupla decres-

cunt; ita ut verbi gratia terminus decimus jam multo minor futurus sit quam $\frac{1}{1024}$, unde si ad partes millionesimas certi esse velimus, sufficeret calculum nequidem ad vigesimum usque terminum extendere.

§. 107. Cum deinde posuerimus

$$Q = \int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1-x^n)^{n-q}}} \begin{bmatrix} \text{ab } x^n = \frac{1}{2} \\ \text{ad } x = 1 \end{bmatrix},$$

statuamus $1 - x^n = y^n$, ut sit $Q = \int \frac{x^{p-1}\,\partial x}{y^{n-q}}$, tum vero erit $x^n = 1 - y^n$, ideoque $x^p = \sqrt[n]{(1-y^n)^p}$, unde differentiando colligitur

$$x^{p-1}\,\partial x = -\,y^{n-1}\,\partial y\,(1-y^n)^{\frac{p-n}{n}},$$

quo valore substituto erit

$$Q = -\int y^{q-1}\,\partial y\,(1-y^n)^{\frac{p-n}{n}} \begin{bmatrix} \text{ab } y^n = \frac{1}{2} \\ \text{ad } y = 0 \end{bmatrix}.$$

Quando enim fit $x^n = \frac{1}{2}$, tum etiam erit $y^n = \frac{1}{2}$, at facto $x = 1$, manifesto fit $y = 0$; quare si terminos integrationis permutemus, etiam signum ipsius formulae immutari debet, sicque fiet

$$Q = \int y^{q-1}\,\partial y\,(1-y^n)^{\frac{p-n}{n}} \begin{bmatrix} \text{ab } y = 0 \\ \text{ad } y^n = \frac{1}{2} \end{bmatrix}.$$

§. 108. Haec autem formula pro Q inventa omnino similis est illi, quam pro P invenimus, hoc tantum discrimine, quod litterae p et q inter se sunt permutatae; quocirca, si integratio per seriem instituatur, proveniet sequens

$$Q = \frac{1}{\sqrt{2^q}}\left(\frac{1}{q} + \frac{n-p}{2n}\cdot\frac{1}{n+q} + \frac{n-p}{2n}\cdot\frac{2n-p}{4n}\cdot\frac{1}{2n+q} + \frac{n-p}{2n}\cdot\frac{2n-p}{4n}\cdot\frac{3n-p}{6n}\cdot\frac{1}{3n+q} + \text{etc.}\right),$$

quae series aeque converget, ac praecedens pro P inventa. His autem duabus seriebus ad calculum revocatis semper erit valor quaesitus $S = P + Q$.

Corollarium 1.

§. 109. Iste calculus plurimum contrahetur iis casibus, quibus est $p = q$, tum enim fiet $P = Q$, hisque casibus, quibus $S = \int \frac{x^{p-1}\partial x}{\sqrt[n]{(1-x^n)^{n-p}}}$, valor istius formulae ab $x = 0$ ad $x = 1$ extensae erit

$$S = \frac{2}{\sqrt[n]{2^p}}\left(\frac{1}{p} + \frac{n-p}{2n}\cdot\frac{1}{n+p} + \frac{n-p}{2n}\cdot\frac{2n-p}{4n}\cdot\frac{1}{2n+p} + \frac{n-p}{2n}\cdot\frac{2n-p}{4n}\cdot\frac{3n-p}{6n}\cdot\frac{1}{3n+p} + \text{etc.}\right).$$

Corollarium 2.

§. 110. Quoniam igitur in singulis ordinibus nonnullae hujusmodi formulae (p, p) occurrunt, statim atque valores aliquot hujusmodi formularum fuerint ad calculum decimalem revocati, quoniam formulae circulares per se sunt notae, ex iis valores omnium reliquarum formularum ejusdem ordinis assignare licebit.

Exemplum.

§. 111. Proposita sit formula ordinis primi, ubi $p = q = 2$ et $S = \int \frac{x \partial x}{\sqrt[3]{(1 - x^3)}}$. Series igitur pro S inventa erit

$$S = \sqrt[3]{2} \left(\tfrac{1}{2} + \tfrac{1}{6} \cdot \tfrac{1}{5} + \tfrac{1}{6} \cdot \tfrac{4}{12} \cdot \tfrac{1}{8} + \tfrac{1}{6} \cdot \tfrac{4}{12} \cdot \tfrac{7}{18} \cdot \tfrac{1}{11} + \tfrac{1}{6} \cdot \tfrac{4}{12} \cdot \tfrac{7}{18} \cdot \tfrac{10}{24} \cdot \tfrac{1}{14} + \text{etc.} \right).$$

Subducta autem calculo reperitur

$$S = 0,54325 \times \sqrt[3]{2} = 0,68445,$$

qui ergo est valor formulae (2, 2) in ordine 1^{mo} §. 22. ubi invenimus $(2, 2) = \frac{A}{P}$, ita ut jam sit $P = \frac{A}{(2,2)}$. Est vero $A = \frac{2\pi}{3\sqrt{3}} = 1,20918$, hinc erit $P = 2,22582 = (1, 1)$: unde in fractionibus decimalibus ternae formulae ordinis primi erunt $(1, 1) = 2,22582$, $(1, 2) = 1,20918$, $(2, 2) = 0,68445$. Hocque modo etiam omnes formulas sequentium ordinum evolvere licebit.

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3) Additamentum ad Dissertationem praecedentem, de valoribus formulae integralis

$$\int \frac{x^{p-1} \partial x}{\sqrt[n]{(1 - x^n)^{n-q}}},$$

ab $x = 0$ ad $x = 1$ extensae. *Nova Acta Acad. Imp. Scient. Petropolitanae. Tom. V. Pag.* 118 — 129.

§. 112. Si methodum in praecedente dissertatione traditam ad altiores ordines quam $n = 7$ transferre vellemus, ob

ingentem aequationum considerandarum numerum labor fieret nimis molestus. Quoniam autem vidimus, non omnes istas aequationes concurrere ad valores singularum formularum determinandos, opus non mediocriter sublevabitur, si quovis casu eas tantum aequationes in computum ducamus, quae immediate ad determinationes formularum perducant, quemadmodum hic pro casu $n = 10$ sum ostensurus.

Determinatio

harum formularum pro casu $n = 10$, ubi formula

$$(p, q) = \int \frac{x^{p-1}\,\partial x}{\sqrt[10]{(1-x^{10})^{10-q}}} = \int \frac{x^{q-1}\,\partial x}{\sqrt[10]{(1-x^{10})^{10-p}}}.$$

§. 113. Hoc casu ergo formulae valorem absolutum recipientes sunt $(10, 1) = 1$, $(10, 2) = \frac{1}{2}$, $(10, 3) = \frac{1}{3}$ et in genere $(10, \alpha) = \frac{1}{\alpha}$. Deinde omnes formulae, in quibus est $p + q = 10$, a circulo pendent, ideoque pro cognitis haberi possunt, quas ergo propriis litteris designemus

$(1, 9) = \dfrac{\pi}{10 \sin. \frac{1}{10}\pi} = A,$	$(6, 4) = \dfrac{\pi}{10 \sin. \frac{6}{10}\pi} = D,$
$(2, 8) = \dfrac{\pi}{10 \sin. \frac{2}{10}\pi} = B,$	$(7, 3) = \dfrac{\pi}{10 \sin. \frac{7}{10}\pi} = C,$
$(3, 7) = \dfrac{\pi}{10 \sin. \frac{3}{10}\pi} = C,$	$(8, 2) = \dfrac{\pi}{10 \sin. \frac{8}{10}\pi} = B,$
$(4, 6) = \dfrac{\pi}{10 \sin. \frac{4}{10}\pi} = D,$	$(9, 1) = \dfrac{\pi}{10 \sin. \frac{9}{10}\pi} = A,$
$(5, 5) = \dfrac{\pi}{10 \sin. \frac{5}{10}\pi} = E,$	

§. 114. Per has autem formulas circulares reliquas in forma generali contentas neutiquam determinare licet; sed insuper aliquot formulas transcendentes in subsidium vocari oportet, ex quibus cum circularibus illis conjunctis reliquarum omnium valores assignare licebit. Nostro autem casu, quo $n = 10$, sequentes formulas tanquam cognitas spectari conveniet, quae in ordine praecedenti, ubi $n = 9$, erant circulares, nunc autem in ordinem transcendentium transeunt. Eas igitur sequenti modo designemus

$$(1, 8) = P, \ (2, 7) = Q, \ (3, 6) = R, \ (4, 5) = S,$$
$$(5, 4) = S, \ (6, 3) = R, \ (7, 2) = Q, \ (8, 1) = P.$$

Scilicet si valores harum litterarum quoque tanquam cognitos spectemus, per eos cum circularibus junctos reliquas formulas omnes in hoc ordine contentas determinare poterimus. Cum igitur numerus omnium formularum integralium in hoc ordine $n = 10$ contentarum sit 45, ex iis autem novem ut cognitae spectentur, reliquae 36 per has litteras majusculas determinari debebunt.

§. 115. Istas autem determinationes ex aequatione generali supra demonstrata peti oportet, quae hac forma continetur

$$(a, b) \ (a + b, c) = (a, c) \ (a + c, b),$$

ubi assumere licebit, semper esse $b > c$, quoniam, si foret $c = b$, aequatio foret identica. Primo igitur ut hinc aequationes, quae immediate determinationes praebeant, nanciscamur, sumamus $a + b = 10$, ut sit $(10, c) = \frac{1}{c}$; tum vero capiatur $c = b - 1$, quo facto pro a ordine scribendo numeros 1, 2, 3, etc. sequentes prodibunt determinationes

$$(1, 9) \ (10, 8) = (1, 8) \ (9, 9), \text{ sive } \tfrac{1}{8} A = P \ (9, 9), \text{ ergo}$$
$$(9, 9) = \frac{A}{8P}.$$

$$(2, 8) \ (10, 7) = (2, 7) \ (9, 8), \text{ sive } \tfrac{1}{7} B = Q \ (9, 8), \text{ ergo}$$
$$(9, 8) = \frac{B}{7Q}.$$

$(3, 7)\,(10, 6) = (3, 6)\,(9, 7)$, sive $\frac{1}{6}C = R\,(9, 7)$, ergo
$$(9, 7) = \frac{C}{6R}.$$
$(4, 6)\,(10, 5) = (4, 5)\,(9, 6)$, sive $\frac{1}{5}D = S\,(9, 6)$, ergo
$$(9, 6) = \frac{D}{5S}.$$
$(5, 5)\,(10, 4) = (5, 4)\,(9, 5)$, sive $\frac{1}{4}E = S\,(9, 5)$, ergo
$$(9, 5) = \frac{E}{4S}.$$
$(6, 4)\,(10, 3) = (6, 3)\,(9, 4)$, sive $\frac{1}{3}D = R\,(9, 4)$, ergo
$$(9, 4) = \frac{D}{3R}.$$
$(7, 3)\,(10, 2) = (7, 2)\,(9, 3)$, sive $\frac{1}{2}C = Q\,(9, 3)$, ergo
$$(9, 3) = \frac{C}{2Q}.$$
$(8, 2)\,(10, 1) = (8, 1)\,(9, 2)$, sive $B = P\,(9, 2)$, ergo
$$(9, 2) = \frac{B}{P}.$$

§. 116. Ex formulis igitur incognitis illis numero 36 jam octo determinavimus, quae nobis viam sternent ad novas determinationes, quas primo derivabimus ex aequatione generali sumendo $a = 1$, $b = 9$, et pro c scribendo ordine numeros 1, 2, 3 . . . . . 8, unde calculus ita se habebit

$(1, 9)\,(10, 1) = (1, 1)\,(2, 9)$	$A = (1, 1)\frac{B}{P}$, ergo $(1, 1) = \frac{AP}{B}$
$(1, 9)\,(10, 2) = (1, 2)\,(3, 9)$	$\frac{1}{2}A = (1, 2)\frac{C}{2Q}$, ergo $(1, 2) = \frac{AQ}{C}$
$(1, 9)\,(10, 3) = (1, 3)\,(4, 9)$	$\frac{1}{3}A = (1, 3)\frac{D}{3R}$, ergo $(1, 3) = \frac{AR}{D}$
$(1, 9)\,(10, 4) = (1, 4)\,(5, 9)$	$\frac{1}{4}A = (1, 4)\frac{E}{4S}$, ergo $(1, 4) = \frac{AS}{E}$
$(1, 9)\,(10, 5) = (1, 5)\,(6, 9)$	$\frac{1}{5}A = (1, 5)\frac{D}{5S}$, ergo $(1, 5) = \frac{AS}{D}$
$(1, 9)\,(10, 6) = (1, 6)\,(7, 6)$	$\frac{1}{6}A = (1, 6)\frac{C}{6R}$, ergo $(1, 6) = \frac{AR}{C}$
$(1, 9)\,(10, 7) = (1, 7)\,(8, 9)$	$\frac{1}{7}A = (1, 7)\frac{B}{7Q}$, ergo $(1, 7) = \frac{AQ}{B}$
$(1, 9)\,(10, 8) = (1, 8)\,(9, 9)$	$\frac{1}{8}A = (1, 8)\frac{A}{8P}$, ergo $(1, 8) = \frac{AP}{A}$

hocque modo septem novas determinationes sumus adepti.

§. 117. His autem inventis consideremus aequationes ex valoribus $a = 1$, $b = 8$, $c = 1, 2, \ldots \ldots 7$ ortas, eritque

$(1,8)\,(9,1) = (1,1)\,(2,8)$	$AP = (1,1)\,B$	Identica.
$(1,8)\,(9,2) = (1,2)\,(3,8)$	$B = (3,8)\,\frac{AQ}{C}$	$(3,8) = \frac{BC}{AQ}$
$(1,8)\,(9,3) = (1,3)\,(4,8)$	$\frac{CP}{2Q} = (4,8)\,\frac{AR}{D}$	$(4,8) = \frac{CDP}{2AQR}$
$(1,8)\,(9,4) = (1,4)\,(5,8)$	$\frac{DP}{3R} = (5,8)\,\frac{AS}{E}$	$(5,8) = \frac{DEP}{3ARS}$
$(1,8)\,(9,5) = (1,5)\,(6,8)$	$\frac{EP}{4S} = (6,8)\,\frac{AS}{D}$	$(6,8) = \frac{DEP}{4ASS}$
$(1,8)\,(9,6) = (1,6)\,(7,8)$	$\frac{DP}{5S} = (7,8)\,\frac{AR}{C}$	$(7,8) = \frac{CDP}{5ARS}$
$(1,8)\,(9,7) = (1,7)\,(8,8)$	$\frac{CP}{6R} = (8,8)\,\frac{AQ}{B}$	$(8,8) = \frac{BCP}{6AQR}$

§. 118. Novas determinationes reperiemus ponendo $a = 1$, $b = 7$, $c = 3, 4, 5, 6$; hinc enim nanciscimur sequentes determinationes

$(1,7)\,(8,3) = (1,3)\,(4,7)$	$C = (4,7)\,\frac{AR}{D}$	$(4,7) = \frac{CD}{AR}$
$(1,7)\,(8,4) = (1,4)\,(5,7)$	$\frac{CDP}{2BR} = (5,7)\,\frac{AS}{E}$	$(5,7) = \frac{CDEP}{2ABRS}$
$(1,7)\,(8,5) = (1,5)\,(6,7)$	$\frac{DBPQ}{3BRS} = (6,7)\,\frac{AS}{D}$	$(6,7) = \frac{DDEPQ}{3ABRSS}$
$(1,7)\,(8,6) = (1,6)\,(7,7)$	$\frac{DEPQ}{4BSS} = (7,8)\,\frac{AR}{C}$	$(7,7) = \frac{CDEPQ}{4ABRSS}$

§. 119. Sumamus nunc $a = 1$, $b = 6$, $c = 4, 5$, eritque

$(1,6)\,(7,4) = (1,4)\,(5,6)$	$D = (5,6)\,\frac{AS}{E}$	$(5,6) = \frac{DE}{AS}$
$(1,6)\,(7,5) = (1,5)\,(6,6)$	$\frac{DEP}{2BS} = (6,6)\,\frac{AS}{D}$	$(6,6) = \frac{DDEP}{2ABSS}$

Hactenus igitur omnes formulas (p, q) determinavimus, in quibus $p + q > 10$. Ex reliquis autem, ubi $p + q > 9$, jam nacti sumus istas

(1, 1), (1, 2), (1, 3), (1, 4), (1, 5), (1, 6), (1, 7),

ita ut adhuc determinandae relinquantur istae

(2, 2), (2, 3), (2, 4), (2, 5), (2, 6),
(3, 3), (3, 4), (3, 5),
(4, 4).

§. 120. Pro his inveniendis sumamus $a = 1$ et $c = 1$, pro b autem ordine capiamus numeros 2, 3, etc. atque consequemur has aequationes

$(1,2)\ (3,1) = (1,1)\ (2,2)$	$\frac{AAQR}{CD} = (2,2)\frac{AP}{B}$	$(2,2) = \frac{ABQR}{CDP}$
$(1,3)\ (4,1) = (1,1)\ (2,3)$	$\frac{AARS}{DE} = (2,3)\frac{AP}{B}$	$(2,3) = \frac{ABRS}{DEP}$
$(1,4)\ (5,1) = (1,1)\ (2,4)$	$\frac{AASS}{DE} = (2,4)\frac{AP}{B}$	$(2,4) = \frac{ABSS}{DEP}$
$(1,5)\ (6,1) = (1,1)\ (2,5)$	$\frac{AARS}{CD} = (2,5)\frac{AP}{B}$	$(2,5) = \frac{ABRS}{CDP}$
$(1,6)\ (7,1) = (1,1)\ (2,6)$	$\frac{AAQR}{BC} = (2,6)\frac{AP}{B}$	$(2,6) = \frac{ABQR}{BCP}$

sicque etiamnunc determinandae restant formulae (3, 3), (3, 4), (3, 5) et (4, 4).

§. 121. Pro his sumatur $a = 1$, $c = 2$, et $b = 3, 4, 5$, etc. tum enim prodibunt hae aequationes

$(1,3)\ (4,2) = (1,2)\ (3,3)$	$\frac{AABRSS}{DDEP} = (3,3)\frac{AQ}{C}$	$(3,3) = \frac{ABCRSS}{DDEPQ}$
$(1,4)\ (5,2) = (1,2)\ (3,4)$	$\frac{AABRSS}{CDEP} = (3,4)\frac{AQ}{C}$	$(3,4) = \frac{ABRSS}{DEPQ}$
$(1,5)\ (6,2) = (1,2)\ (3,5)$	$\frac{AAQRS}{CDP} = (3,5)\frac{AQ}{C}$	$(3,5) = \frac{ARS}{DP}$

Unica ergo formula restat determinanda, scilicet (4, 4), quae ex hac aequatione $(1, 4)\ (5, 3) = (1, 3)\ (4, 4)$ definietur; erit enim $\frac{AARSS}{DEP} = (4, 4)\frac{AR}{D}$, ideoque $(4, 4) = \frac{ASS}{EP}$.

§. 122. Ut nunc omnes has determinationes simul aspectui exponamus, quoniam in hoc ordine $n = 10$ omnino 45 formulae integrales occurrunt, si ex iis ut cognitae spectentur novem sequentes

$$(1,9) = A,\ (2,8) = B,\ (3,7) = C,\ (4,6) = D,\ (5,5) = E,$$
$$(1,8) = P,\ (2,7) = Q,\ (3,6) = R,\ (4,5) = S,$$

reliquae triginta sex ex his sequenti modo determinabuntur

1. $(9, 9) = \frac{A}{8P}$	19. $(2, 6) = \frac{AQR}{CP}$
2. $(9, 8) = \frac{B}{7Q}$	20. $(3, 5) = \frac{ARS}{DP}$
3. $(9, 7) = \frac{C}{6R}$	21. $(4, 4) = \frac{ASS}{EP}$
4. $(9, 6) = \frac{D}{5S}$	22. $(4, 8) = \frac{CDP}{2AQR}$
5. $(9, 5) = \frac{E}{4S}$	23. $(5, 8) = \frac{DEP}{3ARS}$
6. $(9, 4) = \frac{D}{3R}$	24. $(6, 8) = \frac{DEP}{4ASS}$
7. $(9, 3) = \frac{C}{2Q}$	25. $(7, 8) = \frac{CDP}{5ARS}$
8. $(9, 2) = \frac{B}{P}$	26. $(8, 8) = \frac{BCP}{6AQR}$
9. $(1, 1) = \frac{AP}{B}$	27. $(2, 2) = \frac{ABQR}{CDP}$
10. $(1, 2) = \frac{AQ}{C}$	28. $(2, 3) = \frac{ABRS}{DEP}$
11. $(1, 3) = \frac{AR}{D}$	29. $(2, 4) = \frac{ABSS}{DEP}$
12. $(1, 4) = \frac{AS}{E}$	30. $(2, 5) = \frac{ABRS}{CDP}$
13. $(1, 5) = \frac{AS}{D}$	31. $(5, 7) = \frac{CDEP}{2ABRS}$
14. $(1, 6) = \frac{AR}{C}$	32. $(6, 6) = \frac{DDEP}{2ABSS}$
15. $(1, 7) = \frac{AQ}{B}$	33. $(3, 4) = \frac{ABRSS}{DEPQ}$
16. $(3, 8) = \frac{BC}{AQ}$	34. $(6, 7) = \frac{DDEPQ}{3ABRSS}$
17. $(4, 7) = \frac{CD}{AR}$	35. $(7, 7) = \frac{CDEPQ}{4ABRSS}$
18. $(5, 6) = \frac{DE}{AS}$	36. $(3, 3) = \frac{ABCRSS}{DDEPQ}$

§. 123. Eadem methodo, qua hic usi sumus pro casu $n = 10$, haud difficile erit ordines altiores evolvere; neque tamen hinc adhuc elucet, quanam lege omnes determinationes progrediantur, quandoquidem valores certarum formularum continuo magis evadunt complicati. Ceterum valores, quos hic invenimus, omnibus aequationibus in forma generali

$$(a, b)\ (a + b, c) = (a, c)\ (a + c, b)$$

contentis satisfacere deprehenduntur, ita ut perpetuo aequatio identica resultet, neque idcirco inde ulla nova relatio inter litteras nostras majusculas deduci queat. Tandem probe hic notasse juvabit, quod in omnibus ordinibus, praeter formulas a circulo pendentes, commodissime eae formulae, quae in ordine proxime praecedente erant circulares, hic etiam tanquam cognitae accipi queant, quippe quibus determinationes omnes optimo successu perfici possunt.

Methodus generalis determinandi valores formulae

$$(p, q) = \int \frac{x^{p-1}\,\partial x}{\sqrt[n]{(1 - x^n)^{n-q}}} = \int \frac{x^{q-1}\,\partial x}{\sqrt[n]{(1 - x^n)^{n-p}}},$$

a termino $x = 0$ usque ad $x = 1$ extensa: ubi praeter formulas circulum involventes, in quibus est $p + q = n$, etiam illae pro cognitis accipiuntur, in quibus est $p + q = n - 1$.

I. Cum aequatio generalis, unde omnes hae determinationes sunt petendae, sit

$$(a, b)\ (a + b, c) = (a, c)\ (a + c, b),$$

sumatur primo $a = n - \alpha$, $b = \alpha$, et $c = \alpha - 1$, eritque aequatio

$$(n-\alpha, \alpha)\,(n, \alpha-1) = (n-\alpha, \alpha-1)\,(n-1, \alpha),$$

ubi est $(n, \alpha - 1) = \frac{1}{\alpha-1}$. In primo autem factore, ob $p = n - \alpha$ et $q = \alpha$, est $p + q = n$, ideoque datur. In tertio porro factore, ubi $p = n - \alpha$ et $q = \alpha - 1$, est $p + q = n - 1$, ideoque pariter datur. Hinc ergo colligimus

$$(n-1, \alpha) = \frac{1}{\alpha-1} \cdot \frac{(n-\alpha, \alpha)}{(n-\alpha, \alpha-1)},$$

ubi esse debet $\alpha > 1$, ita ut pro α accipi queant omnes numeri a 2 usque ad $n - 1$; at vero casu $\alpha = 1$ valor formulae per se est notus.

II. In aequatione generali jam sumatur $a = \beta$, $b = n - \beta - 1$, et $c = 1$, eritque nostra aequatio

$$(\beta, n-\beta-1)\,(n-1, 1) = (\beta, 1)\,(\beta+1, n-\beta-1),$$

ex qua aequatione colligitur

$$(\beta, 1) = \frac{(\beta, n-\beta-1)\,(n-1, 1)}{(\beta+1, n-\beta-1)},$$

ubi esse debet $\beta < n - 1$, ita ut hinc omnes formulae $(\beta, 1)$ definiantur, a valore $\beta = 1$ usque ad $\beta = n - 1$, quo posteriore casu formula $(n - 1, 1)$ per se cognoscitur.

III. Ut hinc etiam alias formas eliciamus, sumamus $a = 1$, $b = n - 2$, $c = \gamma$, ut oriatur haec aequatio

$$(1, n-2)\,(n-1, \gamma) = (1, \gamma)\,(1+\gamma, n-2),$$

ubi primus factor ac tertius dantur per N°. II. secundus vero per N°. I. unde quartus derivatur, scilicet

$$(1+\gamma, n-2) = \frac{(1, n-2)\,(n-1, \gamma)}{(1, \gamma)},$$

ubi valores ipsius $1 + \gamma$ a 2 usque ad $n - 2$ augeri possunt.

Cum igitur per N°. I. sit

$$(n-1, \gamma) = \frac{1}{\gamma-1} \cdot \frac{(n-\gamma, \gamma)}{(n-\gamma, \gamma-1)},$$

tum vero per N°. II. fit

$$(\gamma, 1) = \frac{(\gamma, n-\gamma-1)\,(n-1, 1)}{(\gamma+1, n-\gamma-1)},$$

his valoribus substitutis fiet

$$(n-2, 1+\gamma) = \frac{1}{\gamma-1} \cdot \frac{(1, n-2)\,(n-\gamma, \gamma)\,(\gamma+1, n-\gamma-1)}{(n-\gamma, \gamma-1)\,(\gamma, n-\gamma-1)\,(n-1, 1)}.$$

IV. Sumamus nunc $a = 1$, $b = n-3$, $c = \delta$, prodibitque haec aequatio

$$(1, n-3)\,(n-2, \delta) = (1, \delta)\,(1+\delta, n-3),$$

unde colligitur

$$(n-3, 1+\delta) = \frac{(n-3, 1)\,(n-2, \delta)}{(\delta, 1)},$$

ubi ergo $1+\delta$ continet numeros $2, 3, 4 \ldots n-3$, ita ut hinc excludatur $n-3, 1$, quae autem per N°. I. datur. At si valores ante reperti substituantur, fiet

$$(n-3, 1+\delta) = \frac{1}{\delta-2} \cdot \frac{(n-3, 2)\,(n-2, 1)\,(n-\delta+1, \delta-1)\,(\delta, n-\delta)\,(\delta+1, n-\delta-1)}{(n-2, 2)\,(n-\delta+1, \delta-2)\,(\delta-1, n-\delta)\,(n-1, 1)\,(\delta, n-\delta-1)},$$

unde patet esse debere $\delta > 2$, eodemque modo pro praecedente formula $\gamma > 1$, ita ut hic excludantur casus $(n-3, 1)$, $(n-3, 2)$, quorum quidem prior per N°. I. datur, alter vero per se.

V. Statuamus nunc $a = 1$, $b = n-4$ et $c = \varepsilon$, prodibitque haec aequatio

$$(1, n-4)\,(n-3, \varepsilon) = (1, \varepsilon)\,(1+\varepsilon, n-4),$$

unde concluditur

$$(n-4, 1+\varepsilon) = \frac{(n-4, 1)\,(n-3, \varepsilon)}{(1, \varepsilon)};$$

ubi si loco $(n-3, \varepsilon)$ valor ante inventus substitueretur, factor

absolutus ingrederetur $\frac{1}{\varepsilon - 3}$, ita ut esse debeat $\varepsilon > 3$, ideoque $1 + \varepsilon > 4$, unde hic excluduntur casus $(n - 4, 1)$, $(n - 4, 2)$, $(n - 4, 3)$, quorum quidem primus ex N°. II. tertius autem per se datur, medius vero revera manet incognitus.

VI. Statuamus porro $a = 1$, $b = n - 5$, $c = \zeta$, et aequatio erit

$$(1, n - 5)\ (n - 4, \zeta) = (1, \zeta)\ (1 + \zeta, n - 5),$$

unde fit

$$(n - 5, 1 + \zeta) = \frac{(n-5, 1)\ (n-4, \zeta)}{(1, \zeta)},$$

ubi ob formulam $(n - 4, \zeta)$ debet esse $\zeta > 4$, ideoque $1 + \zeta > 5$, unde hinc excluduntur casus $(n - 5, 1)$, $(n - 5, 2)$, $(n - 5, 3)$, $(n - 5, 4)$, quorum quidem primus ex N°. II. constat, quartus vero per se datur, ita ut hic occurrant duo casus etiamnunc incogniti $(n - 5, 2)$ et $(n - 5, 3)$.

VII. Simili modo si ulterius sumamus $a = 1$, $b = n - 6$ et $c = \eta$, prodibit

$$(n - 6, 1 + \eta) = \frac{(n-6, 1)\ (n-5, \eta)}{(1, \eta)},$$

ubi revera occurrunt tres sequentes casus $(n - 6, 2)$, $(n - 6, 3)$, $(n - 6, 4)$, qui adhuc manent incogniti, atque hoc modo progredi licebit, quousque necesse fuerit; unde patet numerum casuum incognitorum continuo augeri, ita ut terminorum p et q alter futurus sit vel 2, vel 3, vel 4, etc. qui igitur casus adhuc definiendi restant.

VIII. Sumamus nunc primo $a = 1$, $b = \theta$, $c = 1$, ut aequatio nostra fiat

$$(1, \theta)\ (1 + \theta, 1) = (1, 1)\ (2, \theta),$$

unde concludimus

$$(2, \theta) = \frac{(1, \theta)\,(1+\theta, 1)}{(1, 1)},$$

quae formula jam omnes casus exclusos suppeditat, in quibus alter terminus erat 2.

IX. Deinde sumamus $a = 2$, $b = \varkappa$ et $c = 1$, ut aequatio prodeat $(2, \varkappa)\,(2+\varkappa, 1) = (2, 1)\,(3, \varkappa)$, unde fit

$$(3, \varkappa) = \frac{(2, \varkappa)\,(2+\varkappa, 1)}{(2, 1)},$$

ubi cum $(2, \varkappa)$ per praecedentem N^{rum} detur, nunc etiam ii casus innotescunt, ubi alter terminus erat 3.

X. Sumatur porro $a = 3$, $b = \varkappa$, $c = 1$, eritque $(3, \varkappa)\,(3+\varkappa, 1) = (3, 1)\,(4, \varkappa)$, unde fit

$$(4, \varkappa) = \frac{(3, \varkappa)\,(3+\varkappa, 1)}{(3, 1)},$$

unde igitur ii casus eliciuntur, ubi alter terminus erat 4. Eodem modo pro reliquis proceditur; sicque omnes plane casus in formula proposita contenti plene sunt determinati.

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4) De valoribus integralium a termino variabilis $x = 0$ usque ad $x = \infty$ extensorum. *M. S. Academiae exhib. d.* 30 *Aprilis* 1781.

§. 124. Talium formularum, quae a termino $x = 0$ usque ad terminum $x = \infty$ extensae finitum sortiuntur valorem, simplicissima est circularis $\int \frac{\partial x}{1+xx}$, cujus valor est $\frac{\pi}{2}$ denotante π peripheriam pro diametro $= 1$. Deinde etiam methodo prorsus singulari inveni esse

$$\int \frac{x^{m-1}\,\partial x}{(1+x)^n} \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\pi}{n \sin. \frac{m\pi}{n}}.$$

Praeterea vero hoc modo plures alias formulas hujus generis expedivi, in quarum differentialia non solum functiones algebraicae ipsius x sed etiam lx ingrediatur.

§. 125. Obtulerunt se mihi autem quondam aliae hujusmodi formulae etiam functiones transcendentes involventes, quarum valores desiderati omnes methodos adhuc cognitas respuere videantur. Quaesiveram scilicet eam lineam curvam in qua radius osculi ubique reciproce esset proportionalis arcui curvae, ita ut posito arcu $= s$ et radio osculi $= r$, esset $rs = aa$. Hinc enim haud difficile est, figuram curvae libero quasi manus ductu describere, quandoquidem ea talem habere
Fig. 2. debet figuram. Initio nimirum curvae in A constituto inde curva continuo magis incurvabitur et tandem post infinitas spiras in certum punctum O glomerabitur, quod polum hujus curvae appellare licebit. Propositum igitur mihi fuerat locum hujus poli accuratius investigare, pro eoque quantitatem coordinatarum A C et C O perscrutari.

§. 126. Hunc in finem, introducta in calculum portionis cujusvis AM $= s$ amplitudine $= \Phi$, ut sit $r = \frac{\partial s}{\partial \Phi}$, fit $s\partial s = aa\partial\Phi$, hincque

$$ss = 2aa\Phi, \text{ et } s = a\sqrt{2}\Phi = 2c\sqrt{\Phi}.$$

Hinc jam prodit $\partial s = \frac{c\partial\Phi}{\sqrt{\Phi}}$, unde posita abscissa pro hoc arcu AP $= x$ et applicata PM $= y$, colligitur fore

$$x = c\int\frac{\partial\Phi \cos.\Phi}{\sqrt{\Phi}} \text{ et } y = c\int\frac{\partial\Phi \sin.\Phi}{\sqrt{\Phi}}.$$

§. 127. Hinc ergo pro polo O determinando requiruntur valores harum duarum formularum integralium, postquam a termino $\Phi = 0$ usque ad $\Phi = \infty$ fuerint extensae. Initio quidem sum arbitratus, hos valores aliter obtineri non

posse nisi approximando, dum utraque formula successive per partes evolvatur; primo scilicet a $\Phi = 0$ usque ad $\Phi = \pi$; deinde a $\Phi = \pi$ usque ad $\Phi = 2\pi$; porro a $\Phi = 2\pi$ usque ad $\Phi = 3\pi$; etc. quippe quo pacto series prodibunt satis prompte convergentes. Verum evidens est hanc operationem longos calculos satis taediosos requirere, quos quidem evolvere non sum ausus. Nuper autem forte fortuna per methodum prorsus singularem perspexi esse tam

$$\int \frac{\partial \Phi \cos.\Phi}{\sqrt{\Phi}} \left[\begin{smallmatrix} a & \Phi = 0 \\ ad & \Phi = \infty \end{smallmatrix}\right] = \sqrt{\tfrac{\pi}{2}} \text{ quam}$$

$$\int \frac{\partial \Phi \sin.\Phi}{\sqrt{\Phi}} \left[\begin{smallmatrix} a & \Phi = 0 \\ ad & \Phi = \infty \end{smallmatrix}\right] = \sqrt{\tfrac{\pi}{2}};$$

ita ut pro loco poli quaesito O sit

$$AC = c\sqrt{\tfrac{\pi}{2}} \text{ et } CO = c\sqrt{\tfrac{\pi}{2}}.$$

§. 128. Quoniam igitur methodus, qua huc sum perductus, non parum polliceri videtur, Geometris haud ingratum fore arbitror, si eam omni cura hic exposuero. Et quia multo latius quam ad istas formulas patet, eam etiam omni extensione sum propositurus, quae omnia ex consideratione hujus formulae satis simplicis $\int x^{n-1} \partial x e^{-x}$ deduxi, cujus ergo integrale pro variis valoribus exponentis n investigare convenit.

§. 129. Ac primo quidem, pro casu $n = 1$ hujus formulae $\int \partial x e^{-x}$ integrale manifestum est $1 - e^{-x}$, quod casu $x = 0$ evanescit, facto autem $x = \infty$ abit in unitatem. Praeterea, cum hujus formulae $x^\lambda . e^{-x}$ differentiale sit

$$\lambda x^{\lambda-1} \partial x . e^{-x} - x^\lambda \partial x . e^{-x},$$

erit vicissim

$$\int x^\lambda \partial x . e^{-x} = \lambda \int x^{\lambda-1} \partial x . e^{-x} - x^\lambda . e^{-x},$$

quod postremum membrum tam pro casu $x = 0$ quam $x = \infty$ evanescit, si modo fuerit $\lambda > 0$. Tum igitur pro nostris ter-

minis integrationis erit

$$\int x^\lambda \partial x \,.\, e^{-x} = \lambda \int x^{\lambda-1} \partial x \,.\, e^{-x},$$

cujus formulae ope, ob $\int \partial x e^{-x} = 1$, sequentes integralium valores deducuntur

$$\int x \partial x e^{-x} = 1$$
$$\int x^2 \partial x \,.\, e^{-x} = 1.2$$
$$\int x^3 \partial x \,.\, e^{-x} = 1.3.3$$
$$\int x^4 \partial x \,.\, e^{-x} = 1.2.3.4$$

sicque in genere

$$\int x^{n-1} \partial x e^{-x} = 1.2.3.4 \,.\,.\,.\,.\,.\,.\,.\,. (n-1),$$

cujus producti valores quoties n fuerit numerus integer positivus sponte se produnt; quando autem n est numerus fractus olim ostendi, quomodo valores per quadraturas curvarum algebraicarum exhiberi queant. Sic pro casu $n = \frac{1}{2}$ constat, istum valorem esse $= \sqrt{\pi}$.

§. 130. Cum igitur omnes valores hujus producti infiniti $1.2.3.4 \ldots . (n-1)$ tanquam cogniti spectari queant, eos littera Δ designabo, ita ut sit $\Delta = 1.2.3.4 \ldots . (n-1)$, sicque jam adepti sumus hanc insignem formulam integralem

$$\int x^{n-1} \partial x \,.\, e^{-x} = \Delta,$$

integrali scilicet ab $x = 0$ ad $x = \infty$ extenso; atque ex hac ipsa formula omnia deduxi, quae ad casum ante memoratum pertinent, ubi quidem ratiocinia penitus singularia adhiberi debent, quae igitur hic diligentius sum expositurus.

§. 131. Posui autem primo $x = ky$, et quoniam ambo termini integralis iidem manent, erit etiam

$$k^n \int y^{n-1} \partial y \,.\, e^{-ky} = \Delta,$$

quandoquidem haec formula etiam ab $y = 0$ ad $y = \infty$ usque

extenditur; quamobrem per k^n dividendo habebimus

$$\int y^{n-1}\partial y \,.\, e^{-ky} = \frac{\Delta}{k^n},$$

ubi autem notari oportet, pro k nullos numeros negativos accipi posse, quia alioquin formula e^{-ky} non amplius evanesceret casu $x = 0$, atque hic isti soli valores sunt excludendi, ita ut etiam valores imaginarii loco k adhiberi queant, atque hinc illas arduas integrationes sum assecutus.

§. 132. Ponamus ergo $k = p + q\sqrt{-1}$, et cum sit

$$e^{-qy\sqrt{-1}} = \cos. qy - \sqrt{-1} \sin. qy, \text{ et}$$
$$e^{+qy\sqrt{-1}} = \cos. qy + \sqrt{-1} \sin. qy,$$

nostra formula nunc induet hanc formam

$$\int y^{n-1}\partial y \,.\, e^{-py}(\cos. qy - \sqrt{-1} \sin. qy) = \frac{\Delta}{(p+q\sqrt{-1})^n}.$$

Quamobrem si formulae imaginariae signum mutemus, erit simili modo

$$\int y^{n-1}\partial y \,.\, e^{-py}(\cos. qy + \sqrt{-1} \sin. qy) = \frac{\Delta}{(p-q\sqrt{-1})^n}.$$

§. 133. Quo valores inventos commodius exprimere liceat, ponamus $p = f\cos.\theta$ et $q = f\sin.\theta$, eritque

$$(p + q\sqrt{-1})^n = f^n(\cos. n\theta + \sqrt{-1} \sin. n\theta) \text{ et}$$
$$(p - q\sqrt{-1})^n = f^n(\cos. n\theta - \sqrt{-1} \sin. n\theta);$$

ubi notasse juvabit fore tang. $\theta = \frac{q}{p}$, unde ex valoribus p et q assumtis erit etiam $f = \sqrt{(pp + qq)}$. Hoc ergo modo fit priore casu

$$\frac{\Delta}{(p+q\sqrt{-1})^n} = \frac{\Delta}{f^n(\cos. n\theta + \sqrt{-1} \sin. n\theta)},$$

pro altero

$$\frac{\Delta}{(p-q\sqrt{-1})^n} = \frac{\Delta}{f^n\,(\cos.\ n\theta - \sqrt{-1}\ \sin.\ n\theta)}.$$

Quamobrem si hae duae formulae addantur prodibit

$$\frac{2\,\Delta\cos.\ n\theta}{f^n}.$$

Differentia autem harum formularum dat

$$\frac{2\,\Delta\sqrt{-1}\ \sin.\ n\theta}{f^n}.$$

§. 134. Addamus igitur quoque ipsas formulas integrales, et habebimus

$$\int y^{n-1}\,\partial y\ .\ e^{-py}\cos.\ qy = \frac{\Delta\cos.\ n\theta}{f^n}.$$

Sin autem subtrahamus et per $2\sqrt{-1}$ dividamus, oritur

$$\int y^{n-1}\,\partial y\ .\ e^{-py}\sin.\ qy = \frac{\Delta\sin.\ n\theta}{f^n}.$$

quae jam duae formulae integrales latissime patent, cum numeri p et q prorsus arbitrio nostro relinquuntur, id tantum observando, ne pro p numeri negativi accipiantur. Operae igitur pretium erit, has duas formulas integrales sequentibus binis theorematibus complecti.

Theorema I.

Posito $\Delta = 1.2.2\ldots\ldots(n-1)$, et pro litteris p et q numeros quoscunque positivos accipiendo, fiat inde $\sqrt{(pp+qq)} = f$, et quaeratur angulus θ, ut fit tang. $\theta = \frac{q}{p}$, et habebitur ista integratio memorabilis

$$\int x^{n-1}\,\partial x\ .\ e^{-px}\cos.\ qx \begin{bmatrix}\text{ab } x = 0\\ \text{ad } x = \infty\end{bmatrix} = \frac{\Delta\cos.\ n\theta}{f^n}.$$

Theorema II.

Posito $\Delta = 1.2.3 \ldots\ldots (n-1)$, et pro litteris p et q numeros quoscunque positivos accipiendo, fiat inde $\sqrt{(pp+qq)} = f$, et quaeratur angulus θ, ut sit tang. $\theta = \frac{q}{p}$, atque habebitur ista integratio memorabilis

$$\int x^{n-1}\partial x \,.\, e^{-px} \sin. qx \begin{bmatrix} \text{ab } x = 0 \\ \text{ad } x = \infty \end{bmatrix} = \frac{\Delta \sin. n\theta}{f^n}.$$

§. 135. Cum igitur pro casu curvae supra consideratae pervenerimus ad has formulas integrales

$$\int \frac{\partial\Phi \cos.\Phi}{\sqrt{\Phi}} \text{ et } \int \frac{\partial\Phi \sin.\Phi}{\sqrt{\Phi}},$$

facta applicatione erit $n = \frac{1}{2}$, ideoque $\Delta = \sqrt{\pi}$, tum vero erit $p = 0$ et $q = 1$, unde fit $f = 1$ et tang. $\theta = \frac{q}{p} = \infty$, ideoque $\theta = \frac{\pi}{2}$, ergo cos. $n\theta = \frac{1}{\sqrt{2}} = $ sin. $n\theta$. Hinc igitur fiet

$$\int \frac{\partial\Phi \cos.\Phi}{\sqrt{\Phi}} \begin{bmatrix} a\ \Phi = 0 \\ ad\ \Phi = \infty \end{bmatrix} = \sqrt{\frac{\pi}{2}}, \text{ simulque}$$

$$\int \frac{\partial\Phi \sin.\Phi}{\sqrt{\Phi}} \begin{bmatrix} a\ \Phi = 0 \\ ad\ \Phi = \infty \end{bmatrix} = \sqrt{\frac{\pi}{2}}.$$

§. 136. Operae autem pretium erit, hunc casum quo $n = \frac{1}{2}$ et $\Delta = \sqrt{\pi}$ in genere evolvere, et cum posuerimus

$$\sqrt{(pp+qq)} = f \text{ et } \frac{q}{p} = \text{tang. } \theta, \text{ erit}$$

$$\sin.\theta = \frac{q}{f} \text{ et } \cos.\theta = \frac{p}{f}.$$

Hinc ergo primo

$$\sin.\tfrac{1}{2}\theta = \sqrt{\frac{1-\cos.\theta}{2}} = \sqrt{\frac{f-p}{2f}} \text{ et}$$

$$\cos.\tfrac{1}{2}\theta = \sqrt{\frac{1+\cos.\theta}{2}} = \sqrt{\frac{f+p}{2f}};$$

unde fit pro valoribus integralibus

$$\frac{\Delta \sin.\frac{1}{2}\theta}{\sqrt{f}} = \frac{\sqrt{\pi}}{f}\sqrt{\frac{f-p}{2}} \text{ et}$$

$$\frac{\Delta \cos. \frac{1}{2}\theta}{\sqrt{f}} = \frac{\sqrt{\pi}}{f} \cdot \sqrt{\frac{f+p}{2}}.$$

Quamobrem habebimus binas sequentes formulas integrales

$$\int \frac{\partial x}{\sqrt{x}} e^{-px} \sin. qx = \frac{\sqrt{\pi}}{f} \cdot \sqrt{\frac{f-p}{2}}$$

$$\int \frac{\partial x}{\sqrt{x}} e^{-px} \cos. qx = \frac{\sqrt{\pi}}{f} \cdot \sqrt{\frac{f+p}{2}}.$$

§. 137. Casus autem, quibus pro n sumitur numerus integer positivus, ideoque Δ absolute per numeros integros exhiberi potest, ita sunt comparati, ut etiam per methodos cognitas, ope scilicet formularum integralium reductionis satis notae expediri queant, atque adeo integralia in genere exhiberi. Haec autem operatio postulat calculos non parum prolixos, quamobrem formulae nostrae satis simplices pro casu scilicet $x = \infty$ nihilo minus omni attentione sunt dignae. Quando autem exponenti n valores negativos tribuere voluerimus, hi casus statim in initio integrationis additionem constantis infinitae postulant, ut scilicet integralia evanescant casu $x = 0$, sicque adeo valores integralium, quae hic quaerimus, manebunt infiniti, ideoque ad institutum nostrum non sunt referendi.

§. 138. Casus autem maxime memorabilis hic occurrit, quo $n = 0$, et qui prorsus singularem sollertiam postulat, quem igitur accuratius evolvamus. Quoniam posuimus

$$\Delta = 1 . 2 . 3 . 4 \ldots . (n-1),$$

statuamus simili modo

$$\Delta' = 1 . 2 . 3 \ldots . n, \text{ et } \Delta'' = 1 . 2 . 3 \ldots\ldots (n+1),$$

eritque manifesto

$$\Delta = \frac{\Delta'}{n}, \text{ et } \Delta' = \frac{\Delta''}{n+1}, \text{ ideoque } \Delta = \frac{\Delta''}{n(n+1)}.$$

Sumamus nunc $n = \omega$, existente ω infinite parvo, et cum sit

$\Delta'' = 1$, inde fit $\Delta = \frac{1}{\omega}$, ideoque ejus valor erit infinitus. Cum autem pro formula integrali priore sit sin. $n\theta = \omega\theta$, evidens est fore Δ sin. $n\theta = \theta$; quamobrem ista prior formula integralis erit $\int \frac{\partial x}{x} e^{-px}$ sin. $qx = \theta$, dum nempe integrale a termino $x = 0$ usque ad terminum $x = \infty$ extenditur. Alterius autem formulae nostrae integralis $\int \frac{\partial x}{x} e^{-px}$ cos. qx valor erit infinite magnus. Ille autem casus omnino meretur ut eum singulari theoremate complectamur.

Theorema III.

§. 139. Si litterae p et q denotent numeros positivos quoscunque, atque hinc quaeratur angulus θ, ut sit tang. $\theta = \frac{q}{p}$, habebitur sequens integratio maxime memorabilis

$$\int \frac{\partial x}{x} e^{-px} \text{ sin. } qx \begin{bmatrix} ab\ x = 0 \\ ad\ x = \infty \end{bmatrix} = \theta$$

cujus theorematis demonstratio dubito quin alio modo quam per approximationes investigari queat.

§. 140. Casus autem simplicissimus quo $p = 0$ et $q = 1$ jam omnia calculi artificia adhuc cognita superare videtur, quia autem hoc casu fit tang. $\theta = \frac{1}{0} = \infty$, erit $\theta = \frac{\pi}{2}$, unde oritur haec integratio $\int \frac{\partial x}{x}$ sin. $x = \frac{\pi}{2}$. Interim tamen de ejus veritate eo minus dubitare licet, quod approximationes adhibitae ad eundem valorem propemodum perducant. Quodsi hunc casum cum initio memorato $\int \frac{\partial x}{\sqrt{x}}$ sin. $x = \sqrt{\frac{\pi}{2}}$ comparemus, ingens similitudo summam attentionem meretur, cum hujus integrale sit praecise radix quadrata illius.

5) Investigatio formulae integralis $\int \frac{x^{m-1}\,\partial x}{(1+x^k)^n}$, casu quo post integrationem statuitur $x = \infty$. *Opuscula Analytica. Tom. II. Pag.* 42 — 54.

§. 141. Jam satis notum est, hujus formulae integrale partim logarithmos, partim arcus circulares complecti, et partes logarithmicas hanc progressionem constituere

$$
\begin{aligned}
&-\tfrac{2}{k}\cos.\tfrac{m\pi}{k}\; l\sqrt{(1-2x\cos.\tfrac{\pi}{k}+xx)}\\
&-\tfrac{2}{k}\cos.\tfrac{3m\pi}{k}\; l\sqrt{(1-2x\cos.\tfrac{3\pi}{k}+xx)}\\
&-\tfrac{2}{k}\cos.\tfrac{5m\pi}{k}\; l\sqrt{(1-2x\cos.\tfrac{5\pi}{k}+xx)}\\
&-\tfrac{2}{k}\cos.\tfrac{7m\pi}{k}\; l\sqrt{(1-2x\cos.\tfrac{7\pi}{k}+xx)}\\
&-\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\\
&-\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\;.\\
&-\tfrac{2}{k}\cos.\tfrac{im\pi}{k}\; l\sqrt{(1-2x\cos.\tfrac{i\pi}{k}+xx)}
\end{aligned}
$$

ubi i denotat numerum imparem non majorem quam k. Hinc si k fuerit numerus par, erit $i = k-1$; ac si k fuerit numerus impar, hanc progressionem continuari oportet usque ad $i = k$, ejus vero coëfficiens duplo minor capi debet, seu loco $-\frac{2}{k}$ tantum scribi debet $-\frac{1}{k}$, cujus irregularitatis ratio in Tomo I est exposita.

§. 142. Cum hae partes sponte jam evanescant posito $x = 0$, statuamus statim $x = \infty$, et cum in genere sit

$$
\begin{aligned}
&\sqrt{(1-2x\cos.\omega+xx)} = x - \cos.\omega, \text{ erit}\\
&l\sqrt{(1-2x\cos.\omega+xx)} = l(x-\cos.\omega)\\
&\qquad = lx - \tfrac{\cos.\omega}{x} = lx, \text{ ob } \tfrac{\cos.\omega}{x} = 0;
\end{aligned}
$$

omnes ergo illi logarithmi reducuntur ad eandem formam lx,

quae multiplicanda est per hanc seriem

$$-\frac{2}{k}\cos.\frac{m\pi}{k}-\frac{2}{k}\cos.\frac{3m\pi}{k}-\frac{2}{k}\cos.\frac{5m\pi}{k}\dots\dots-\frac{2}{k}\cos.\frac{im\pi}{k},$$

ubi, ut diximus, i denotat maximum numerum imparem ipso k non majorem, hac tamen restrictione, ut, si k fuerit impar, ideoque $i = k$, ultimum membrum ad dimidium reduci debeat. Quamobrem, si hujus progressionis summam investigare velimus, duo casus erunt constituendi: alter quo k est numerus par et $i = k - 1$, alter vero quo k est impar et $i = k$.

Evolutio casus prioris, quo k est numerus par et $i = k - 1$.

§. 143. Hoc ergo casu, posito $x = \infty$, formula $-\frac{2}{klx}$ multiplicatur per hanc cosinuum seriem

$$\cos.\frac{m\pi}{k}+\cos.\frac{3m\pi}{k}+\cos.\frac{5m\pi}{k}+\cos.\frac{7m\pi}{k}+\dots+\cos.\frac{(k-1)m\pi}{k},$$

cujus summam statuamus $=$ S. Ducamus hanc seriem in $\sin.\frac{m\pi}{k}$, et cum in genere sit

$$\sin.\frac{m\pi}{k}\cos.\frac{im\pi}{k}=\tfrac{1}{2}\sin.\frac{(i+1)m\pi}{k}-\tfrac{1}{2}\sin.\frac{(i-1)m\pi}{k},$$

facta hac reductione habebimus

$$\begin{aligned}\mathrm{S}\sin.\frac{m\pi}{k}=\tfrac{1}{2}\sin.\frac{2m\pi}{k}+\tfrac{1}{2}\sin.\frac{4m\pi}{k}+\tfrac{1}{2}\sin.\frac{6m\pi}{k}\dots+\tfrac{1}{2}\sin.\frac{(k-2)m\pi}{k}+\tfrac{1}{2}\sin.m\pi\\ -\tfrac{1}{2}\sin.\frac{2m\pi}{k}-\tfrac{1}{2}\sin.\frac{4m\pi}{k}-\tfrac{1}{2}\sin.\frac{6m\pi}{k}\dots-\tfrac{1}{2}\sin.\frac{(k-2)m\pi}{k};\end{aligned}$$

ubi omnes termini praeter ultimum manifesto se destruunt, ita ut sit

$$\mathrm{S}\sin.\frac{m\pi}{k}=\tfrac{1}{2}\sin.m\pi.$$

Jam vero quia nostri coëfficientes m et k supponuntur integri, utique erit $\sin. m\pi = 0$, ideoque etiam $\mathrm{S} = 0$, nisi forte etiam fuerit $\sin.\frac{m\pi}{k} = 0$, qui autem casus locum habere nequit, quoniam in integratione formulae propositae $\frac{x^{m-1}\partial x}{1+x^k}$,

semper assumi solet esse $m < k$. Hoc igitur modo evictum est, casu quo post integrationem statuitur $x = \infty$, omnes partes logarithmicas integralis se destruere.

Evolutio casus alterius, quo est k numerus impar et $i = k$.

§. 144. Hoc ergo casu, sumto $x = \infty$, formula lx multiplicatur per hanc seriem

$$-\frac{2}{k}\cos.\frac{m\pi}{k} - \frac{2}{k}\cos.\frac{3m\pi}{k} - \frac{2}{k}\cos.\frac{5m\pi}{k} \ldots\ldots - \frac{1}{k}\cos.\frac{km\pi}{k},$$

ubi terminus penultimus est $-\frac{2}{k}\cos.\frac{(k-2)m\pi}{k}$, pro ultimo vero termino erit $\cos. m\pi = \pm 1$, signo superiore valente si m sit numerus par, inferiore si impar; quare remoto termino ultimo pro reliquis ponamus

$$\cos.\frac{m\pi}{k} + \cos.\frac{3m\pi}{k} + \cos.\frac{5m\pi}{k} + \ldots\ldots + \cos.\frac{(k-2)m\pi}{k} = S,$$

ita ut multiplicator ipsius logarithmi x sit

$$-\frac{2S}{k} - \frac{1}{k}\cos. m\pi.$$

Hinc procedendo ut ante fiet

$$S\sin.\frac{m\pi}{k} = \tfrac{1}{2}\sin.\frac{2m\pi}{k} + \tfrac{1}{2}\sin.\frac{4m\pi}{k} + \tfrac{1}{2}\sin.\frac{6m\pi}{k} \ldots\ldots + \tfrac{1}{2}\sin.\frac{(k-3)m\pi}{k} + \tfrac{1}{2}\sin.\frac{(k-1)m\pi}{k}$$
$$-\tfrac{1}{2}\sin.\frac{2m\pi}{k} - \tfrac{1}{2}\sin.\frac{4m\pi}{k} - \tfrac{1}{2}\sin.\frac{6m\pi}{k} \ldots\ldots - \tfrac{1}{2}\sin.\frac{(k-3)m\pi}{k};$$

ubi iterum omnes termini praeter ultimum se mutuo tollunt, ita ut hinc prodeat

$$S\sin.\frac{m\pi}{k} = \tfrac{1}{2}\sin.\frac{(k-1)m\pi}{k} = \tfrac{1}{2}\sin.\left(m\pi - \frac{m\pi}{k}\right);$$

at vero est

$$\sin.\left(m\pi - \frac{m\pi}{k}\right) = \sin. m\pi\cos.\frac{m\pi}{k} - \cos. m\pi\sin.\frac{m\pi}{k},$$

ubi notetur esse $\sin. m\pi = 0$, ob m numerum integrum; habebimus ergo

$$S\sin.\frac{m\pi}{k} = -\tfrac{1}{2}\cos. m\pi\sin.\frac{m\pi}{k}, \text{ sive } S = -\tfrac{1}{2}\cos. m\pi,$$

consequenter multiplicator ipsius lx erit

$$=\tfrac{1}{k}\cos. m\pi - \tfrac{1}{k}\cos. m\pi = 0,$$

sicque manifestum est, sive k sit numerus par sive impar, omnia membra logarithmica in nostro integrali se mutuo destruere, siquidem post integrationem statuamus $x=\infty$, quemadmodum hic semper supponimus.

§. 145. Consideremus nunc etiam partes a circulo pendentes, ex quibus integrale nostrae formulae componitur. Hae autem partes sequentem progressionem constituere sunt compertae

$$\frac{2}{k}\sin.\frac{m\pi}{k}\text{ Arc. tang. }\frac{x\sin.\frac{\pi}{k}}{1-x\cos.\frac{\pi}{k}}+\frac{2}{k}\sin.\frac{3m\pi}{k}\text{ Arc. tang. }\frac{x\sin.\frac{3\pi}{k}}{1-x\cos.\frac{3\pi}{k}}$$

$$+\frac{2}{k}\sin.\frac{5m\pi}{k}\text{ Arc. tang. }\frac{x\sin.\frac{5\pi}{k}}{1-x\cos.\frac{5\pi}{k}}+\frac{2}{k}\sin.\frac{7m\pi}{k}\text{ Arc. tang. }\frac{x\sin.\frac{7\pi}{k}}{1-x\cos.\frac{7\pi}{k}}$$

$$+\ldots\ldots\ldots\ldots+\frac{2}{k}\sin.\frac{im\pi}{k}\text{ Arc. tang. }\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}}$$

ubi in ultimo membro est vel $i=k-1$, vel $i=k$; prius scilicet valet si i est numerus par, posterius si impar.

§. 146. Cum etiam omnia haec membra evanescant posito $x=0$, faciamus pro instituto nostro $x=\infty$. In genere igitur fiet

$$\text{Arc. tang. }\frac{x\sin.\frac{i\pi}{k}}{1-x\cos.\frac{i\pi}{k}}=\text{Arc. tang. }\left(-\text{tang. }\frac{i\pi}{k}\right).$$

Est vero

$$-\text{tang. }\frac{i\pi}{k}=+\text{tang. }\frac{(k-i)\pi}{k},$$

ex quo hic arcus fit $=\frac{(k-i)\pi}{k}$. Hinc ergo loco i scribendo

successive numeros 1, 3, 5, 7 etc. istae partes nostri integralis quaesiti erunt

$$\frac{2(k-1)\pi}{kk}\sin.\frac{2m\pi}{k}+\frac{2(k-3)\pi}{kk}\sin.\frac{3m\pi}{k}+\frac{2(k-5)\pi}{kk}\sin.\frac{5m\pi}{k}$$
$$+\frac{2(k-7)\pi}{kk}\sin.\frac{7m\pi}{k}+\frac{2(k-9)\pi}{kk}\sin.\frac{9m\pi}{k}+\dots\frac{2(k-i)\pi}{kk}\sin.\frac{7m\pi}{k}$$

ubi casu, quo k est numerus par, progredi oportet usque ad $i=k-1$: ac si k sit numerus impar, usque ad $i=k$.

§. 147. Statuamus brevitatis gratia

$$(k-1)\sin.\frac{m\pi}{k}+(k-3)\sin.\frac{3m\pi}{k}+(k-5)\sin.\frac{5m\pi}{k}+\dots\dots$$
$$+(k-i)\sin.\frac{im\pi}{k}=S$$

ita ut integrale quaesitum sit $\frac{2\pi S}{kk}$, quandoquidem partes logarithmicae se mutuo destruxerunt. Multiplicemus nunc utrinque per $2\sin.\frac{m\pi}{k}$, et cum in genere sit

$$2\sin.\frac{m\pi}{k}\sin.\frac{im\pi}{k}=\cos.\frac{(i-1)m\pi}{k}-\cos.\frac{(i+1)m\pi}{k},$$

facta substitutione erit

$$2S\sin.\frac{m\pi}{k}=(k-1)\cos.\frac{0m\pi}{k}+(k-3)\cos.\frac{2m\pi}{k}+(k-5)\cos.\frac{4m\pi}{k}\dots$$
$$-(k-1)\cos.\frac{2m\pi}{k}-(k-3)\cos.\frac{4m\pi}{k}-(k-5)\cos.\frac{6m\pi}{k}\dots$$
$$\dots+(k-i)\cos.\frac{(i-1)m\pi}{k}.$$
$$-(k-i)\cos.\frac{(i+1)m\pi}{k}$$

quae series manifesto contrahitur in sequentem

$$2S\sin.\frac{m\pi}{k}=(k-1)-2\cos.\frac{2m\pi}{k}-2\cos.\frac{4m\pi}{k}-2\cos.\frac{6m\pi}{k}\dots$$
$$-2\cos.\frac{(i-1)m\pi}{k}-(k-i)\cos.\frac{(i+1)m\pi}{k}$$

ubi, primo et ultimo membro sublatis, regularem termini intermedii constituunt seriem, pro cujus valore investigando ponamus

$$T=\cos.\frac{2m\pi}{k}+\cos.\frac{4m\pi}{k}+\cos.\frac{6m\pi}{k}+\dots\dots+\cos.\frac{(i-1)m\pi}{k},$$

ita ut sit

$$2\,S\sin.\frac{m\pi}{k}=k-1-2\,T-(k-i)\cos.\frac{(i+1)m\pi}{k}.$$

Hic autem iterum convenit duos casus perpendere, prout k fuerit par vel impar.

Evolutio casus prioris, quo k est numerus par et $i=k-1$.

§. 148. Hoc ergo casu habebimus

$$T=\cos.\frac{2m\pi}{k}+\cos.\frac{4m\pi}{k}+\cos.\frac{6m\pi}{k}+\ldots\ldots\cos.\frac{(k-2)m\pi}{k}.$$

Multiplicemus denuo per $2\sin.\frac{m\pi}{k}$, et per reductiones supra indicatas habebimus

$$\begin{aligned}2\,T\sin.\frac{m\pi}{k}= & \quad\quad\quad +\sin.\frac{3m\pi}{k}+\sin.\frac{5m\pi}{k}\ldots\ldots+\sin.\frac{(k-3)m\pi}{k}+\sin.\frac{(k-1)m\pi}{k}\\ & -\sin.\frac{m\pi}{k}-\sin.\frac{3m\pi}{k}-\sin.\frac{5m\pi}{k}\ldots\ldots-\sin.\frac{(k-3)m\pi}{k}\end{aligned}$$

deletis igitur terminis se mutuo tollentibus erit

$$2\,T\sin.\frac{m\pi}{k}=-\sin.\frac{m\pi}{k}+\sin.\frac{(k-1)m\pi}{k}.$$

Est vero

$$\sin.\frac{(k-1)m\pi}{k}=\sin.\left(m\pi-\frac{m\pi}{k}\right)=\sin.m\pi\cos.\frac{m\pi}{k}-\cos.m\pi\sin.\frac{m\pi}{k},$$

ubi $\sin.m\pi=0$, quamobrem fiet $2\,T=-1-\cos.m\pi$.

§. 149. Invento valore pro T colligitur fore

$$2\,S\sin.\frac{m\pi}{k}=k,\ \text{ideoque}\ S=\frac{k}{2\sin.\frac{m\pi}{k}}.$$

Denique vero ipse valor formulae nostrae integralis, quem quaerimus, erit $\frac{2\pi S}{kk}$, et nunc manifestum est, integrale nostrae formulae, casu quo S est numerus par, fore $\frac{\pi}{k\sin.\frac{m\pi}{k}}$, siquidem post integrationem statuatur $x=\infty$.

Evolutio alterius casus, quo k est numerus impar et $i = k$.

§. 150. Hoc ergo casu est

$$T = \cos.\frac{2m\pi}{k} + \cos.\frac{4m\pi}{k} + \cos.\frac{6m\pi}{k} + \ldots + \cos.\frac{(k-1)m\pi}{k},$$

quae series multiplicata per $2 \sin.\frac{m\pi}{k}$ producet ut ante

$$2T \sin.\frac{m\pi}{k} = \begin{aligned} &+\sin.\frac{3m\pi}{k} + \sin.\frac{5m\pi}{k} \ldots + \sin.\frac{(k-2)m\pi}{k} + \sin.\frac{km\pi}{k} \\ -\sin.\frac{m\pi}{k} &- \sin\frac{3m\pi}{k} - \sin.\frac{5m\pi}{k} \ldots - \sin.\frac{(k-2)m\pi}{k} \end{aligned}$$

unde deletis terminis se mutuo tollentibus reperietur

$$2T \sin.\frac{m\pi}{k} = -\sin.\frac{m\pi}{k} + \sin. m\pi$$

ideoque

$$2T = -1 + \frac{\sin. m\pi}{\sin.\frac{m\pi}{k}} = 1, \text{ ob } \sin. m\pi = 0,$$

hincque porro fiet

$$2S \sin.\frac{m\pi}{k} = k;$$

quare cum valor integralis quaesitus sit $\frac{2\pi S}{kk}$, erit etiam hoc casu integrale nostrum $= \frac{\pi}{k \sin.\frac{m\pi}{k}}$, prorsus uti praecedente casu. Hinc ergo deducimus sequens theorema.

Theorema.

§. 151. Si haec formula differentialis $\frac{x^{m-1}\partial x}{1+x^k}$ ita integretur, ut, posito $x = 0$, integrale evanescat, tum vero statuatur $x = \infty$, valor inde resultans semper erit $\frac{\pi}{k \sin.\frac{m\pi}{k}}$, sive k sit numerus par, sive impar. Hujus theorematis demonstratio ex praecedentibus est manifesta.

§. 152. In evolutione hujus formulae assumsimus esse $m < k$, quia alioquin membra logarithmica se non destruissent; at vero ne hac quidem limitatione nunc amplius est opus. Casu enim quo foret $m = k$, integrale formulae $\frac{x^{m-1}\partial x}{1+x^k}$ esset $\frac{1}{k} l(1+x^k)$, quod facto $x = \infty$ fieret etiam ∞; verum hoc idem indicat, nostrum integrale esse $\frac{\pi}{k\, sin.\, \pi} = \infty$. Dummodo ergo m non fuerit majus quam k, nostra formula veritati semper est consentanea

§. 153. Quin etiam ne quidem necesse est ut exponentes m et k sint numeri integri, dummodo non fuerit $m > k$; si enim fuerit $m = \frac{\mu}{\lambda}$ et $k = \frac{\varkappa}{\lambda}$, erit valor per nostram formulam $\frac{\lambda\pi}{\varkappa \sin. \frac{\mu\pi}{\varkappa}}$, cujus veritas ita ostenditur. Quia hoc casu formula integranda est $\int \frac{x^{\frac{\mu}{\lambda}}}{1+x^{\frac{\varkappa}{\lambda}}} \cdot \frac{\partial x}{x}$, statuatur $x = y^\lambda$, erit $\frac{\partial x}{x} = \frac{\lambda \partial y}{y}$, et formula fiet

$$\int \frac{y^\mu}{1+y^\varkappa} \cdot \frac{\lambda \partial y}{y} = \lambda \int \frac{y^{\mu-1}\partial y}{1+y^\varkappa},$$

cujus valor utique erit $\frac{\lambda\pi}{\varkappa \sin. \frac{\mu\pi}{\varkappa}}$.

Alia demonstratio theorematis.

§. 154. Denotet P valorem integralis $\int \frac{x^m}{1+x^k} \cdot \frac{\partial x}{x}$ a termino $x = 0$ usque ad $x = 1$; at Q valorem ejusdem integralis a termino $x = 1$ usque ad $x = \infty$, ita ut $P + Q$ praebeat eum ipsum valorem, qui in theoremate continetur.

Nunc pro valore Q inveniendo statuatur $x = \frac{1}{y}$, unde fit $\frac{\partial x}{x} = -\frac{\partial y}{y}$, fietque

$$Q = \int \frac{y^{-m}}{1 + y^{-k}} \cdot \frac{-\partial y}{y} = -\int \frac{y^{k-m}\,\partial y}{1 + y^{k}} \cdot \frac{\partial y}{y}$$

a termino $y = 1$ usque ad $y = 0$. Hinc igitur commutatis terminis erit

$$Q = +\int \frac{y^{k-m}}{1 + y^{k}} \cdot \frac{\partial y}{y}$$

a termino $y = 0$ usque ad $y = 1$. Jam quia hoc integrali expedito littera y ex calculo egreditur, loco y scribere licebit x, ita ut sit

$$Q = \int \frac{x^{k-m}}{1 + x^{k}} \cdot \frac{\partial x}{x},$$

quo facto habebimus

$$P + Q = \int \frac{x^{m} + x^{k-m}}{1 + x^{k}} \cdot \frac{\partial x}{x}$$

a termino $x = 0$ usque ad terminum $x = 1$. Verum non ita pridem demonstravi, valorem hujus formulae integralis intra terminos $x = 0$ et $x = 1$ contentum esse $= \frac{\pi}{k \sin. \frac{m\pi}{k}}$. Hinc igitur nascitur sequens theorema non minus notatu dignum.

Theorema.

§. 155. Valor hujus formulae integralis

$$\int \frac{x^{m} + x^{k-m}}{1 + x^{k}} \cdot \frac{\partial x}{x}$$

intra terminos $x = 0$ et $x = 1$ contentus, aequalis est valori

istius integralis $\int \frac{x^m}{1+x^k} \cdot \frac{\partial x}{x}$, intra terminos $x = 0$ et $x = \infty$ contento.

§. 156. His expensis formulam integralem in titulo propositam aggrediamur, et quo eam ad formam hactenus tractatam reducamus, in subsidium vocemus sequentem reductionem

$$\int \frac{x^{m-1}\,\partial x}{(1+x^k)^{\lambda+1}} = \frac{A x^m}{(1+x^k)^\lambda} + B \int \frac{x^{m-1}\,\partial x}{(1+x^k)^\lambda},$$

unde facta differentiatione prodit sequens aequatio

$$\frac{x^{m-1}\,\partial x}{(1+x^k)^{\lambda+1}} = \frac{m A x^{m-1}\,\partial x}{(1+x^k)^\lambda} - \frac{\lambda k A x^{m+k-1}\,\partial x}{(1+x^k)^{\lambda+1}} + \frac{B x^{m-1}\,\partial x}{(1+x^k)^\lambda},$$

quae aequatio per $x^{m-1}\,\partial x$ divisa ac per $(1+x^k)^\lambda$ multiplicata, terminum negativum a dextra ad sinistram transponendo, erit

$$\frac{1 + \lambda k A x^k}{1+x^k} = m A + B,$$

quae aequatio manifesto subsistere nequit, nisi sit $\lambda k A = 1$, sive $A = \frac{1}{\lambda k}$, unde erit $1 = m A + B = \frac{m}{\lambda k} + B$, sicque $B = 1 - \frac{m}{\lambda k}$.

§. 157. Inventis his valoribus pro litteris A et B, primum assumimus, integralia ita capi, ut evanescant posito $x = 0$; tum vero posito $x = \infty$, quia exponens n minor supponitur quam k, membrum absolutum littera A affectum sponte evanescit, ita ut hoc casu $x = \infty$ fiat

$$\int \frac{x^{m-1}\,\partial x}{(1+x^k)^{\lambda+1}} = \left(1 - \frac{m}{\lambda k}\right) \int \frac{x^{m-1}\,\partial x}{(1+x^k)^\lambda}.$$

Quod si jam primo capiamus $\lambda = 1$, quia ante invenimus pro eodem casu $x = \infty$ esse

$$\int \frac{x^{m-1}\partial x}{1+x^k} = \frac{\pi}{k \sin . \frac{m\pi}{k}},$$

habebimus valorem istius integralis

$$\int \frac{x^{m-1}\partial x}{(1+x^k)^2} = \left(1-\frac{m}{k}\right) \frac{\pi}{k \sin . \frac{m\pi}{k}},$$

si quidem integrale etiam a termino $x = 0$ usque ad terminum $x = \infty$ extendatur.

§. 158. Quod si jam simili modo ponamus $\lambda = 2$, reperietur pro iisdem terminis integrationis

$$\int \frac{x^{m-1}\partial x}{(1+x^k)^3} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right) \frac{\pi}{k \sin . \frac{m\pi}{k}};$$

eodem modo si litterae λ continuo majores valores tribuantur, reperientur sequentes integralium formae omni attentione dignae

$$\int \frac{x^{m-1}\partial x}{(1+x^k)^4} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right) \frac{\pi}{k \sin . \frac{m\pi}{k}}$$

$$\int \frac{x^{m-1}\partial x}{(1+x^k)^5} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right) \frac{\pi}{k \sin . \frac{m\pi}{k}}$$

$$\int \frac{x^{m-1}\partial x}{(1+x^k)^6} = \left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\left(1-\frac{m}{5k}\right) \frac{\pi}{k \sin . \frac{m\pi}{k}}$$

etc. etc.

§. 159. Quare si littera n denotet numerum quemcunque integrum, pro formula in titulo expressa, si ejus integrale a termino $x = 0$ usque ad $x = \infty$ extendatur, ejus valor sequenti modo se habebit

$$\left(1-\frac{m}{k}\right)\left(1-\frac{m}{2k}\right)\left(1-\frac{m}{3k}\right)\left(1-\frac{m}{4k}\right)\cdots\left(1-\frac{m}{(n-1)k}\right)\frac{\pi}{k\ \sin.\frac{m\pi}{k}}$$

qui ergo conveniet huic formulae integrali $\int\frac{x^{m-1}\partial x}{(1+x^k)^n}$.

§. 160. Hic quidem necessario pro n alii numeri praeter integros accipi non licet: at vero per methodum interpolationum, quae fusius jam passim est explicata, hanc integrationem etiam ad casus, quibus exponens n est numerus fractus, extendere licet. Quod si enim sequentes formulae integrales a termino $y=0$ usque ad $y=1$ extendantur, in genere valor nostrae formulae propositae ita repraesentari poterit

$$\int\frac{x^{m-1}\partial x}{(1+x^k)^n}=\frac{\pi}{k\ \sin.\frac{m\pi}{k}}\cdot\frac{\int y^{nk-m-1}\partial y\,(1-y^k)^{\frac{m}{k}-1}}{\int y^{k-m-1}\partial y\,(1-y^k)^{\frac{m}{k}-1}}.$$

Unde si fuerit $m=1$ et $k=2$, sequitur fore

$$\int\frac{\partial x}{(1+xx)^n}=\frac{\pi}{2}\int\frac{y^{2(n-1)}\partial y}{\sqrt{(1-yy)}}:\int\frac{\partial y}{\sqrt{(1-yy)}}=\int\frac{y^{2(n-1)}\partial y}{\sqrt{(1-yy)}}.$$

Ita si $n=\frac{3}{2}$ erit

$$\int\frac{\partial x}{(1+xx)^{\frac{3}{2}}}=\int\frac{y\partial y}{\sqrt{(1-yy)}}$$

cujus veritas sponte elucet, quia integrale prius generatim est $\frac{x}{\sqrt{(1+xx)}}$, posterius vero $=1-\sqrt{(1-yy)}$, quae facto $x=\infty$ et $y=1$, utique fiunt aequalia. Caeterum pro hac integratione generali notasse juvabit, exponentem unitate minorem accipi non posse, quia alioquin valores amborum integralium in infinitum excrescerent.

6) Investigatio valoris integralis

$$\int \frac{x^{m-1}\,\partial x}{1-2x^k\cos.\theta+x^{2k}}$$

a termino $x=0$ usque ad $x=\infty$ extensi. *Opuscula analytica. Tom. II. Pag. 55 — 75.*

§. 161. Quaeramus primo integrale formulae propositae indefinitum, atque adeo omnes operationes ex primis analyseos principiis repetamus. Ac primo quidem, quoniam denominator in factores reales simplices resolvi nequit, sit in genere ejus factor duplicatus quicunque $1-2x\cos.\omega+xx$; evidens enim est, denominatorem fore productum ex k hujusmodi factoribus duplicatis. Cum igitur, posito hoc factore $=0$, fiat $x=\cos.\omega \pm\sqrt{-1}\sin.\omega$, etiam ipse denominator duplici modo evanescere debebit, sive si ponatur

$$x=\cos.\omega+\sqrt{-1}\sin.\omega,\ \text{sive}$$
$$x=\cos.\omega+\sqrt{-1}\sin.\omega.$$

Constat autem omnes potestates harum formularum ita commode exprimi posse, ut sit

$$(\cos.\omega\pm\sqrt{-1}\sin.\omega)^\lambda=\cos.\lambda\omega\pm\sqrt{-1}\sin.\lambda\omega,$$

hinc igitur erit

$$x^k=\cos.k\omega\pm\sqrt{-1}\sin.k\omega \ \text{et}$$
$$x^{2k}=\cos.2k\omega\pm\sqrt{-1}\sin.2k\omega.$$

Substituamus ergo hos valores, et denominator noster evadet

$$1-2\cos.\theta\cos.k\omega+\cos.2k\omega$$
$$\pm2\sqrt{-1}\cos.\theta\sin.k\omega\pm\sqrt{-1}\sin.2k\omega.$$

§. 162. Perspicuum igitur est hujus aequationis tam terminos reales quam imaginarios seorsim se mutuo tollere de-

bere, unde nascuntur hae duae aequationes

$$\text{I.} \quad 1 - 2 \cos.\theta \cos. k\omega + \cos. 2k\omega = 0,$$

$$\text{II.} \quad -2 \cos.\theta \sin. k\omega + \sin. 2k\omega = 0.$$

Cum igitur sit

$$\sin. 2k\omega = 2 \sin k\omega \cos. k\omega,$$

posterior aequatio induet hanc formam

$$-2 \cos.\theta \sin. k\omega + 2 \sin. k\omega \cos. k\omega = 0,$$

quae per $2 \sin. k\omega$ divisa dat $+\cos. k\omega = \cos.\theta$, ideoque

$$\cos. 2k\omega = \cos. 2\theta = \cos.\theta^2 - \sin.\theta^2 = 2 \cos.\theta^2 - 1,$$

qui valores in aequatione priore substituti praebent aequationem identicam, ita ut utrique aequationi satisfiat sumendo $\cos. k\omega = \cos.\theta$.

§. 163. Pro ω igitur ejusmodi angulum assumi oportet, ut fiat cos. $k\omega = \cos.\theta$, unde quidem statim deducitur $k\omega = \theta$, ideoque $\omega = \frac{\theta}{k}$. Verum quia infiniti dantur anguli eundem cosinum habentes, qui praeter ipsum angulum θ sunt $2\pi \pm \theta$, $4\pi \pm \theta$, $6\pi \pm \theta$, etc. atque adeo in genere $2i\pi \pm \theta$, denotante i omnes numeros integros, quaesito nostro satisfiet, faciendo $k\omega = 2i\pi \pm \theta$, unde colligitur angulus $\omega = \frac{2i\pi \pm \theta}{k}$, sicque pro ω nancisceremur innumerabiles angulos satisfacientes, quorum autem sufficiet tot assumsisse, quot exponens k continet unitates; successive igitur angulo ω sequentes tribuamus valores

$$\frac{\theta}{k},\ \frac{2\pi+\theta}{k},\ \frac{4\pi+\theta}{k},\ \frac{6\pi+\theta}{k},\ \frac{8\pi+\theta}{k},\ \ldots\ldots\ \frac{2(k-1)\pi+\theta}{k}.$$

Quodsi ergo angulo ω successive singulos istos valores, quorum numerus est $= k$, tribuamus, formula $1 - 2x \cos.\omega + xx$ omnes suppeditabit factores duplicatos nostri denominatoris $1 - 2x^k \cos.\theta + x^{2k}$, quorum numerus erit $= k$.

§. 164. Inventis jam omnibus factoribus duplicatis nostri denominatoris, fractio $\frac{x^{m-1}}{1-2x^k\cos.\theta+x^{2k}}$ resolvi debet in tot fractiones partiales, quarum denominatores sint ipsi isti factores duplicati, quorum numerus est k, ita ut in genere talis fractio partialis habitura sit talem formam $\frac{A+Bx}{1-2x\cos.\omega+xx}$, quam insuper resolvamus in binas simplices, etsi imaginarias, et cum sit

$$xx-2x\cos.\omega+1=(x-\cos.\omega+\sqrt{-1}\sin.\omega)(x-\cos.\omega-\sqrt{-1}\sin.\omega),$$

statuantur ambae istae fractiones partiales

$$\frac{f}{x-\cos.\omega-\sqrt{-1}\sin.\omega}+\frac{g}{x-\cos.\omega+\sqrt{-1}\sin.\omega},$$

ita ut totum resolutionis negotium huc redeat, ut ambo numeratores f et g determinentur; iis enim inventis habebitur summa ambarum fractionum

$$=\frac{fx+gx-(f+g)\cos.\omega+\sqrt{-1}(f-g)\sin.\omega}{xx-2x\cos.\omega+1},$$

ubi igitur erit

$$B=f+g \text{ et } A=(f-g)\sqrt{-1}\sin.\omega-(f+g)\cos.\omega.$$

§. 165. Per methodum igitur fractiones quascunque in fractiones simplices resolvendi statuamus

$$\frac{x^{m-1}}{1-2x^k\cos.\theta+x^{2k}}=\frac{f}{x-\cos.\omega-\sqrt{-1}\sin.\omega}+R,$$

ubi R complectatur omnes reliquas fractiones partiales. Hinc per $x-\cos.\omega-\sqrt{-1}\sin.\omega$ multiplicando habebitur

$$\frac{x^m-x^{m-1}(\cos.\omega+\sqrt{-1}\sin.\omega)}{1-2x^k\cos.\theta+x^{2k}}=f+R(x-\cos.\omega-\sqrt{-1}\sin.\omega),$$

quae aequatio cum vera esse debeat, quicunque valor ipsi x tribuatur, statuamus $x=\cos.\omega+\sqrt{-1}\sin.\omega$, ut membrum postremum prorsus e calculo tollatur; tum vero in parte sinistra,

quia formula $x - \cos.\omega - \sqrt{-1}\sin.\omega$ simul est factor denominatoris, facta hac substitutione tam numerator quam denominator in nihilum abibunt, ita ut hinc nihil concludi posse videatur.

§. 166. Hinc igitur utamur regula notissima, et loco tam numeratoris quam denominatoris eorum differentialia scribamus, unde nostra aequatio accipiet sequentem formam

$$\frac{mx^{m-1} - (m-1)x^{m-2}(\cos.\omega + \sqrt{-1}\sin.\omega)}{-2kx^{k-1}\cos.\theta + 2kx^{2k-1}} =$$

$$\frac{mx^m - (m-1)x^{m-1}(\cos.\omega + \sqrt{-1}\sin.\omega)}{-2kx^k\cos.\theta + 2kx^{2k}} = f,$$

posito scilicet $x = \cos.\omega + \sqrt{-1}\sin.\omega$. Tum autem erit

$$x^m = \cos.m\omega + \sqrt{-1}\sin.m\omega \quad \text{et}$$

$$x^{m-1}(\cos.\omega + \sqrt{-1}\sin.\omega) = x^m = \cos.m\omega + \sqrt{-1}\sin.m\omega,$$

et pro denominatore

$$x^k = \cos.k\omega + \sqrt{-1}\sin.k\omega \quad \text{et}$$

$$x^{2k} = \cos.2k\omega + \sqrt{-1}\sin.2k\omega;$$

unde fit numerator

$$x^m = \cos.m\omega + \sqrt{-1}\sin.m\omega$$

et denominator

$$-2k\cos.\theta\cos.k\omega + 2k\cos.2k\omega$$
$$-2k\sqrt{-1}\cos.\theta\sin.k\omega + 2k\sqrt{-1}\sin.2k\omega.$$

§. 167. Pro denominatore reducendo recordemur, jam supra inventum esse $\cos.k\omega = \cos.\theta$, unde fit $\sin.k\omega = \sin.\theta$, tum vero

$$\cos.2k\omega = \cos.2\theta = 2\cos.\theta^2 - 1 \quad \text{et}$$

$$\sin.2k\omega = \sin.2\theta = 2\sin.\theta\cos.\theta,$$

quibus valoribus adhibitis denominator noster erit

$$2k\cos.\theta^2-2k+2k\sqrt{-1}\sin.\theta\cos.\theta=-2k\sin.\theta^2+2k\sqrt{-1}\sin.\theta\cos.\theta$$
$$=-2k\sin.\theta(\sin.\theta-\sqrt{-1}\cos.\theta),$$

quamobrem hoc valore adhibito habebimus

$$f=\frac{\cos.m\omega+\sqrt{-1}\sin.m\omega}{2k\sin.\theta(\sqrt{-1}\cos.\theta-\sin.\theta)}.$$

Simul vero hinc sine novo calculo deducemus valorem g, quippe qu: ab f ratione signi $\sqrt{-1}$ tantum discrepat, sicque erit

$$g=\frac{\cos.m\omega-\sqrt{-1}\sin.m\omega}{-2k\sin.\theta(\sin.\theta+\sqrt{-1}\cos.\theta)}.$$

§. 168. Inventis autem his litteris f et g, pro litteris A et B colligemus primo

$$f+g=\frac{\cos.\theta\sin.m\omega-\sin.\theta\cos.m\omega}{k\sin.\theta}=\frac{\sin.(m\omega-\theta)}{k\sin.\theta}$$

tum vero erit

$$f-g=-\frac{\sqrt{-1}\cos.(m\omega-\theta)}{k\sin.\theta}.$$

Ex his igitur reperiemus

$$B=\frac{\sin.(m\omega-\theta)}{k\sin.\theta}\text{ et}$$
$$A=\frac{\sin.\omega\cos.(m\omega-\theta)-\cos.\omega\sin.(m\omega-\theta)}{k\sin.\theta}=-\frac{\sin.[(m\omega-\theta)-\omega]}{k\sin.\theta},$$

ubi ergo imaginaria sponte se mutuo destruxerunt.

§. 169. Inventis his valoribus A et B, investigari oportet integrale partiale $\int\frac{(A+Bx)\partial x}{1-2x\cos.\omega+xx}$, ubi, cum denominatoris differentiale sit

$$2x\partial x-2\partial x\cos.\omega=2\partial x(x-\cos.\omega),$$

statuamus

$$A+Bx=B(x-\cos.\omega)+C,\text{ eritque}$$
$$C=A+B\cos.\omega,$$

hinc igitur erit

$$C = \frac{\cos.\omega\,\sin.(m\omega - \theta) - \sin.[(m\omega - \theta) - \omega]}{k\,\sin.\theta}.$$

Quia vero

$$-\sin.(m\omega - \theta - \omega) = -\sin.(m\omega - \theta)\cos.\omega + \cos.(m\omega - \theta)\sin.\omega, \text{ erit}$$

$$C = \frac{\sin.\omega\,\cos.(m\omega - \theta)}{k\,\sin.\theta}.$$

Hac ergo forma adhibita, formula integranda $\frac{(A + Bx)\partial x}{1 - 2x\cos.\omega + xx}$ discerpatur in has duas partes

$$\frac{B(x - \cos.\omega)\partial x}{1 - 2x\cos.\omega + xx} + \frac{C\partial x}{1 - 2x\cos.\omega + xx}.$$

Hic igitur prioris partis integrale manifesto est

$$B\,l\sqrt{(1 - 2x\cos.\omega + xx)},$$

alterius vero partis facile patet integrale per arcum circuli expressum iri, cujus tangens sit $\frac{x\sin.\omega}{1 - x\cos.\omega}$. Ad hoc integrale inveniendum ponamus

$$\int \frac{C\partial x}{1 - 2x\cos.\omega + xx} = D\cdot \text{Arc. tang. } \frac{x\sin.\omega}{1 - x\cos.\omega},$$

et sumtis differentialibus, quia ∂. Arc. tang. t aequale est $\frac{\partial t}{1 + tt}$, habebimus

$$\frac{C\partial x}{1 - 2x\cos.\omega + xx} = D\cdot\frac{\partial x\sin.\omega}{1 - 2x\cos.\omega + xx},$$

unde manifesto fit

$$D = \frac{C}{\sin.\omega} = \frac{\cos.(m\omega - \theta)}{k\,\sin.\theta}.$$

§. 170. Substituamus igitur loco B et D valores modo inventos, et ex singulis factoribus denominatoris

$$1 - 2x^k\cos.\theta + x^{2k},$$

quorum forma est $1 - 2x\cos.\omega + xx$, oritur pars integralis constans ex membro logarithmico et arcu circulari, quae erit

$$\frac{\sin.(m\omega - \theta)}{k\,\sin.\theta}\,l\sqrt{(1 - 2x\cos.\omega + xx)} + \frac{\cos.(m\omega - \theta)}{k\,\sin.\theta}\text{ Arc. tang. }\frac{x\sin.\omega}{1 - x\cos.\omega}$$

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quae evanescit sumto $x = 0$. In hac igitur forma tantum opus est, ut loco ω successive scribamus valores supra indicatos, scilicet

$$\omega = \frac{\theta}{k},\ \frac{2\pi+\theta}{k},\ \frac{4\pi+\theta}{k},\ \frac{6\pi+\theta}{k},\ \text{etc.}$$

donec perveniatur ad $\frac{2(k-1)\pi+\theta}{k}$; tum enim summa omnium harum formarum praebebit totum integrale indefinitum formulae propositae.

§. 171. Postquam igitur integrale indefinitum elicuimus, nihil aliud superest, nisi ut in eo faciamus $x = \infty$, quo facto pars logarithmica, ob

$$\sqrt{(1 - 2x\cos.\omega + xx)} = x - \cos.\omega,$$

erit $Bl(x - \cos.\omega)$. Est vero

$$l(x - \cos.\omega) = lx - \frac{\cos.\omega}{x} = lx, \text{ ob } \frac{\cos.\omega}{k} = 0,$$

quamobrem facto $x = \infty$ quaelibet pars logarithmica habebit hanc formam $\frac{\sin.(m\omega-\theta)}{k\sin.\theta}\, lx$. Deinde pro partibus a circulo pendentibus, facto $x = \infty$ fit

$$\frac{x\sin.\omega}{1 - x\cos.\omega} = -\tan g.\,\omega = \tan g.(\pi - \omega),$$

sicque arcus, cujus haec est tangens, erit $= \pi - \omega$, hincque pars circularis quaecunque fiet $\frac{\cos.(m\omega-\theta)}{k\sin.\theta}(\pi - \omega)$.

§. 172. Cum quilibet valor anguli ω in genere hanc habeat formam $\frac{2i\pi+\theta}{k}$, erit angulus

$$m\omega - \theta = \frac{2im\pi - \theta(k-m)}{k} \text{ et } \pi - \omega = \frac{\pi(k-2i)-\theta}{k}.$$

Ponamus brevitatis gratia

$$\frac{\theta(k-m)}{k} = \zeta \text{ et } \frac{m\pi}{k} = \alpha, \text{ ut sit } m\omega - \theta = 2i\alpha - \zeta,$$

ubi loco i scribi debent successive numeri 0, 1, 2, 3, etc. usque ad $k - 1$. Hinc igitur si omnes partes logarithmicas in unam

summam colligamus, ea ita repraesentari poterit

$$\frac{lx}{k\,\sin.\theta}\,[-\sin.\zeta+\sin.(2\alpha-\zeta)+\sin.(4\alpha-\zeta)+\sin.(6\alpha-\zeta)$$
$$+\sin.(8\alpha-\zeta)\,.\,.\,.\,.\,.+\sin.[2(k-1)\alpha-\zeta]];$$

ubi quidem ex iis, quae hactenus sunt tradita, facile suspicari licet, totam hanc progressionem ad nihilum redigi. Verum hoc ipsum firma demonstratione muniri necesse est.

§. 173. Ad hoc ostendendum ponamus

$$S=-\sin.\zeta+\sin.(2\alpha-\zeta)+\sin.(4\alpha-\zeta)+\ldots+\sin.[2(k-1)\alpha-\zeta],$$

multiplicemus utrinque per $2\sin.\alpha$, et cum sit

$$2\sin.\alpha\sin.\Phi=\cos.(\alpha-\Phi)-\cos.(\alpha+\Phi),$$

hujus reductionis ope obtinebimus sequentem expressionem

$$2S\sin.\alpha=\cos.(\alpha+\zeta)+\cos.(\alpha-\zeta)+\cos.(3\alpha-\zeta)+\cos.(5\alpha-\zeta)\ldots$$
$$-\cos.(\alpha-\zeta)-\cos.(3\alpha-\zeta)-\cos.(5\alpha-\zeta)\ldots$$
$$\ldots\ldots+\cos.[(2k-3)\alpha-\zeta]-\cos.[(2k-1)\alpha-\zeta]\ldots$$
$$\ldots\ldots-\cos.[(2k-3)\alpha-\zeta],$$

unde deletis terminis se mutuo destruentibus habebitur

$$2S\sin.\alpha=\cos.(\alpha+\zeta)-\cos.[(2k-1)\alpha-\zeta].$$

§. 174. Ponamus hos duos angulos, qui sunt relicti, $\alpha+\zeta=p$ et $(2k-1)\alpha-\zeta=q$; eritque eorum summa $p+q=2\alpha k$. Quia porro est $\alpha=\frac{m\pi}{k}$, erit $p+q=2m\pi$, hoc est multiplo totius circuli peripheriae; ob m numerum integrum. Quare cum sit $q=2m\pi-p$, erit $\cos.q=\cos.p$; unde patet summam inventam nihilo esse aequalem, sicque manifestum est, omnes partes logarithmicas, quae in integrale formulae nostrae ingrediuntur, casu $x=\infty$ se mutuo destruere.

§. 175. **Progrediamur igitur ad partes circulares, quarum forma generalis, ut vidimus, est** $\frac{\cos.(m\omega-\theta)}{k\sin.\theta}(\pi-\omega)$, quae $\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}\left(\pi-\frac{2i\pi-\theta}{k}\right)=\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}\left(\pi-\frac{2i\pi}{k}-\frac{\theta}{k}\right)$.

Hic ponatur porro $\frac{\pi}{k}=\beta$ et $\pi-\frac{\theta}{k}=\gamma$, ut forma generalis sit $\frac{\cos.(2i\alpha-\zeta)}{k\sin.\theta}(\gamma-2i\beta)$. Quare si loco i scribamus ordine valores, 0, 1, 2, 3, 4, usque ad $k-1$, omnes partes circulares hanc progressionem constituent

$$\frac{1}{k\sin.\theta}[\gamma\cos.\zeta+(\gamma-2\beta)\cos.(2\alpha-\zeta)+(\gamma-4\beta)\cos.(4\alpha-\zeta)\\ \ldots\ldots+[\gamma-2(k-1)\beta]\cos.[2(k-1)\alpha-\zeta]].$$

Ponamus igitur

$$S=\gamma\cos.\zeta+(\gamma-2\beta)\cos.(2\alpha-\zeta)+(\gamma-4\beta)\cos.(4\alpha-\zeta)\\ \ldots\ldots+[\gamma-2(k-1)\beta]\cos.[2(k-1)\alpha-\zeta]$$

ut summa omnium partium circularium sit $\frac{S}{k\sin.\theta}$, quae ergo praebebit valorem quaesitum formulae integralis propositae, casu quo post integrationem statuitur $x=\infty$, ita ut totum negotium in investigando valore ipsius S versetur.

§. 176. Hunc in finem multiplicemus utrinque per $2\sin.\alpha$, et cum in genere sit

$$2\sin.\alpha\cos.\Phi=\sin.(\alpha+\Phi)-\sin.(\Phi-\alpha),$$

hac reductione in singulis terminis facta, perveniemus ad hanc aequationem

$$2S\sin.\alpha=\gamma\sin.(\alpha+\zeta)+\gamma\sin.(\alpha-\zeta)+(\gamma-2\beta)\sin.(3\alpha-\zeta)\\ -(\gamma-2\beta)\sin.(\alpha-\zeta)-(\gamma-4\beta)\sin.(3\alpha-\zeta)\\ +(\gamma-4\beta)\sin.(5\alpha-\zeta)\ldots\ldots+[\gamma-2(k-1)\beta]\sin.[(2k-1)\alpha-\zeta]\\ -(\gamma-6\beta)\sin.(5\alpha-\zeta)$$

ubi praeter primum et ultimum terminum omnes reliqui con-

trahi possunt, ita ut prodeat

$$2S\sin.\alpha = \gamma\sin.(\alpha+\zeta)+2\beta\sin.(\alpha-\zeta)+2\beta\sin.(3\alpha-\zeta)$$
$$+2\beta\sin.(5\alpha-\zeta)\ldots+2\beta\sin.[(2k-3)\alpha-\zeta]$$
$$+[\gamma-2(k-1)\beta]\sin.[(2k-1)\alpha-\zeta].$$

§. 177. Jam pro hac serie summanda ponamus porro

$$T = 2\sin.(\alpha-\zeta)+2\sin.(3\alpha-\zeta)+2\sin.(5\alpha-\zeta)+\ldots\ldots$$
$$\ldots\ldots+2\sin.[(2k-3)\alpha-\zeta]$$

ut habeamus

$$2S\sin.\alpha = \gamma\sin.(\alpha+\zeta)+[\gamma-2(k-1)\beta]\sin.[(2k-1)\alpha-\zeta]+\beta T.$$

Jam multiplicemus, ut hactenus, per $\sin.\alpha$, et cum sit

$$2\sin.\alpha\sin.\Phi = \cos.(\Phi-\alpha)-\cos.(\Phi+\alpha),$$

facta hac reductione nanciscimur

$$T\sin.\alpha = +\cos.\zeta+\cos.(2\alpha-\zeta)+\cos.(4\alpha-\zeta)+\ldots+\cos.[2(k-2)\alpha-\zeta]$$
$$-\cos.(2\alpha-\zeta)-\cos.(4\alpha-\zeta)-\ldots-\cos.[2(k-2)\alpha-\zeta]$$
$$-\cos.[2(k-1)\alpha-\zeta]$$

unde deletis terminis, quae se mutuo destruunt, remanebit tantum ista expressio

$$T\sin.\alpha = \cos.\zeta-\cos.[2(k-1)\alpha-\zeta].$$

Cum igitur sit $\alpha = \frac{m\pi}{k}$ erit

$$2(k-1)\alpha = 2m\pi-\frac{2m\pi}{k},$$

cujus loco scribere licet $-\frac{2m\pi}{k}$, unde ob $\zeta = \frac{\theta(k-m)}{k}$, erit

$$T\sin.\alpha = \cos.\frac{\theta(k-m)}{k}-\cos.\left(\frac{2m\pi+\theta(k-m)}{k}\right).$$

§. 178. Nunc vero notetur in genere esse

$$\cos.p-\cos.q = 2\sin.\frac{q+p}{2}\sin.\frac{q-p}{2},$$

quare cum sit

$$p = \frac{\theta(k-m)}{k} \text{ et } q = \frac{2m\pi+\theta(k-m)}{k}, \text{ erit}$$
$$\frac{q+p}{2} = \frac{m\pi+\theta(k-m)}{k} \text{ et } \frac{q-p}{2} = \frac{m\pi}{k}.$$

unde sequitur fore

$$T \sin. \alpha = 2 \sin. \left(\frac{m\pi + \theta(k-m)}{k}\right) \sin. \frac{m\pi}{k},$$

ideoque

$$T = 2 \sin. \left(\frac{m\pi + \theta(k-m)}{k}\right), \text{ ob } \alpha = \frac{m\pi}{k}.$$

§. 179. Hoc igitur valore T invento reperiemus porro

$$2 S \sin. \alpha = \gamma \sin. (\alpha + \zeta) + [\gamma - 2(k-1)\beta] \sin. [(2k-1)\alpha - \zeta] + 2\beta \sin. \left(\frac{m\pi + \theta(k-m)}{k}\right),$$

quae ob $\frac{m\pi + \theta(k-m)}{k} = \alpha + \zeta$ reducitur ad hanc formam

$$2 S \sin. \alpha = (\gamma + 2\beta) \sin. (\alpha + \zeta) + [\gamma - 2(k-1)\beta] \sin. [(2k-1)\alpha - \zeta],$$

quae ita repraesentari potest

$$2 S \sin. \alpha = (\gamma + 2\beta) [\sin. (\alpha + \zeta) + \sin. [(2k-1)\alpha - \zeta]] - 2\beta k \sin. [(2k-1)\alpha - \zeta],$$

ubi pro parte priore, ob

$$\sin. p + \sin. q = 2 \sin. \frac{p+q}{2} \cos. \frac{p-q}{2}, \text{ erit}$$

$$\frac{p+q}{2} = \alpha k \text{ et } \frac{p-q}{2} = (k-1)\alpha - \zeta,$$

unde pars ipsa prior fit

$$2(\gamma + 2\beta) \sin. \alpha k \cos. [(k-1)\alpha - \zeta],$$

ubi cum sit $\alpha k = m\pi$, erit $\sin. \alpha k = 0$, ita ut tantum supersit

$$2 S \sin. \alpha = - 2\beta k \sin. [(2k-1)\alpha - \zeta],$$

hincque

$$S = - \frac{\beta k \sin. [(2k-1)\alpha - \zeta]}{\sin. \alpha}.$$

Est vero

$$(2k-1)\alpha - \zeta = 2m\pi - \frac{m\pi}{k} - \frac{\theta(k-m)}{k};$$

omisso igitur termino $2m\pi$, erit

$$S = + \frac{\pi \sin. \left[\frac{m\pi + \theta(k-m)}{k}\right]}{\sin. \frac{m\pi}{k}},$$

ideoque valor quaesitus

$$\frac{S}{k \sin. \theta} = + \frac{\pi \sin. \left[\frac{m\pi + \theta(k-m)}{k}\right]}{k \sin. \theta \sin. \frac{m\pi}{k}},$$

quae forma reducitur ad hanc

$$\frac{\pi \sin. \left[\frac{m(\pi-\theta) + \theta k}{k}\right]}{k \sin. \theta \sin. \frac{m\pi}{k}}.$$

§. 180. Contemplemur hic ante omnia casum quo $\theta = \frac{\pi}{2}$, et formula integralis proposita abit in hanc $\int \frac{x^{m-1} \partial x}{1+x^{2k}}$, cujus ergo valor, si post integrationem ponatur $x = \infty$, evadet

$$\frac{\pi \sin. \left(\frac{\pi}{2} + \frac{m\pi}{2k}\right)}{k \sin. \frac{m\pi}{k}} = \frac{\pi \cos. \frac{m\pi}{2k}}{k \sin. \frac{m\pi}{k}}.$$

Quia igitur est

$$\sin. \frac{m\pi}{k} = 2 \sin. \frac{m\pi}{2k} \cos. \frac{m\pi}{2k},$$

prodibit iste valor $= \frac{\pi}{2k \sin. \frac{m\pi}{k}}$, qui valor egregie convenit cum eo, quem non ita pridem pro formula $\int \frac{x^{m-1} \partial x}{1+x^k}$ assignavimus, si quidem loco k scribatur $2k$.

§. 181. Evolvamus etiam casum quo $\theta = \pi$, et formula nostra integralis $\int \frac{x^{m-1} \partial x}{(1+x^k)^2}$, cujus ergo, facto $x = \infty$,

valor erit

$$\frac{\pi \sin. \left[\frac{m(\pi-\theta)}{k}+\theta\right]}{k \sin. \theta \sin. \frac{m\pi}{k}} = \frac{\pi}{k \sin. \frac{m\pi}{k}} \cdot \frac{\sin. \left[\frac{m(\pi-\theta)}{k}+\theta\right]}{\sin. \theta}.$$

Hujus autem posterioris fractionis, casu $\theta = \pi$, tam numerator quam denominator evanescit; quare, ut ejus verus valor eruatur, loco utriusque ejus differentiale scribamus, quo facto ista fractio abibit in hanc

$$\frac{\partial \theta \left(1-\frac{m}{k}\right) \cos. \left[\frac{m(\pi-\theta)}{k}+\theta\right]}{\partial \theta \cos. \theta},$$

cujus valor facto $\theta = \pi$ nunc manifesto est $1-\frac{m}{k}$; sicque valor integralis quaesitus erit $\left(1-\frac{m}{k}\right)\frac{\pi}{k \sin. \frac{m\pi}{k}}$, prorsus uti in superiore dissertatione invenimus.

§. 182. Quo autem valorem generalem inventum commodiorem reddamus, ponamus $\pi - \theta = \eta$, fietque

$$\sin. \theta = \sin. \eta \text{ et } \cos. \theta = -\cos. \eta;$$

tum vero erit angulus

$$\frac{m(\pi-\theta)}{k}+\theta = \frac{m\eta}{k}+\pi-\eta,$$

cujus sinus est $\sin. \left(1-\frac{m}{k}\right)\eta$, unde valor quaesitus nostrae formulae erit $\frac{\pi \sin. \left(1-\frac{m}{k}\right)\eta}{k \sin. \eta \sin. \frac{m\pi}{k}}$, atque hinc tandem sequens adepti sumus theorema.

Theorema.

§. 183. Si haec formula integralis

$$\int \frac{x^{m-1} \partial x}{1+2x^k \cos. \eta + x^{2k}}$$

a termino $x = 0$ usque ad terminum $x = \infty$ extendatur, ejus valor erit $= \frac{\pi \sin. (1 - \frac{m}{k}) \eta}{k \sin. \eta \sin. \frac{m\pi}{k}}$, sive cum sit

$$\sin. (1 - \tfrac{m}{k}) \eta = \sin. \eta \cos. \tfrac{m\eta}{k} - \cos. \eta \sin. \tfrac{m\eta}{k},$$

iste valor etiam hoc modo exprimi potest

$$\frac{\pi \cos. \frac{m\eta}{k}}{k \sin. \frac{m\pi}{k}} - \frac{\pi \sin. \frac{m\eta}{k}}{k \operatorname{tang}. \eta \sin. \frac{m\pi}{k}}.$$

§. 184. Consideremus nunc alio modo hanc formulam integralem

$$\int \frac{x^{m-1} \partial x}{1 + 2x^k \cos. \eta + x^{2k}},$$

cujus valor a termino $x = 0$ usque ad $x = 1$ ponatur $= P$, ejusdem vero valor ab $x = 1$ usque ad $x = \infty$ ponatur $= Q$, ita ut $P + Q$ exhibere debeat ipsum valorem ante inventum. Nunc vero pro valore Q inveniendo ponamus $x = \frac{1}{y}$, et formula nostra ita repraesentata

$$\frac{x^m}{1 + 2x^k \cos. \eta + x^{2k}} \cdot \frac{\partial x}{x},$$

ob $\frac{\partial x}{x} = - \frac{\partial y}{y}$ induet hanc formam

$$-\int \frac{y^{-m}}{1 + 2y^{-k} \cos. \eta + y^{-2k}} \cdot \frac{\partial y}{y} = -\int \frac{y^{2k-m-1} \partial y}{y^{2k} + 2y^k \cos. \eta + 1},$$

cujus valor a termino $y = 1$ usque ad $y = 0$ extendi debet. Commutatis igitur his terminis habebimus

$$Q = + \int \frac{y^{2k-m-1} \partial y}{y^{2k} + 2y^k \cos. \eta + 1},$$

a termino $y = 0$ usque ad $y = 1$.

§. 185. Quia in utraque forma pro P et Q eadem conditio integrationis praescribitur, a termino 0 usque ad 1, nihil impedit quo minus in posteriore loco y scribamus x, unde pro $P + Q$ habebimus hanc formam integralem

$$\int \frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k \cos.\eta + x^{2k}} \partial x,$$

cujus valor, a termino $x = 0$ usque ad $x = 1$ extensus, aequabitur huic expressioni $\frac{\pi \sin.\left(1 - \frac{m}{k}\right)\eta}{k \sin.\eta \sin.\frac{m\pi}{k}}$. Comparatis igitur his binis formulis integralibus nanciscemur sequens theorema notatu maxime dignum.

Theorema.

§. 186. Haec formula integralis

$$\int \frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k \cos.\eta + x^{2k}} \partial x,$$

a termino $x = 0$ usque ad terminum $x = 1$ extensa, aequalis est huic formulae integrali

$$\int \frac{x^{m-1}\partial x}{1 + 2x^k \cos.\eta + x^{2k}},$$

a termino $x = 0$ usque ad terminum $x = \infty$ extensae: utriusque enim valor erit $\frac{\pi \sin.\left(1 - \frac{m}{k}\right)\eta}{k \sin.\eta \sin.\frac{m\pi}{k}}$.

§. 187. Quod si hanc fractionem

$$\frac{\sin.\eta}{1 + 2x^k \cos.\eta + x^{2k}}$$

in seriem infinitam evolvamus, quae sit

$$\sin.\eta + Ax^k + Bx^{2k} + Cx^{3k} + Dx^{4k} + Ex^{5k} + \text{etc.}$$

per denominatorem multiplicando perveniemus ad hanc expressionem infinitam

$$\begin{aligned}\sin.\eta = \sin.\eta &+ A x^k + B x^{2k} + C x^{3k} + D x^{4k} + E x^{5k} + F x^{6k} + \text{etc.}\\ &+ 2\sin.\eta\cos.\eta + 2A\cos.\eta + 2B\cos.\eta + 2C\cos.\eta + 2D\cos.\eta + 2E\cos.\eta + \text{etc.}\\ &+ \sin.\eta + A + B + C + D + \text{etc.}\end{aligned}$$

unde singulis terminis ad nihilum reductis reperiemus

1°. $A + 2\sin.\eta\cos.\eta = 0$, hincque $A = -\sin.2\eta$

2°. $B + 2A\cos.\eta + \sin.\eta = 0$, unde fit $B = \sin.3\eta$

3°. $C + 2B\cos.\eta + A = 0$, unde fit $C = -\sin.4\eta$

4°. $D + 2C\cos.\eta + B = 0$, unde fit $D = \sin.5\eta$

etc. etc.

ita ut nostra fractio $\dfrac{\sin.\eta}{1 + 2x^k\cos.\eta + x^{2k}}$ resolvatur in hanc seriem

$$\sin.\eta - x^k\sin.2\eta + x^{2k}\sin.3\eta - x^{3k}\sin.4\eta + x^{4k}\sin.5\eta - \text{etc.}$$

§. 188. Multiplicemus nunc hanc seriem per

$$x^{m-1}\partial x + x^{2k-m-1}\partial x,$$

et post integrationem faciamus $x = 1$, ut obtineamus valorem hujus formulae

$$\sin.\eta\int\frac{x^{m-1} + x^{2k-m-1}}{1 + 2x^k\cos.\eta + x^{2k}}\partial x$$

pro casu $x = 1$, hocque modo perveniemus ad geminas sequentes series

$$\frac{\sin.\eta}{m} - \frac{\sin.2\eta}{m+k} + \frac{\sin.3\eta}{m+2k} - \frac{\sin.4\eta}{m+3k} + \frac{\sin.5\eta}{m+4k} - \text{etc.}$$

$$\frac{\sin.\eta}{2k-m} - \frac{\sin.2\eta}{3k-m} + \frac{\sin.3\eta}{4k-m} - \frac{\sin.4\eta}{5k-m} + \frac{\sin.5\eta}{6k-m} - \text{etc.}$$

Aggregatum igitur harum duarum serierum junctim sumtarum

aequabitur huic valori $\frac{\pi \sin. \left(1-\frac{m}{k}\right)\eta}{k \sin. \frac{m\pi}{k}}$, unde subjungamus adhuc istud theorema.

Theorema.

§. 189. Si η denotet angulum quemcunque, litterae vero m et k pro lubitu accipiantur, ex iisque binae sequentes series formentur

$$P = \frac{\sin. \eta}{m} - \frac{\sin. 2\eta}{m+k} + \frac{\sin. 3\eta}{m+2k} - \frac{\sin. 4\eta}{m+3k} + \frac{\sin. 5\eta}{m+4k} - \text{etc.}$$

$$Q = \frac{\sin. \eta}{2k-m} - \frac{\sin. 2\eta}{3k-m} + \frac{\sin. 3\eta}{4k-m} - \frac{\sin. 4\eta}{5k-m} + \frac{\sin. 5\eta}{6k-m} - \text{etc.}$$

neutrius quidem summa exhiberi potest, utriusque autem junctim sumtae summa erit

$$P + Q = \frac{\pi \sin. \left(1-\frac{m}{k}\right)\eta}{k \sin. \frac{m\pi}{k}}.$$

Corollarium.

§. 190. Quod si ergo angulum η infinite parvum capiamus, ut fiat

$$\sin. \eta = \eta, \ \sin. 2\eta = 2\eta, \ \sin. 3\eta = 3\eta, \text{ etc.}$$

quia in formula summae fiet

$$\sin. \left(1-\frac{m}{k}\right)\eta = \left(1-\frac{m}{k}\right)\eta;$$

si utrinque per η dividamus, obtinebimus sequentem seriem geminatam

$$\frac{1}{m} - \frac{2}{m+k} + \frac{3}{m+2k} - \frac{4}{m+3k} + \frac{5}{m+4k} - \text{etc.}$$

$$\frac{1}{2k-m} - \frac{2}{3k-m} + \frac{3}{4k-m} - \frac{4}{5k-m} + \frac{5}{6k-m} - \text{etc.}$$

cujus ergo summa erit $\left(1-\frac{m}{k}\right)\frac{\pi}{k \sin. \frac{m\pi}{k}}$, ubi notetur, ambas istas

series non incongrue in hanc simplicem contrahi posse

$$\frac{2k}{m(2k-m)} - \frac{8k}{(m+k)(3k-m)} + \frac{18k}{(m+2k)(4k-m)} - \frac{32k}{(m+3k)(5k-m)} + \text{etc.}$$

ubi numeratores sunt numeri quadrati duplicati.

§. 191. Formulae autem, quarum valores hactenus invenimus, multo concinnius et elegantius exprimi possunt, si loco exponentis m scribamus $k-n$, tum enim in valore integrali invento fiet $\left(1-\frac{m}{k}\right)\eta = \frac{n\eta}{k}$; at vero pro denominatore fiet $\frac{m\pi}{k} = \pi - \frac{n\pi}{k}$, cujus sinus erit sin. $\frac{n\pi}{k}$; sicque nostra formula inventa hanc induet formam $\dfrac{\pi \sin. \frac{n\eta}{k}}{k \sin. \eta \sin. \frac{n\pi}{k}}$, quae ergo exprimet valorem hujus formulae integralis

$$\int \frac{x^{k-n-1}\,\partial x}{1+2x^k \cos. \eta + x^{2k}},$$

ab $x=0$ usque ad $x=\infty$, ut et hujus formulae

$$\int \frac{x^{k-n-1}+x^{k+n-1}}{1+2x^k \cos. \eta + x^{2k}}\,\partial x,$$

a termino $x=0$ usque ad terminum $x=1$; et quia utriusque valor est $\dfrac{\pi \sin. \frac{n\eta}{k}}{k \sin. \eta \sin. \frac{n\pi}{k}}$, perspicuum est eum manere eundem, etsi loco n scribatur $-n$, ex quo prior formula ita repraesentari poterit

$$\int \frac{x^{k\pm n-1}}{1+x^k \cos. \eta + x^{2k}}\,\partial x;$$

at posterior formula ob hanc ambiguitatem nullam plane mutationem patitur.

§. 192. Ponendo $m = k - n$ etiam series nostra geminata pulchriorem accipiet faciem; habebitur enim

$$\frac{\sin.\eta}{k-n} - \frac{\sin.2\eta}{2k-n} + \frac{\sin.3\eta}{3k-n} - \frac{\sin.4\eta}{4k-n} + \text{etc.}$$
$$\frac{\sin.\eta}{k+n} - \frac{\sin.2\eta}{2k+n} + \frac{\sin.3\eta}{3k+n} - \frac{\sin.4\eta}{4k+n} + \text{etc.}$$

cujus ergo summa erit $\dfrac{\pi\sin.\frac{n\eta}{k}}{k\sin.\frac{n\pi}{k}}$. Tum vero si hae geminae series in unam contrahantur, et utrinque per $2k$ dividatur, obtinebitur sequens summatio memoratu digna

$$\frac{\pi\sin.\frac{n\eta}{k}}{2kk\sin.\frac{n\pi}{k}} = \frac{\sin.\eta}{kk-nn} - \frac{2\sin.2\eta}{4kk-nn} + \frac{3\sin.3\eta}{9kk-nn} - \frac{4\sin.4\eta}{16kk-nn} + \text{etc.}$$

§. 193. Quodsi haec postrema series differentietur, sumendo solum angulum η variabilem, ob

$$\partial.\sin.\frac{n\eta}{k} = \frac{n\partial\eta}{k}\cos.\frac{n\eta}{k}$$

habebimus

$$\frac{\pi n\cos.\frac{n\eta}{k}}{2k^3\sin.\frac{n\pi}{k}} = \frac{\cos.\eta}{kk-nn} - \frac{4\cos.2\eta}{4kk-nn} + \frac{9\cos.3\eta}{9kk-nn} - \frac{16\cos.4\eta}{16kk-nn} + \text{etc.}$$

Unde si sumatur $\eta = 0$, orietur ista summatio

$$\frac{\pi n}{2k^3\sin.\frac{n\pi}{k}} = \frac{1}{kk-nn} - \frac{4}{4kk-nn} + \frac{9}{9kk-nn} - \frac{16}{16kk-nn} + \text{etc.}$$

Sin autem sumatur $\eta = 90^\circ = \frac{\pi}{2}$, erit

$$\cos.\eta = 0,\ \cos.2\eta = -1,\ \cos.3\eta = 0,\ \cos.4\eta = +1 \text{ etc.}$$

unde nascitur sequens series

$$\frac{n\pi\cos.\frac{n\pi}{2k}}{2k^3\sin.\frac{n\pi}{k}} = \frac{4}{4kk-nn} - \frac{16}{16kk-nn} + \frac{36}{36kk-nn} - \frac{64}{64kk-nn} + \text{etc.}$$

Quia autem $\sin.\frac{n\pi}{k} = 2\sin.\frac{n\pi}{2k}\cos.\frac{n\pi}{2k}$, erit ejusdem seriei summa $\frac{n\pi}{4k^3\sin.\frac{n\pi}{2k}}$.

§. 194. At si series illa §. 192. exhibita in $\partial\eta$ ducatur et integretur, ob

$$\int\partial\eta\sin.\frac{n\eta}{k} = -\frac{k}{n}\cos.\frac{n\eta}{k},\text{ erit}$$

$$C - \frac{\pi\cos.\frac{n\eta}{k}}{2nk\sin.\frac{n\pi}{k}} = -\frac{\cos.\eta}{kk-nn} + \frac{\cos.2\eta}{4kk-nn} - \frac{\cos.3\eta}{9kk-nn} + \frac{\cos.4\eta}{16kk-nn} + \text{etc.}$$

Ut autem hic constantem addendam C definiamus, sumamus $\eta = 0$, fietque

$$C - \frac{\pi}{2nk\sin.\frac{n\pi}{k}} = -\frac{1}{kk-nn} + \frac{1}{4kk-nn} - \frac{1}{9kk-nn} + \text{etc.}$$

quare si hujus seriei summa aliunde pateat, constans C definiri poterit. Series autem haec in sequentem geminatam resolvi potest

$$2nC - \frac{\pi}{k\sin.\frac{n\pi}{k}} = \frac{1}{k+n} - \frac{1}{2k+n} + \frac{1}{3k+n} - \frac{1}{4k+n} + \text{etc.}$$
$$- \frac{1}{k-1} + \frac{1}{2k-n} - \frac{1}{3k-n} + \frac{1}{4k-n} - \text{etc.}$$

§. 195. Cum igitur in *Introductione in Analysin Infinitorum* pag. 142. ad hanc pervenissem seriem

$$\frac{1}{kk-nn} - \frac{1}{4kk-nn} + \frac{1}{9kk-nn} - \frac{1}{16kk-nn} + \text{etc.}$$
$$= \frac{\pi}{2kn\sin.\frac{n\pi}{k}} - \frac{1}{2nn},$$

(hic scilicet loco litterarum ibi adhibitarum m et n scripsi n et k), hoc valore adhibito nostra aequatio erit

$$C - \frac{\pi}{2nk\sin.\frac{n\pi}{k}} = \frac{1}{2nn} - \frac{\pi}{2nk\sin.\frac{n\pi}{k}},$$

unde fit $C = \frac{1}{2nn}$. Hinc ergo habebimus istam summationem

$$\frac{\pi\cos.\frac{n\eta}{k}}{2nk\sin.\frac{n\pi}{k}} - \frac{1}{2nn} = \frac{\cos.\eta}{kk-nn} - \frac{\cos.2\eta}{4kk-nn}$$

$$+ \frac{\cos.3\eta}{9kk-nn} - \frac{\cos.4\eta}{16kk-nn} + \text{etc.}$$

quae series utique omni attentione digna videtur.

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7) Methodus inveniendi formulas integrales quae certis casibus datam inter se teneant rationem. *Opuscula Analytica. Tom. II. Pag.* 178—216.

§. 196. Quemadmodum in seriebus recurrentibus quilibet terminus ex uno pluribusve praecedentibus secundum legem quandam constantem determinatur, ita hic ejusmodi series sum consideraturus, in quibus quilibet terminus ex uno pluribusve praecedentibus secundum quampiam legem variabilem determinatur. Quoniam autem in talibus seriebus formula generalis singulos terminos exprimens plerumque non est algebraica, sed transcendens, singulos terminos per formulas integrales exhiberi conveniet, quae ut valores determinatos praebeant, post integrationem quantitati variabili valorem determinatum tribui assumo, ita ut singuli termini prodeant quantitates determinatae; atque

nunc quaestio principalis huc redit, quemadmodum istae formulae integrales debeant esse comparatae, ut quilibet terminus secundum datam legem ex uno pluribusve praecedentibus determinetur.

§. 197. Quod quo clarius perspiciatur, contemplemur seriem notissimam harum formularum integralium

$$\int \frac{\partial x}{\surd(1-xx)}, \int \frac{xx\partial x}{\surd(1-xx)}, \int \frac{x^4\partial x}{\surd(1-xx)}, \int \frac{x^6\partial x}{\surd(1-xx)}, \text{ etc.}$$

quae si singulae ita integrentur, ut evanescant posito $x=0$, tum vero variabili x tribuatur valor $=1$, quilibet terminus a praecedente ita pendet, ut sit

$$\int \frac{xx\partial x}{\surd(1-xx)} = \frac{1}{2}\int \frac{\partial x}{\surd(1-xx)},$$
$$\int \frac{x^4\partial x}{\surd(1-xx)} = \frac{3}{4}\cdot\int \frac{xx\partial x}{\surd(1-xx)},$$
$$\int \frac{x^6\partial x}{\surd(1-xx)} = \frac{5}{6}\cdot\int \frac{x^4\partial x}{\surd(1-xx)},$$

atque in genere

$$\int \frac{x^n\partial x}{\surd(1-xx)} = \frac{n-1}{n}\int \frac{x^{n-2}\partial x}{\surd(1-xx)}.$$

Unde patet, hanc formulam generalem spectari posse tanquam terminum generalem illius seriei, atque quemlibet terminum ex praecedente oriri, si iste multiplicetur per $\frac{n-1}{n}$.

§. 198. Ad similitudinem igitur hujus casus seriem formularum integralium ita in genere constituamus,

$$\int\partial v, \int x\partial v, \int xx\partial v, \int x^3\partial v, \int x^4\partial v, \text{ etc.}$$

ita ut terminus indici n respondens sit $\int x^{n-1}\partial v$, quae singula integralia ita accipi sumamus, ut evanescant posito $x=0$, post integrationem autem quantitati variabili x tribuamus quempiam valorem constantem, veluti $x=1$, vel alio cuipiam numero. Quibus positis quaestio huc redit, qualis pro v assumi debeat functio ipsius x, ut quilibet terminus per unum, vel duos pluresve

praecedentes, secundum legem quandam datam utcunque variabilem, sive ab indice n pendentem, determinetur; ubi quidem imprimis eo erit respiciendum, ad quot dimensiones index n in scala relationis proposita ascendat: plerumque autem non ultra primam dimensionem assurgere erit opus. Hinc igitur sequentia problemata pertractemus.

Problema I.

§. 199. *Invenire functionem v, ut ista relatio inter binos terminos sibi succedentes locum habeat:*

$$\int x^n \partial v = \frac{\alpha n + a}{\beta n + b} \int x^{n-1} \partial v.$$

Solutio.

Requiritur igitur hic, ut sit

$$(\alpha n + a) \int x^{n-1} \partial v = (\beta n + b) \int x^n \partial v,$$

si scilicet post integrationem variabili x certus valor tribuatur. Quoniam igitur ista conditio tum demum locum habere debet, postquam variabili x iste valor constans fuerit datus, ponamus in genere, dum x est variabilis, hanc aequationem locum habere

$$(\alpha n + a) \int x^{n-1} \partial v = (\beta n + b) \int x^n \partial v + V,$$

quantitatem autem V ita esse comparatam, ut evanescat postquam variabili ille valor determinatus fuerit assignatus. Praeterea vero, quia ambo integralia ita capi assumimus, ut evanescant posito $x = 0$, necesse est ut etiam ista quantitas V eodem quoque casu evanescat.

§. 200. Quoniam haec aequalitas subsistere debet pro omnibus indicibus n, quos quidem semper ut positivos spectamus, facile intelligitur, quantitatem istam V factorem habere debere x^n;

quo pacto jam isti conditioni satisfit, ut posito $x = 0$ etiam fiat $V = 0$. Quamobrem statuamus $V = x^n Q$, ubi Q denotet functionem ipsius x proposito accommodatam, et quam simul ita comparatam esse desideramus, ut evanescat si ipsi x certus quidem valor tribuatur.

§. 201. Cum igitur esse debeat

$$(\alpha n + a) \int x^{n-1} \partial v = (\beta n + b) \int x^n \partial v + x^n Q,$$

differentietur ista aequatio, ac differentiali per x^{n-1} diviso pervenietur ad hanc aequationem differentialem

$$(\alpha n + a) \partial v = (\beta n + b) x \partial v + n Q \partial x + x \partial Q,$$

quae cum subsistere debeat pro omnibus valoribus ipsius n, termini ista littera affecti seorsim se tollere debent, unde nanciscimur has duas aequalitates

$$\text{I. } (\alpha - \beta x) \partial v = Q \partial x \text{ et}$$

$$\text{II. } (a - b x) \partial v = x \partial Q.$$

Ex priore fit $\partial v = \frac{Q \partial x}{\alpha - \beta x}$, ex altera vero $\partial v = \frac{x \partial Q}{a - b x}$, qui duo valores inter se aequati suppeditant hanc aequationem $\frac{\partial Q}{Q} = \frac{\partial x}{x} \cdot \frac{a - b x}{\alpha - \beta x}$, quae aequatio resolvitur in has partes

$$\frac{\partial Q}{Q} = \frac{a}{\alpha} \cdot \frac{\partial x}{x} + \frac{\alpha\beta - b\alpha}{\alpha} \cdot \frac{\partial x}{\alpha - \beta x},$$

cujus ergo integrale erit

$$l Q = \frac{a}{\alpha}\, l x - \frac{\alpha\beta - b\alpha}{\alpha\beta}\, l(\alpha - \beta x);$$

unde deducitur

$$Q = C x^{\frac{a}{\alpha}} . (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta}}.$$

§. 202. Ex hoc valore pro Q invento statim patet, eum evanescere casu $x = \frac{\alpha}{\beta}$, si modo fuerit $\frac{b\alpha - a\beta}{\alpha\beta} > 0$; sin autem secus eveniat, non patet quomodo haec quantitas ullo casu

evanescere queat. Invento autem hoc valore Q, inde reperietur

$$\partial v = C x^{\frac{a}{\alpha}} \partial x \,(\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta} - 1}$$

hincque nostrae seriei terminus indici n respondens erit

$$\int x^{n-1} \partial v = C \int x^{n + \frac{a}{\alpha} - 1} \partial x (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta} - 1},$$

tum vero erit

$$V = C x^{n + \frac{a}{\alpha}} (\alpha - \beta x)^{\frac{b\alpha - a\beta}{\alpha\beta}}.$$

Ubi res imprimis eo redit, ut ista quantitas praeter casum $x = 0$ insuper alio casu evanescat.

Corollarium 1.

§. 203. Hic duo casus occurrunt, qui peculiarem evolutionem postulant; prior est, quo $\alpha = 0$; tum autem inchoandum erit ab aequatione $\frac{\partial Q}{Q} = -\frac{(a - bx)\partial x}{\beta x x}$, unde integrando elicitur $l Q = \frac{a}{\beta x} + \frac{b}{\beta} l x$, hincque sumendo e pro numero cujus logarithmus hyperbolicus $= 1$, colligitur

$$Q = e^{\frac{a}{\beta x}} \cdot x^{\frac{b}{\beta}}$$

quae formula in nihilum abire nequit, nisi fiat $\frac{a}{\beta x} = -\infty$, ideoque $x = 0$, sicque non duo haberentur casus, quibus fieret $V = 0$, cum tamen duo desiderentur. Interim autem hinc fiet

$$\partial v = \frac{e^{\frac{a}{\beta x}} x^{\frac{b}{\beta}} \partial x}{\alpha - \beta x}.$$

Corollarium 2.

§. 204. Alter casus peculiarem integrationem postulans erit quo $\beta = 0$; tum autem erit $\frac{\partial Q}{Q} = \frac{\partial x(a - bx)}{\alpha x}$, unde fit $lQ = \frac{a}{\alpha} lx - \frac{bx}{\alpha}$, ideoque $Q = x^{\frac{a}{\alpha}} \cdot e^{\frac{-bx}{\alpha}}$, quae formula casu $x = \infty$ evanescit, si modo fuerit $\frac{b}{\alpha}$ numerus positivus, sin autem $\frac{b}{\alpha}$ fuerit numerus negativus, tum Q evanescit casu $x = -\infty$. Porro vero hoc casu fiet

$$\partial v = \frac{x^{\frac{a}{\alpha}} \cdot e^{\frac{-bx}{\alpha}} \partial x}{\alpha - \beta x}.$$

Scholion.

§. 205. His in genere observatis aliquot casus speciales evolvamus, quibus litteris α, β et a, b certos valores tribuemus, qui ad casus jam satis cognitos perducant.

Exemplum 1.

§. 206. Quaerantur formulae integrales, ut fiat

$$\int x^n \partial v = \frac{(2n-1)}{2n} \int x^{n-1} \partial v.$$

Cum igitur hic esse debeat

$$(2n-1) \int x^{n-1} \partial v = 2n \int x^n \partial v,$$

erit hoc casu $\alpha = 2$ et $a = -1$, tum vero $\beta = 2$ et $b = 0$; hinc fit

$$\frac{\partial Q}{Q} = -\frac{\partial x}{2x(1-x)} = -\frac{\partial x}{2x} - \frac{\partial x}{2(1-x)},$$

unde integrando

$$lQ = -\tfrac{1}{2} lx + \tfrac{1}{2} l(1-x),$$

ideoque

$$Q = C\sqrt{\frac{1-x}{x}}, \text{ ergo } V = C x^n \sqrt{\frac{1-x}{x}}.$$

Porro cum hic sit $\partial v = \frac{Q \partial x}{2(1-x)}$, erit

$$\partial v = \frac{C \partial x \sqrt{\frac{1-x}{x}}}{2(1-x)} = \frac{C \partial x}{2\sqrt{(x-xx)}},$$

sumto ergo $C = 2$ erit $\partial v = \frac{\partial x}{\sqrt{(x-xx)}}$, et formula nostra generalis

$$\int x^{n-1} \partial v = \int \frac{x^{n-1} \partial x}{\sqrt{(x-xx)}},$$

unde cum sit $V = x^n \sqrt{\frac{1-x}{x}}$, haec quantitas manifesto evanescit sumto $x = 1$, ita ut nostra formula, si post integrationem statuatur $x = 1$, quaesito satisfaciat. Quod si jam ponamus $x = yy$, ista formula induet hanc formam $2\int \frac{y^{2n-2} \partial y}{\sqrt{(1-yy)}}$, quae, posito post integrationem $y = 1$, praebet hanc relationem

$$\int \frac{y^{2n} \partial y}{\sqrt{(1-yy)}} = \frac{2n-1}{2n} \int \frac{y^{2n-2} \partial y}{\sqrt{(1-yy)}},$$

quae continet relationes supra §. 197. commemoratas; hinc enim fiet

$$\int \frac{yy \partial y}{\sqrt{(1-yy)}} = \frac{1}{2} \int \frac{\partial y}{\sqrt{(1-yy)}},$$
$$\int \frac{y^4 \partial y}{\sqrt{(1-yy)}} = \frac{3}{4} \int \frac{yy \partial y}{\sqrt{(1-yy)}},$$
$$\int \frac{y^6 \partial y}{\sqrt{(1-yy)}} = \frac{5}{6} \int \frac{y^4 \partial y}{\sqrt{(1-yy)}} \text{ etc.}$$

Exemplum 2.

§. 207. Quaerantur formulae integrales, ut fiat

$$\int x^n \partial v = \frac{\alpha n-1}{\alpha n} \int x^{n-1} \partial v.$$

Cum igitur hic esse debeat

$$(\alpha n-1) \int x^{n-1} \partial v = \alpha n \int x^n \partial v,$$

erit hoc casu $a = -1$, $\beta = \alpha$ et $b = 0$, unde per formulas supra datas colligitur

$$Q = Cx^{\frac{-1}{\alpha}}(\alpha - \alpha x)^{\frac{-+\alpha}{\alpha}} = Cx^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}}$$

quae quantitas manifesto evanescit posito $x = 1$. Tum autem erit

$$\partial v = \frac{x^{\frac{-1}{\alpha}}(1-x)^{\frac{+1}{\alpha}}\partial x}{(1-x)},$$

unde formula nostra generalis erit

$$\int x^{n-1}\partial v = \int x^{n-\frac{1}{\alpha}-1}(1-x)^{+\frac{1}{\alpha}-1}\partial x = \int \frac{x^{n-\frac{1}{\alpha}-1}\partial x}{(1-x)^{1-\frac{1}{\alpha}}},$$

quae concinnior redditur, faciendo $x = y^{\alpha}$, tum enim ea induet hanc formam $\int \frac{y^{\alpha n-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}}$, ubi iterum post integrationem statui debet $y = 1$. Erit hinc

$$\int \frac{y^{\alpha n+\alpha-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha n - 1}{\alpha n}\int \frac{y^{\alpha n-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}},$$

atque hinc orientur sequentes casus speciales

$$\int \frac{y^{2\alpha-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}} = \frac{\alpha-1}{\alpha}\int \frac{y^{\alpha-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}} \text{ et}$$

$$\int \frac{y^{3\alpha-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}} = \frac{2\alpha-1}{2\alpha}\int \frac{y^{2\alpha-2}\partial y}{(1-y^{\alpha})^{\frac{\alpha-1}{\alpha}}}.$$

§. 208. Hinc igitur si sumatur $\alpha = 1$, ut fieri debeat

$$\int x^{n}\partial v = \frac{n-1}{n}\int x^{n-1}\partial v,$$

formula nostra generalis jam in y expressa erit $\int y^{n-2}\partial y$, cujus ergo valor est $\frac{1}{n-1}y^{n-1} = \frac{1}{n-1}$, unde tota series nostrarum formularum integralium abibit in hanc

$$\tfrac{1}{0},\ \tfrac{1}{1},\ \tfrac{1}{2},\ \tfrac{1}{3},\ \tfrac{1}{4},\ \tfrac{1}{5},\ \tfrac{1}{6},\ \tfrac{1}{7},\ \text{etc.}$$

§. 209. Sumamus etiam $\alpha = \frac{1}{2}$, et jam non amplius opus erit ad y procedere. Hoc igitur casu erit

$$Q = \frac{(1-x)^2}{xx} \text{ et } \partial v = \frac{(1-x)\partial x}{xx},$$

unde formula nostra generalis fit

$$\int x^{n-1}\partial v = \int x^{n-3}(1-x)\partial x,$$

cujus ergo valor algebraice expressus erit

$$\frac{1}{n-2}x^{n-2} - \frac{1}{n-1}x^{n-1} = \frac{1}{(n-1)(n-2)};$$

unde series nostrarum formularum evadet

$$\frac{1}{0\cdot -1},\ \frac{1}{0\cdot 1},\ \frac{1}{1\cdot 2},\ \frac{1}{2\cdot 3},\ \frac{1}{3\cdot 4},\ \frac{1}{4\cdot 5},\ \text{etc.}$$

Exemplum 3.

§. 210. Quaerantur formulae integrales, ut sit

$$\int x^n\partial v = n\int x^{n-1}\partial v.$$

Cum igitur esse debeat

$$n\int x^{n-1}\partial v = 1\,.\int x^n\partial v, \text{ erit}$$

$$\alpha = 1,\ a = 0,\ b = 1,\ \beta = 0.$$

Cum igitur sit $\beta = 0$, casus Coroll. 2. hic locum habet, indeque erit $Q = e^{-x}$, ideoque $V = e^{-x}\,.\,x^n$, quae quantitas his duobus casibus evanescit $x = 0$ et $x = \infty$. Porro vero erit $\partial v = e^{-x}\partial x$, hincque formula nostra generalis fiet $\int x^{n-1}\partial x\,.\,e^{-x}$, unde ipsi seriei termini ab initio sequenti modo se habebunt

$$\int e^{-x}\partial x,\ \int e^{-x}x\partial x,\ \int e^{-x}xx\partial x,\ \int e^{-x}x^3\partial x \text{ etc.}$$

quibus integratis ita ut evanescant posito $x = 0$, tum vero posito $x = \infty$, orietur sequens series satis simplex

$$1.1,\ 1.2,\ 1.2.3,\ 1.2.3.4,\ 1.2.3.4.5,\ \text{etc.}$$

quae est series hypergeometrica Wallisii, cujus ergo terminus generalis est

$$\int x^{n-1}\, e^{-x}\, \partial x = 1.2.3.4 \ldots\ldots (n-1).$$

§. 211. Ope ergo hujus termini generalis hanc seriem interpolare licebit. Ita si quaeratur terminus medius inter duos primos, poni debet $n = \frac{3}{2}$, ac valor hujus termini erit $\int e^{-x}\,\partial x \sqrt{x}$, cujus autem valor nullo modo algebraice exprimi potest. Inventi autem singulari modo hunc ipsum terminum aequari $\frac{1}{2}\sqrt{\pi}$, denotante π peripheriam circuli cujus diameter $= 1$, unde hic vicissim cognoscimus esse $\int e^{-x}\,\partial x \sqrt{x} = \frac{\sqrt{\pi}}{2}$, posito scilicet post integrationem $x = \infty$. Terminus autem hunc praecedens, indici $\frac{1}{2}$ respondens, erit $= \sqrt{\pi}$, cui ergo aequatur formula $\int \frac{e^{-x}\,\partial x}{\sqrt{x}}$. Quod si hic ponamus $e^{x} = y$, ita ut posito $x = 0$ sit $y = 1$, at posito $x = \infty$ fiat $y = \infty$, tum ergo ista formula $\int \frac{e^{-x}\,\partial x}{\sqrt{x}}$ abit in hanc $\int \frac{\partial y}{yy\sqrt{ly}}$, quae formula si ita integretur ut evanescat posito $y = 1$, tum vero fiat $y = \infty$, praebet valorem ipsius $\sqrt{\pi}$. Si porro fiat $y = \frac{1}{z}$, erunt termini integrationis $z = 1$, et $z = 0$, et formula integralis erit

$$-\int \frac{\partial z}{\sqrt{-lz}} \begin{bmatrix} \text{a} & z = 1 \\ \text{ad} & z = 0 \end{bmatrix} = \sqrt{\pi},$$

sive permutatis terminis integrationis erit

$$\int \frac{\partial z}{\sqrt{-lz}} \begin{bmatrix} \text{a} & z = 0 \\ \text{ad} & z = 1 \end{bmatrix} = \sqrt{\pi},$$

quemadmodum jam olim observavi.

Exemplum 4.

§. 212. Quaerantur formulae integrales, ut sit

$$\int x^n \partial v = \frac{1}{n}\int x^{n-1}\partial v, \text{ sive}$$

$$\int x^{n-1}\partial v = n\int x^n \partial v.$$

Hic est $\alpha = 0$ et $a = 1$, $\beta = 1$ et $b = 0$; qui ergo est casus in Coroll. 1. tractatus, unde colligitur fore $Q = e^{\frac{1}{x}}$, ideoque $V = x^n e^{\frac{1}{x}}$, quae formula nequidem evanescit sumto $x = 0$, quandoquidem formula $e^{\frac{1}{0}}$ aequivalet infinitò infinitesimae potestatis. Hic autem miro modo evenit, ut casus $x = -0$ reddat formulam $e^{\frac{-1}{0}}$ subito evanescentem. Scilicet, si ω denotet quantitatem infinite parvam, erit $e^{\frac{1}{\omega}} = \infty^{\infty}$, tum vero repente fiet $e^{\frac{-1}{\omega}} = \frac{1}{\infty^{\infty}} = 0$, quam ob causam formulam hinc exhibere non licet scopo nostro respondentem. Reperietur quidem $\partial v = -e^{\frac{1}{x}}\frac{\partial x}{x}$, ita ut formula nostra generalis futura sit $-\int x^{n-2}\partial x\, e^{\frac{1}{x}}$, quae autem nobis nullum usum praestare potest.

§. 213. Quod si hic ponamus $x = y$, formula ista generalis transit in hanc $+\int\frac{e^y \partial y}{y^n}$. At vero nunc erit $V = \frac{e^y}{y^n}$, quae formula evanescit posito $y = -\infty$. Quomodocunque autem hanc expressionem transformemus, semper idem incommodum occurret. Interim tamen etiam hunc casum sequenti modo resolvere licebit. Sit enim seriei, quam quaerimus, primus terminus $= a$,

ex quo per regulam praescriptam sequentes ordine ita procedent

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & & & n \\ \omega, & \frac{\omega}{1}, & \frac{\omega}{1\cdot 2}, & \frac{\omega}{1\cdot 2\cdot 3}, & \frac{\omega}{1\cdot 2\cdot 3\cdot 4}, & \ldots\ldots & & \frac{\omega}{1\cdot 2\cdot 3\cdot\ldots\ldots(n-1)}. \end{array}$$

Supra autem vidimus, hujus formulae $1.2.3.4.5\ldots\ldots(n-1)$ valorem exprimi per hoc integrale $\int x^{n-1}e^{-x}\partial x$, integratione ab $x=0$ ad $x=\infty$ extensa; tantum igitur opus est ut hanc formulam integralem in denominatorem transferamus, et seriei quam quaerimus terminus generalis erit

$$\frac{1}{\int x^{n-1}e^{-x}\partial x},$$

unde satis intelligitur, negotium non per simplicem formulam integralem expediri posse, quod idem quoque tenendum est de aliis casibus, quibus quantitas V non duobus casibus evanescere potest; tum enim tantum opus est fractionem $\frac{\alpha n+a}{\beta n+b}$ invertere, atque formulam integralem in denominatorem transferre.

Scholion.

§. 214. Nisi sit vel $\alpha=0$ vel $\beta=0$, quos casus jam expedivimus, resolutio nostri problematis semper reduci potest ad casum, quo ambae litterae α et β sunt aequales unitati. Cum enim esse debeat

$$\int x^n\partial v=\frac{\alpha n+a}{\beta n+b}\int x^{n-1}\partial v,$$

ponatur $x=\frac{\alpha y}{\beta}$, fietque

$$\frac{\alpha}{\beta}\int y^n\partial v=\frac{\alpha n+a}{\beta n+b}\int y^{n-1}\partial v,$$

quae aequatio reducitur ad hanc formam

$$\int y^n\partial v=\frac{n+a:\alpha}{n+b:\beta}\int y^{n-1}\partial v.$$

Quod si jam nunc loco $\frac{a}{\alpha}$ scribamus a, et b loco $\frac{b}{\beta}$, resolvenda erit haec formula

$$\int y^n \partial v = \frac{n+a}{n+b} \int y^{n-1} \partial v,$$

cujus resolutio, si loco x scribamus y et loco litterarum α et β unitatem, ex superiori solutione praebet primo

$$Q = C y^a (1-y)^{b-a},$$

quod ergo evanescit posito $y = 1$, si modo fuerit $b > a$, tum autem erit ipsa formula

$$\int y^{n-1} \partial v = C \int y^{n+a-1} \partial y \, (1-y)^{b-a-1};$$

sin autem fuerit $b < a$, haec solutio, uti vidimus, locum habere nequit; verum hoc casu pro termino nostrae seriei assumi debet haec forma $\frac{1}{\int y^{n-1} \partial v}$, ita ut tum esse debeat

$$\frac{1}{\int y^n \partial v} = \frac{n+a}{n+b} \cdot \frac{1}{\int y^{n-1} \partial v}, \text{ sive}$$

$$\int y^n \partial v = \frac{n+b}{n+a} \int y^{n-1} \partial v,$$

cujus resolutio permutatis litteris a et b praebet

$$Q = C y^b (1-y)^{a-b},$$

quae jam casu $y = 1$ evanescit, si fuerit $a > b$, atque tum erit formula generalis

$$\int y^{n-1} \partial v = C \int y^{n+b-1} \partial y \, (1-y)^{a-b-1}.$$

Sive igitur sit $b > a$ sive $a > b$, solutio nulla amplius laborat difficultate.

§. 215. Sin autem fuerit vel $\alpha = 0$ vel $\beta = 0$, loco alterius etiam scribi poterit unitas; unde si esse debeat

$$\int x^n \partial v = \frac{n+a}{b} \int x^{n-1} \partial v,$$

ob $\alpha = 1$ et $\beta = 0$, solutio nostra generalis dat

$$\frac{\partial Q}{Q} = \frac{\partial x}{x}(a - bx);$$

unde colligitur $Q = Cx^a . e^{-bx}$; quae formula evanescit posito $x = \infty$, si modo b fuerit numerus positivus; tum autem fit terminus generalis

$$\int x^{n-1} \partial v = C \int x^{n+a-1} \partial x . e^{-x} .$$

At vero numerus b negativus esse nequit, quia alioquin conditio praescripta esset incongrua.

§. 216. Consideremus etiam alterum casum, quo $\alpha = 0$ et $\beta = 1$, ideoque conditio praescripta

$$\int x^n \partial v = \frac{a}{n+b} \int x^{n-1} \partial v,$$

unde fit

$$\frac{\partial Q}{Q} = -\frac{\partial x}{xx}(a - bx).$$

Hinc autem pro Q orietur valor, qui praeter casum $x = 0$ evanescere non posset; quam ob causam formula generalis statui debet $\frac{1}{\int x^{n-1} \partial v}$, ita ut esse debeat

$$\int x^n \partial v = \frac{n+b}{a} \int x^{n-1} \partial v,$$

unde prodit

$$\frac{\partial Q}{Q} = \frac{\partial x}{x}(b - ax), \text{ ideoque } Q = Ce^{-ax} . x^b,$$

quae expressio evanescit posito $x = \infty$, quoniam a necessario debet esse numerus positivus; tum autem erit

$$\partial v = Ce^{-ax} . x^b \partial x,$$

unde formula generalis seriei erit

$$\frac{1}{C \int x^{a+b-1} \partial x . e^{-ax}}.$$

Problema 2.

Denotet T terminum indici n respondentem in serie quam considerandam suscepimus, at vero T' terminum sequentem, atque proponatur haec conditio adimplenda

$$T' = \frac{(\alpha n + a)\,(\alpha' n + a')}{(\beta n + b)\,(\beta' n + b')}\,T.$$

Solutio.

§. 217. Quoniam hic valores geminati occurrunt, huic conditioni commodissime satisfiet, si terminus generalis T tanquam productum ex duobus factoribus spectetur. Statuatur igitur $T = RS$, sitque terminus sequens $= R'S'$, et quaerantur formulae R et S, ut fiat

$$R' = \frac{\alpha n + a}{\beta n + b}\,R \text{ et } S' = \frac{\alpha' n + a'}{\beta' n + b'}\,S,$$

tum enim sumendo $T = RS$ conditioni praescriptae manifesto satisfiet. Hoc igitur modo pro R et S vel hujusmodi formulae $\int x^{n-1}\,\partial v$, vel inversae $\frac{1}{\int x^{n-1}\,\partial v}$ reperientur, id quod pro solutione generali sufficit, unde rem exemplo illustremus.

Exemplum.

§. 218. Quaeratur formula generalis T, ut fiat

$$T' = \frac{nn - cc}{nn}\,T.$$

Resolvamus igitur T in duos factores R et S, ac statuamus

$$R' = \frac{n-c}{n}\,R \text{ et } S' = \frac{n+c}{n}\,S.$$

Pro priore forma si statuamus $R = \int x^{n-1}\,\partial v$, ex solutione generali, ubi erit $\alpha = 1$, $a = -c$, $\beta = 1$ et $b = 0$, fiet

$$Q = Cx^{-c}\,(1-x)^c,$$

quae forma manifesto evanescit posito $x = 1$, hincque quia fit

$$V = C\,x^{n-c}\,(1-x)^c,$$

haec forma etiam casu $x = 0$ evanescit, si modo n fuerit $> c$, id quod tuto assumi potest, quia exponentem n successive in infinitum crescere assumimus, ac plerumque pro c fractiones tantum accipi solent. Hinc ergo erit

$$R = C \int x^{n-c-1} (1-x)^{c-1} \partial x.$$

§. 219. Hinc jam alter valor litterae S deduci posset, scribendo tantum $-c$ loco c, tum autem non amplius fieret $Q = 0$ posito $x = 1$, quamobrem pro S formulam inversam $\frac{1}{\int x^{n-1} \partial v}$ assumi oportet, ut esse debeat

$$\int x^n \partial v = \frac{n}{n+c} \int x^{n-1} \partial v,$$

ubi cum sit $\alpha = 1$, $a = 0$, $\beta = 1$ et $b = c$, reperitur $Q = C (1-x)^c$, quae forma manifesto fit $= 0$ posito $x = 1$, hinc autem prodit

$$\partial v = C (1-x)^{c-1} \partial x,$$

ergo habebimus

$$S = \frac{1}{C \int x^{n-1} (1-x)^{c-1} \partial x};$$

consequenter formula nostra generalis quaesita erit

$$T = \frac{\int x^{n-c-1} (1-x)^{c-1} \partial x}{\int x^{n-1} (1-x)^{c-1} \partial x}.$$

§. 220. Quod si ergo nostrae seriei per factores procedentes primum terminum ponamus $= A$, ipsa series erit

$$\begin{array}{llll} \text{I.} & \text{II.} & \text{III.} & \text{IV.} \\ A, & \frac{1-cc}{1} A, & \frac{1-cc}{1} \cdot \frac{4-cc}{4} A, & \frac{1-cc}{1} \cdot \frac{4-cc}{4} \cdot \frac{9-cc}{9} A, \text{ etc.} \end{array}$$

unde si sumamus $c = \frac{1}{2}$, erit haec series

$$A, \frac{1 \cdot 3}{2 \cdot 2} A, \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} A, \frac{1 \cdot 3}{2 \cdot 2} \cdot \frac{3 \cdot 5}{4 \cdot 4} \cdot \frac{5 \cdot 7}{6 \cdot 6} A, \text{ etc.}$$

cujus ergo terminus indici n respondens est

$$\frac{\int x^{n-\frac{3}{2}}(1-x)^{-\frac{1}{2}}\partial x}{\int x^{n-1}(1-x)^{-\frac{1}{2}}\partial x},$$

qui posito $x = yy$ transit in hanc formam

$$\frac{\int y^{2n-2}(1-yy)^{-\frac{1}{2}}\partial y}{\int y^{2n-1}(1-yy)^{-\frac{1}{2}}\partial y};$$

unde patet, terminum primum fore

$$A = \int \frac{\partial y}{\surd(1-yy)} : \int \frac{y\partial y}{\surd(1-yy)} = \frac{\pi}{2},$$

posito scilicet post integrationem $y = 1$.

Problema 3.

Denotet T terminum seriei indici n respondentem, sintque T' et T'' termini sequentes pro indicibus $n+1$ et $n+2$, si proponatur inter ternos terminos se insequentes talis relatio, ut sit

$$(\alpha n + a)\, T = (\beta n + b)\, T' + (\gamma n + c)\, T'',$$

investigare formulam pro T, qua terminus generalis hujus seriei exprimatur.

Solutio.

§. 221. Assumatur pro T formula integralis $\int x^{n-1}\partial v$, hujusque integrale ita capiatur, ut evanescat posito $x = 0$, eruntque termini sequentes

$$T' = \int x^{n}\partial v \text{ et } T'' = \int x^{n+1}\partial v,$$

siquidem post integrationem variabili x certus valor determinatus tribuatur. Quamdiu autem haec quantitas x ut variabilis spectatur, ponamus esse

$$(\alpha n + a)\, T = (\beta n + b)\, T' + (\gamma n + c)\, T'' + x^{n} Q,$$

ac perspicuum est Q ejusmodi functionem esse debere ipsius x, quae evanescat, si loco x valor ille determinatus substituatur, quem autem a cifra diversum esse oportet, quoniam jam assumsimus, omnes istas formulas in nihilum abire posito $x = 0$. Quodsi vero, absoluto calculo, huic conditioni nullo modo satisfieri poterit, id erit indicio, problema nostrum hac ratione resolvi non posse, ut scilicet ejus terminus generalis T per talem formulam differentialem simplicem $\int x^{n-1}\partial v$ exhibeatur.

§. 222. Differentiemus nunc aequationem modo stabilitam, ac divisione facta per x^{n-1} sequens prodibit aequatio

$$(\alpha n+a)\partial v = (\beta n+b)x\partial v+(\gamma n+c)xx\partial v+nQ\partial x+x\partial Q,$$

quae, quia termini littera n affecti seorsim se destruere debent, discerpetur in binas sequentes aequationes

$$1^\circ.\ \alpha\partial v = \beta x\partial v+\gamma xx\partial v+Q\partial x,$$

$$2^\circ.\ a\partial v = bx\partial v+cxx\partial v+x\partial Q,$$

ex quarum priore fit

$$\partial v = \frac{Q\partial x}{\alpha-\beta x-\gamma xx},$$

ex altera vero fit

$$\partial v = \frac{x\partial Q}{a-bx-cxx},$$

quorum valorum posterior per priorem divisus praebet

$$\frac{\partial Q}{Q} = \frac{\partial x\,(a-bx-cxx)}{x\,(\alpha-\beta x-\gamma xx)},$$

ex cujus ergo integratione valor ipsius Q elici debet, quo facto facile patebit, utrum is certo quodam casu praeter $x = 0$ evanescere possit. Imprimis autem hic notari convenit, si hoc integrale involvat hujusmodi factorem $e^{\frac{\lambda}{x}}$, tum solutionem quoque successu esse cariturum, quandoquidem posito $x = 0$ iste factor tantam involvet infiniti potestatem, ut, etiamsi per x^n multiplicetur, productum etiamnum infinitum maneat.

§. 223. Quodsi igitur his conditionibus praescriptis satisfacere licuerit, tum invento valore litterae Q, quem ponamus fieri $= 0$ posito $x = f$, habebitur

$$\partial v = \frac{Q\partial x}{\alpha - \beta x - \gamma xx},$$

et formula generalis naturam seriei complectens erit

$$T = \int x^{n-1}\partial v = \int \frac{x^{n-1} Q\partial x}{\alpha - \beta x - \gamma xx},$$

quippe cujus integrale, a termino $x = 0$ usque ad terminum $x = f$ extensum, praebebit valorem termini T, indici cuicunque n respondentis.

Scholion

§. 224. Inventa autem tali relatione inter ternos terminos cujuspiam seriei sibi invicem succedentes, inde more solito formari poterit fractio continua, cujus valorem assignare licebit. Si enim characteres T', T'', T''', T^{IV}, etc. denotent ordine omnes terminos post T sequentes in infinitum, ex relationibus, quas inter se tenent, sequentes formulae deducentur. Ex relatione

$$(\alpha n + a)\, T = (\beta n + b)\, T' + (\gamma n + c)\, T''$$

deducitur

$$(\alpha n + a)\frac{T}{T'} = \beta n + b + \frac{(\gamma n + c)\,(\alpha n + \alpha + a)}{(\alpha n + \alpha + a)\; T' : T''}.$$

Ex relatione sequente

$$(\alpha n + \alpha + a)\, T' = (\beta n + \beta + b)\, T'' + (\gamma n + \gamma + c)\, T'''$$

deducitur

$$(\alpha n + \alpha + a)\frac{T'}{T''} = \beta n + \beta + b + \frac{(\gamma n + \gamma + c)\,(\alpha n + 2\alpha + a)}{(\alpha n + 2\alpha + a)\; T'' : T'''}.$$

Simili modo sequentes relationes suppeditabunt

$$(\alpha n + 2\alpha + a)\frac{T''}{T'''} = \beta n + 2\beta + b + \frac{(\gamma n + 2\gamma + c)\,(\alpha n + 3\alpha + a)}{(\alpha n + 3\alpha + a)\; T''' : T''''};$$

$$(\alpha n+3\alpha+a)\frac{T'''}{T''''}=(\beta n+3\beta+b)+\frac{(\gamma n+3\gamma+c)(\alpha n+4\alpha+a)}{(\alpha n+4\alpha+a)T'''':T'''''}, \text{ etc.}$$

unde manifestum est, si in prima formula continuo sequentes valores ordine substituantur, proditurum esse fractionem continuam, cujus valor aequalis erit formulae $(\alpha n+a)\frac{T}{T'}$.

§. 225. Quod si ergo loco n successive scribamus numeros 1, 2, 3, 4, etc. sequens problema circa fractiones continuas resolvere poterimus.

Problema 4.

Proposita fractione continua hujus formae

$$\beta+b+\cfrac{(\gamma+c)(2\alpha+a)}{2\beta+b+\cfrac{(2\gamma+c)(3\alpha+a)}{3\beta+b+\cfrac{(3\gamma+c)(4\alpha+a)}{4\beta+b+\cfrac{(4\gamma+c)(5\alpha+a)}{5\beta+b+\cfrac{(5\gamma+c)(6\alpha+a)}{6\beta+b+\text{etc.}}}}}}$$

ejus valorem investigare.

Solutio.

§. 226. Consideretur in genere ista relatio inter ternas quantitates sibi succedentes T, T′, T″, quae sit

$$(\alpha n+a)T=(\beta n+b)T'+(\gamma n+c)T'',$$

atque ex praecedente problemate quaeratur valor ipsius T, siquidem fieri potest, hoc modo expressus

$$T=\int x^{n-1}\partial v=\int\frac{x^{n-1}Q\partial x}{\alpha-\beta x-\gamma xx},$$

cujus integrale ab $x=0$ usque ad $x=f$ extendatur, qua formula inventa ponatur

$$\int \frac{Q\partial x}{\alpha - \beta x - \gamma xx} = A \text{ et } \int \frac{xQ\partial x}{\alpha - \beta x - \gamma xx} = B,$$

ita ut A et B sint valores ipsius T, pro casibus $n = 1$ et $n = 2$, quibus definitis fractionis continuae propositae valor per praecedentia erit $= \frac{(\alpha + a) A}{B}$. Hanc igitur investigationem ad sequentia exempla accommodemus.

Exemplum 1.

§. 227. Investigare valorem fractionis continuae notissimae, quam olim *Brouncherus* pro quadratura circuli protulit, quae est

$$2 + \cfrac{1.1}{2 + \cfrac{3.3}{2 + \cfrac{5.5}{2 + \text{etc.}}}}$$

Quia omnes partes integrae laevam respicientes sunt constantes $= 2$, pro nostra forma generali fiet

$$\beta + b = 2,\ 2\beta + b = 2,\ 3\beta + b = 2, \text{ etc.}$$

ergo $\beta = 0$ et $b = 2$; at pro numeratoribus sequentium fractionum, quandoquidem constant binis factoribus, erit pro factoribus prioribus

$$\gamma + c = 1,\ 2\gamma + c = 3,\ 3\gamma + c = 5,\ 4\gamma + c = 7, \text{ etc.}$$

unde concluditur $\gamma = 2$ et $c = -1$, pro alteris vero erit

$$2\alpha + a = 1,\ 3\alpha + a = 3,\ 4\alpha + a = 5, \text{ etc.}$$

unde $\alpha = 2$ et $a = -3$. Ex his autem valoribus colligimus hanc aequationem

$$\frac{\partial Q}{Q} = - \frac{\partial x(3 + 2x - xx)}{2x(1 - xx)},$$

quae per $1 + x$ depressa praebet

$$\frac{\partial Q}{Q} = - \frac{\partial x(3 - x)}{2x(1 - x)};$$

unde integrando fit

$$lQ = -\tfrac{3}{2}lx + l(1-x) \text{ et hinc } Q = \frac{1-x}{x^{\frac{3}{2}}},$$

ex quo valore porro sequitur

$$A = \int \frac{(1-x)\partial x}{2x^{\frac{3}{2}}(1-xx)} = \int \frac{\partial x}{2x(1+x)\sqrt{x}}$$

$$B = \int \frac{(1-x)\partial x}{2x^{\frac{1}{2}}(1-xx)} = \int \frac{\partial x}{2(1+x)\sqrt{x}}.$$

§. 228. In his autem valoribus istud incommodum deprehenditur, quod prius integrale evanescens reddi nequit posito $x = 0$. Hoc autem incommodum facile removeri potest, si fractionem continuam supremo membro truncemus et quaeramus valorem istius fractionis

$$2 + \cfrac{3.3}{2 + \cfrac{5.5}{2 + \text{etc.}}}$$

qui si repertus fuerit $= s$, erit ipsius propositae valor $= b + \frac{1}{s}$. Nunc vero, comparatione instituta, fit quidem ut ante $\beta = 0$ et $b = 2$, tum vero $\gamma = 2$ et $c = +1$, $\alpha = 2$ et $a = -1$, unde sequitur

$$\frac{\partial Q}{Q} = -\frac{\partial x(1+2x+xx)}{x(1-xx)} = -\frac{\partial x(1+x)}{2x(1-x)},$$

unde integrando fit

$$lQ = -\tfrac{1}{2}lx + l(1-x), \text{ ideoque } Q = \frac{1-x}{\sqrt{x}},$$

ex quo valore jam habebimus

$$A = \int \frac{(1-x)\partial x}{2(1-xx)\sqrt{x}} = \tfrac{1}{2}\int \frac{\partial x}{(1+x)\sqrt{x}}, \text{ et}$$

$$B = \tfrac{1}{2}\int \frac{\partial x\sqrt{x}}{1+x}$$

ubi cum sit $Q = \frac{1-x}{\sqrt{x}}$, ejus valor manifesto evanescit posito $x = 1$, quamobrem illa integralia a termino $x = 0$ usque ad $x = 1$ sunt extendenda.

§. 229. Quo nunc haec integralia facilius eruamus, statuamus $x = zz$, ita ut termini integrationis etiamnunc sint $z = 0$ et $z = 1$, eritque

$$A = \int \frac{\partial z}{1+zz} = \text{Arc. tang. } z = \frac{\pi}{4}, \text{ et}$$

$$B = \int \frac{zz\,\partial z}{1+zz} = 1 - \frac{\pi}{4},$$

sicque habebimus $s = \frac{\pi}{4-\pi}$, quocirca ipsius fractionis *Brouncherianae* valor est $1 + \frac{4}{\pi}$, omnino uti olim *Brounckerus* jam invenerat.

Exemplum 2.

§. 230. Investigare valorem hujus fractionis continuae *Brouncherianae* latius patentis

$$b + \cfrac{1.1}{b + \cfrac{3.3}{b + \cfrac{5.5}{b + \text{etc.}}}}$$

Ut hic incommodum superius evitemus, omittamus membrum supremum et quaeramus

$$s = b + \cfrac{3.3}{b + \cfrac{5.5}{b + \text{etc.}}}$$

quandoquidem tum erit valor quaesitus $= b + \frac{1}{s}$. Nunc igitur erit $\beta = 0$ et $b = b$, $\gamma = 2$, $c = 1$, $\alpha = 2$ et $a = -1$, unde fit

$$\frac{\partial Q}{Q} = -\frac{\partial x\,(1+bx+xx)}{2x\,(1-xx)},$$

ac proinde

$$lQ = -\tfrac{1}{2}\,lx - \frac{b-2}{4}\,l(1+x) + \frac{b+2}{4}\,l(1-x),$$

hincque

$$Q = \frac{(1-x)^{\frac{b+2}{4}}}{(1+x)^{\frac{b-2}{4}}\sqrt{x}},$$

quae formula manifesto fit $= 0$ ponendo $x = 1$, siquidem $b+2$ fuerit numerus positivus, unde fit

$$\partial v = \frac{(1-x)^{\frac{b-2}{4}}\,\partial x}{2\,(1+x)^{\frac{b+2}{4}}\sqrt{x}}.$$

Hinc autem colligetur

$$A = \tfrac{1}{2}\int\frac{(1-x)^{\frac{b-2}{4}}\,\partial x}{(1+x)^{\frac{b+2}{4}}\sqrt{x}} \quad \text{et}$$

$$B = \tfrac{1}{2}\int\frac{(1-x)^{\frac{b-2}{4}}\,\partial x\sqrt{x}}{(1+x)^{\frac{b+2}{4}}},$$

sive ponendo $x = zz$ habebimus

$$A = \int\frac{(1-zz)^{\frac{b-2}{4}}\,\partial z}{(1+zz)^{\frac{b+2}{4}}} \quad \text{et}$$

$$B = \int\frac{(1-zz)^{\frac{b-2}{4}}\,zz\,\partial z}{(1+zz)^{\frac{b+2}{4}}},$$

quae ambo integralia a $z = 0$ usque ad $z = 1$ sunt extendenda. Ex his autem valoribus A et B erit $s = \frac{A}{B}$; ipsius igitur fractionis propositae valor erit $= b + \frac{1}{s} = b + \frac{B}{A}$.

§. 231. Quod si hic ponatur $b = 2$, prodit casus ante expositus a quadratura circuli pendens, quippe quo casu formula fit rationalis. Quando autem exponentes $\frac{b-2}{4}$ et $\frac{b+2}{4}$ non sunt numeri integri, tum litteras A et B neque per arcus circulares, neque per logarithmos exprimere licet. Veluti si fuerit $b = 4$, erit

$$A = \int \frac{\partial z \sqrt{(1 - zz)}}{(1 + zz)^{\frac{3}{2}}},$$

cujus valor per arcus ellipticos exhiberi posset. At si b fuerit numerus impar, hi valores multo magis evadunt transcendentes, ita ut his ipsis litteris A et B debeamus esse contenti. Contra autem si exponentes illi fiant numeri integri, totum negotium per arcus circulares expedire licebit.

§. 232. Exponentes autem illi $\frac{b-2}{4}$ et $\frac{b+2}{4}$ erunt numeri integri, quoties fuerit b numerus hujus formae $b = 4i + 2$, tum enim erit

$$A = \int \frac{(1 - zz)^i \partial z}{(1 + zz)^{i+1}} \text{ et}$$

$$B = \int \frac{(1 - zz)^i zz \partial z}{(1 + zz)^{i+1}};$$

quos ergo casus quomodo evolvi oporteat, operae pretium erit docere, quoniam *Wallisius* eos jam est contemplatus.

§. 233. Quoniam hoc negotium totum redit ad reductionem hujusmodi formularum integralium ad formas simpliciores, consideremus in genere formam

$$P = \frac{z^m}{(1+zz)^n},$$

cujus differentiale sub sequentibus formis exhiberi potest

$$1^\circ).\ \partial P = \frac{m z^{m-1}\partial z}{(1+zz)^n} - \frac{2n z^{m+1}\partial z}{(1+zz)^{n+1}}$$

$$2^\circ).\ \partial P = \frac{m z^{m-1}\partial z}{(1+zz)^{n+1}} - \frac{(2n-m) z^{m+1}\partial z}{(1+zz)^{n+1}}.$$

$$3^\circ).\ \partial P = -\frac{(2n-m) z^{m-1}\partial z}{(1+zz)^n} - \frac{2n z^{m-1}\partial z}{(1+zz)^{n+1}}$$

unde hanc triplicem reductionem integralium deducimus

$$\text{I.}\quad \int\frac{z^{m+1}\partial z}{(1+zz)^{n+1}} = \frac{m}{2n}\int\frac{z^{m-1}\partial z}{(1+zz)^n} - \frac{1}{2n}\cdot\frac{z^m}{(1+zz)^n}$$

$$\text{II.}\quad \int\frac{z^{m+1}\partial z}{(1+zz)^{n+1}} = \frac{m}{2n-m}\int\frac{z^{m-1}\partial z}{(1+zz)^{n+1}} - \frac{1}{2n-m}\cdot\frac{z^m}{(1+zz)^n}$$

$$\text{III.}\quad \int\frac{z^{m-1}\partial z}{(1+zz)^{n+1}} = \frac{2n-m}{2n}\int\frac{z^{m-1}\partial z}{(1+zz)^n} + \frac{1}{2n}\cdot\frac{z^m}{(1+zz)^n}$$

quarum reductionum ope casibus $b = 4i + 2$ totum negotium absolvi et ad formulam $\frac{\pi}{4}$ reduci poterit, siquidem post integrationem sumatur $z = 1$.

§. 234. Sit $i = 1$ ideoque $b = 6$, eritque

$$A = \int\frac{(1-zz)\partial z}{(1+zz)^2} \quad\text{et}\quad B = \int\frac{(1-zz)zz\partial z}{(1+zz)^2}.$$

Nunc igitur reperiemus per reductionem tertiam,

$$\int\frac{\partial z}{(1+zz)^2} = \tfrac{1}{2}\int\frac{\partial z}{1+zz} + \tfrac{1}{2}\cdot\frac{z}{1+zz} = \frac{\pi}{8} + \frac{1}{4},$$

et per reductionem primam

$$\int \frac{zz\partial z}{(1+zz)^2} = \frac{1}{2}\int \frac{\partial z}{1+zz} - \frac{1}{2}\cdot\frac{z}{1+zz} = \frac{\pi}{8} - \frac{1}{4},$$

porro

$$\int \frac{z^4\partial z}{(1+zz)^2} = \frac{3}{2}\int \frac{zz\partial z}{1+zz} - \frac{1}{2}\cdot\frac{z^3}{1+zz} = \frac{5}{4} - \frac{3\pi}{8}.$$

Ex his jam valoribus colligitur $A = \frac{1}{2}$ et $B = \frac{\pi}{2} - \frac{3}{2}$, ideoque $\frac{B}{A} = \pi - 3$, quocirca orietur ista summatio

$$3 + \pi = 6 + \cfrac{1.1}{6 + \cfrac{3.3}{6 + \cfrac{5.5}{6 + \cfrac{7.7}{6 + \text{etc.}}}}}$$

§. 235. Sit nunc $i = 2$ et $b = 10$, eritque

$$A = \int \frac{(1-zz)^2\partial z}{(1+zz)^3} \quad \text{et} \quad B = \int \frac{zz(1-zz)^2\partial z}{(1+zz)^3}.$$

Quo harum integralium valores investigemus, sequentes evolvamus formulas.

$$\int \frac{\partial z}{(1+zz)^3} = \frac{3}{4}\int \frac{\partial z}{(1+zz)^2} + \frac{1}{4}\cdot\frac{z}{(1+zz)^2} = \frac{3\pi}{32} + \frac{1}{4}$$

$$\int \frac{zz\partial z}{(1+zz)^3} = \frac{1}{4}\int \frac{\partial z}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z}{(1+zz)^2} = \frac{\pi}{32}$$

$$\int \frac{z^4\partial z}{(1+zz)^3} = \frac{3}{4}\int \frac{zz\partial z}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z^3}{(1+zz)^2} = \frac{3\pi}{32} - \frac{1}{4}$$

$$\int \frac{z^6\partial z}{(1+zz)^3} = \frac{5}{4}\int \frac{z^4\partial z}{(1+zz)^2} - \frac{1}{4}\cdot\frac{z^5}{(1+zz)^2} = \frac{3}{2} - \frac{15\pi}{32}.$$

Ex quibus jam valoribus deducitur $A = \frac{\pi}{8}$ et $B = 2 - \frac{5\pi}{8}$, ideoque $\frac{B}{A} = \frac{16 - 5\pi}{\pi}$, unde emergit sequens summatio

$$\frac{5\pi + 16}{\pi} = 10 + \cfrac{1.1}{10 + \cfrac{3.3}{10 + \cfrac{5.5}{10 + \text{etc.}}}}$$

§. 236. Si b esset numerus negativus, investigatio nulla prorsus laboraret difficultate. Si enim in genere fuerit

$$s = -a + \cfrac{\alpha}{-b + \cfrac{\beta}{-c + \cfrac{\gamma}{-d + \cfrac{\delta}{-e + \text{etc.}}}}}$$

semper erit

$$-s = a + \cfrac{\alpha}{b + \cfrac{\beta}{c + \cfrac{\gamma}{d + \cfrac{\delta}{\varepsilon + \text{etc.}}}}}$$

unde si habeatur valor istius expressionis, idem negative sumtus dabit valorem illius.

Exemplum 3.

§. 237. Proposita sit fractio continua, cujus valorem investigari oporteat, ista

$$1 + \cfrac{1.1}{3 + \cfrac{3.3}{5 + \cfrac{5.5}{7 + \cfrac{7.7}{9 + \text{etc.}}}}}$$

Quo fractiones supra allegatae, omisso membro supremo, sint

$$3+\cfrac{3.3}{5+\cfrac{5.5}{7+\cfrac{7.7}{9+\text{etc.}}}}$$

eritque $\beta + b = 3$, $2\beta + b = 5$, ideoque $\beta = 2$ et $b = 1$; tum vero ut ante $\alpha = 2$, $a = -1$, $\gamma = 2$ et $c = +1$; invento autem s erit valor quaesitus $= 1 + \frac{1}{s}$. Nunc igitur habebimus

$$\frac{\partial Q}{Q} = -\frac{\partial x(1+x+xx)}{2x(1-x-xx)}.$$

Est vero

$$\frac{1+x+xx}{x(1-x-xx)} = \frac{1}{x} + \frac{2+2x}{1-x-xx},$$

unde fit

$$lQ = -\tfrac{1}{2} lx - \int \frac{\partial x(1+x)}{1-x-xx}.$$

Porro vero pro formula $\int \frac{\partial x(1+x)}{1-x-xx}$ invenienda, statuamus denominatorem

$$1-x-xx = (1-fx)(1-gx),$$

eritque $f+g = 1$ et $fg = -1$, unde fit

$$f = \frac{1+\sqrt{5}}{2} \text{ et } g = \frac{1-\sqrt{5}}{2}.$$

Nunc statuatur

$$\frac{1+x}{1-x-xx} = \frac{\mathfrak{A}}{1-fx} + \frac{\mathfrak{B}}{1-gx},$$

unde reperietur

$$\mathfrak{A} = \frac{1+f}{f-g} \text{ et } \mathfrak{B} = -\frac{(1+g)}{f-g},$$

sive substitutis pro f et g valoribus supra datis erit

$$\mathfrak{A} = \frac{\sqrt{5}+3}{2\sqrt{5}} \text{ et } \mathfrak{B} = \frac{\sqrt{5}-3}{2\sqrt{5}},$$

quibus inventis erit

$$\int \frac{\partial x\,(1+x)}{1-x-xx} = -\frac{\mathfrak{A}}{f}\, l(1-fx) - \frac{\mathfrak{B}}{g}\, l(1-gx) =$$
$$-\frac{(1+\surd 5)}{2\surd 5}\, l(1-fx) - \frac{(\surd 5-1)}{2\surd 5}\, l(1-gx),$$

quocirca fiet

$$l\,Q = -\tfrac{1}{2}\, lx + \frac{(\surd 5+1)}{2\surd 5}\, l(1-fx) + \frac{(\surd 5-1)}{2\surd 5}\, l(1-gx),$$

consequenter

$$Q = \frac{(1-fx)^{\frac{\surd 5+1}{2\surd 5}}\,(1-gx)^{\frac{\surd 5-1}{2\surd 5}}}{\surd x},$$

qui valor duobus casibus evanescit: altero quo

$$x = \frac{1}{f} = \frac{2}{1+\surd 5} = \frac{\surd 5-1}{2},$$

altero vero quo $x = \frac{1}{g} = -\frac{1-\surd 5}{2}$; utrovis autem utamur, res eodem redibit.

§. 238. Ex hoc autem valore habebimus

$$A = \int \frac{Q\,\partial x}{1-x-xx} \text{ et } B = \int \frac{Q\,x\,\partial x}{1-x-xx},$$

unde porro deducitur

$$s = (\alpha + a)\frac{A}{B} = \frac{A}{B},$$

et propositae fractionis summa erit $1 + \frac{B}{A}$. Hinc autem nihil ulterius concludere licet, ob formulas differentiales non solum irrationales, sed etiam vere transcendentes ob exponentes surdos.

Exemplum 4.

§. 239. Proposita sit haec fractio continua

$$b + \cfrac{1.1}{b + \cfrac{2.2}{b + \cfrac{3.3}{b + \cfrac{4.4}{b + \text{etc.}}}}}$$

ubi est $\beta = 0$, $b = b$. Nunc consideremus hanc formam

$$s = b + \cfrac{2 \,.\, 2}{b + \cfrac{3 \,.\, 3}{b + \text{etc.}}}$$

quippe quo valore invento quaesitus erit $= b + \frac{1}{s}$. Habebimus igitur $\gamma + c = 2$, $2\gamma + c = 3$, ideoque $\gamma = 1$ et $c = 1$, deinde erit $\alpha = \gamma = 1$, $a = 0$ et $c = 1$. Hinc igitur colligimus

$$\frac{\partial Q}{Q} = -\frac{\partial x(bx + xx)}{x(1 - xx)} = -\frac{\partial x(b + xx)}{1 - xx},$$

ideoque

$$lQ = -\tfrac{b}{2} l \frac{1+x}{1-x} + \tfrac{1}{2} l(1 - xx),$$

hincque

$$Q = \frac{(1-x)^{\frac{b}{2}} \sqrt{(1 - xx)}}{(1+x)^{\frac{b}{2}}} = \frac{(1-x)^{\frac{b+1}{2}}}{(1+x)^{\frac{b-1}{2}}},$$

quae quantitas manifesto evanescit posito $x = 1$. Hinc igitur fiet

$$A = \int \frac{Q \partial x}{1 - xx} = \int \frac{(1-x)^{\frac{b+1}{2}} \partial x}{(1+x)^{\frac{b-1}{2}} (1-xx)} = \int \frac{(1-x)^{\frac{b-1}{2}} \partial x}{(1+x)^{\frac{b+1}{2}}}, \text{ etc.}$$

$$B = \int \frac{x(1-x)^{\frac{b-1}{2}} \partial x}{(1+x)^{\frac{b+1}{2}}},$$

tum autem erit $s = (\alpha + a)\frac{A}{B} = \frac{A}{B}$, ideoque summa quaesita $= b + \frac{B}{A}$.

§. 240. Percurramus nunc casus praecipuos: ac primo sit $b = 1$, eritque

$$A = \int \frac{\partial x}{1+x} = l(1+x) = l2, \text{ et}$$
$$B = \int \frac{x\,\partial x}{1+x} = x - \int \frac{\partial x}{1+x} = 1 - l2,$$

ideoque $b + \frac{B}{A} = \frac{1}{l2}$; ergo hinc prodiit ista summatio

$$\frac{1}{l2} = 1 + \cfrac{1.1}{1 + \cfrac{2.2}{1 + \cfrac{3.3}{1 + \text{etc.}}}}$$

§. 241. Sit nunc $b = 2$, eritque

$$A = \int \frac{\partial x \sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}}, \text{ et } B = \int \frac{x\,\partial x \sqrt{(1-x)}}{(1+x)^{\frac{3}{2}}}.$$

Ad has formulas rationales reddendas statuamus

$$\frac{\sqrt{1-x}}{\sqrt{1+x}} = z, \text{ eritque } x = \frac{1-zz}{1+zz},$$

unde terminis integrationis $x = 0$ et $x = 1$ respondebunt $z = 1$ et $z = 0$; tum vero erit

$$1 + x = \frac{2}{1+zz}, \text{ et } \partial x = -\frac{4z\,\partial z}{(1+zz)^2},$$

hincque colligitur

$$A = -2\int \frac{zz\,\partial z}{1+zz} = -2z + 2 \text{ Arc. tang. } z + 2 - \frac{\pi}{2} = 2 - \frac{\pi}{2},$$

porro fit

$$B = -2\int \frac{zz\,\partial z}{(1+zz)^2} + 2\int \frac{z^4\,\partial z}{(1+zz)^2}.$$

Per reductiones igitur supra §. 234. monstratas, si hic scilicet terminos integrationis $z = 1$ et $z = 0$ permutemus, ut habeamus

$$B = +2\int \frac{zz\,\partial z}{(1+zz)^2} - 2\int \frac{z^4\,\partial z}{(1+zz)^2}, \text{ erit}$$
$$B = 2\left(\frac{\pi}{8} - \frac{1}{4}\right) - 2\left(\frac{3}{4} - \frac{\pi}{8}\right) = \pi - 3,$$

unde sequitur ista summatio

$$\frac{2}{4-\pi} = 2 + \cfrac{1.1}{2 + \cfrac{2.2}{2 + \cfrac{3.3}{2 + \text{etc.}}}}$$

quae *Brouncherianae* simplicitate nihil cedit.

§. 242. Si ponamus $b = 0$, fractio continua abit in sequens continuum productum

$$\frac{1\cdot 1}{2\cdot 2} \cdot \frac{3\cdot 3}{4\cdot 4} \cdot \frac{5\cdot 5}{6\cdot 6} \cdot \frac{7\cdot 7}{8\cdot 8} \cdot \text{etc.}$$

hoc autem casu fit

$$A = \int \frac{\partial x}{\sqrt{(1-xx)}} = \frac{\pi}{2} \text{ et } B = \int \frac{x\partial x}{\sqrt{(1-xx)}} = 1;$$

unde istius producti valor colligitur $\frac{2}{\pi}$, id quod egregie convenit cum jam dudum cognitis, quandoquidem hoc productum est ipsa progressio *Wallisiana*.

Exemplum 5.

§. 243. Proposita sit haec fractio continua, ubi $\beta = 0$, $b = b$, et numeratores numeri trigonales

$$b + \cfrac{1}{b + \cfrac{3}{b + \cfrac{6}{b + \cfrac{10}{b + \text{etc.}}}}}$$

Omisso supremo membro statuamus

$$s = b + \cfrac{3}{b + \cfrac{6}{b + \text{etc.}}}$$

et primo numeratores per producta repraesentemus hoc modo

$$3 = 2 \cdot \tfrac{3}{2}, \quad 6 = 3 \cdot \tfrac{4}{2}, \quad 10 = 4 \cdot \tfrac{5}{2}, \text{ etc.}$$

quorum priores comparentur cum formulis

$$\gamma + c, \quad 2\gamma + c, \quad 3\gamma + c, \text{ etc.}$$

posteriores vero cum formulis $2\alpha + a$, $3\alpha + a$, $4\alpha + a$, etc. eritque $\gamma = 1$, $c = 1$, $\alpha = \frac{1}{2}$, $a = \frac{1}{2}$, unde erit

$$\frac{\partial Q}{Q} = \frac{\partial x\,(\frac{1}{2} - bx - xx)}{x(\frac{1}{2} - xx)} = \frac{\partial x\,(1 - 2bx - 2xx)}{x\,(1 - 2xx)}, \text{ sive}$$

$$\frac{\partial Q}{Q} = \frac{\partial x}{x} - \frac{2b\,\partial x}{1 - 2xx},$$

cujus integrale est

$$lQ = lx - \frac{b}{\sqrt{2}}\, l\, \frac{1 + x\sqrt{2}}{1 - x\sqrt{2}}, \text{ ergo}$$

$$Q = \frac{x\,(1 - x\sqrt{2})^{\frac{b}{\sqrt{2}}}}{(1 + x\sqrt{2})^{\frac{b}{\sqrt{2}}}},$$

quae formula evanescit casu $x = \frac{1}{\sqrt{2}}$. Hinc igitur erit

$$\partial v = \frac{2x\,(1 - x\sqrt{2})^{\frac{b}{\sqrt{2}}}\,\partial x}{(1 - 2xx)\,(1 + x\sqrt{2})^{\frac{b}{\sqrt{2}}}}.$$

Sit $\frac{b}{\sqrt{2}} = \lambda$, eritque

$$A = 2\int \frac{x\,(1 - x\sqrt{2})^{\lambda}\,\partial x}{(1 - 2xx)\,(1 + x\sqrt{2})^{\lambda}} = 2\int \frac{x\,(1 - x\sqrt{2})^{\lambda - 1}\,\partial x}{(1 + x\sqrt{2})^{\lambda + 1}}$$

et

$$B = 2\int \frac{xx\,(1 - x\sqrt{2})^{\lambda - 1}\,\partial x}{(1 + x\sqrt{2})^{\lambda + 1}},$$

ubi post integrationem statuitur $x = \frac{1}{\sqrt{2}}$; tum autem fit $s = \frac{A}{B}$, hincque valor fractionis propositae $= b + \frac{B}{A}$.

§. 244. Nisi igitur fuerit $\lambda = \frac{b}{\sqrt{2}}$ numerus rationalis, hos valores commode assignare non licet. Sit igitur $b = \sqrt{2}$, sive $\lambda = 1$, eritque

$$A = 2\int \frac{x\partial x}{(1+x\sqrt{2})^2}, \text{ et } B = 2\int \frac{xx\partial x}{(1+x\sqrt{2})^2}.$$

Hinc integrando colligitur

$$A = l(1+x\sqrt{2}) - \frac{x\sqrt{2}}{1+x\sqrt{2}},$$

ideoque posito $x\sqrt{2} = 1$ fiet $A = l2 - \frac{1}{2}$; tum vero reperitur

$$B = \frac{3}{2\sqrt{2}} - \sqrt{2}\,.\,l2,$$

quare ob $b = \sqrt{2}$ erit

$$b + \frac{B}{A} = \frac{1}{\sqrt{2}(2l2-1)},$$

unde sequitur haec summatio

$$\frac{1}{\sqrt{2}(2l2-1)} = \sqrt{2} + \cfrac{1}{\sqrt{2} + \cfrac{3}{\sqrt{2} + \cfrac{6}{\sqrt{2} + \text{etc.}}}}$$

Scholion.

§. 245. Fractiones autem continuae, ad quas plerumque calculo numerico deducimur, hujusmodi formam habere solent

$$a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{e + \text{etc.}}}}}$$

ubi omnes numeratoros sunt unitates, denominatores vero a, b, c, d, etc. numeri integri. Verum ope nostrae methodi difficulter

talium formarum valores eruere licet, etiamsi numeri $a, b, c, d,$ etc. progressionem arithmeticam constituant, id quod sequenti exemplo ostendamus.

Exemplum 6.

§. 246. Proposita sit ista fractio continua

$$\beta + b + \cfrac{1}{2\beta + b + \cfrac{1}{3\beta + b + \cfrac{1}{4\beta + b + \cfrac{1}{5\beta + b + \text{etc.}}}}}$$

ubi $\alpha = 0$, $\gamma = 0$, $a = 1$, $c = 1$.
Hinc fit

$$\frac{\partial Q}{Q} = -\frac{\partial x\,(1 + bx - xx)}{\beta xx}, \text{ unde}$$

$$lQ = \frac{1}{\beta x} + \frac{b}{\beta}\, lx + \frac{x}{\beta} \text{ et}$$

$$Q = e^{\frac{1 + xx}{\beta x}} \cdot x^{\frac{b}{\beta}},$$

quae autem expressio nullo casu evanescere potest, etiamsi per x^n multiplicetur, siquidem β fuerit numerus positivus. Verum si pro β sumamus numeros negativos, puta $\beta = -m$, tum valor $Q = x^{\frac{-b}{m}} \times e^{\frac{-(1+xx)}{mx}}$ manifesto evanescit, tam si $x = 0$, quam si $x = \infty$. Hinc autem erit

$$\partial v = \frac{x^{\frac{-b}{m}} \cdot e^{\frac{-1-xx}{mx}}\, \partial x}{mxx},$$

quamobrem habebimus

$$A = \frac{1}{m} \int \frac{\partial x}{x^{2 + \frac{b}{m}} \cdot e^{\frac{1+xx}{mx}}}, \text{ et}$$

$$B = \frac{1}{m}\int \frac{\partial x}{x^{1+\frac{b}{m}} \cdot e^{\frac{1+xx}{mx}}}.$$

His valoribus inventis formula $\frac{A}{B}$ exprimet summam hujus fractionis continuae

$$-m+b+\cfrac{1}{-2m+b+\cfrac{1}{-3m+b+\cfrac{1}{-4m+b+\cfrac{1}{-5m+b+\text{etc.}}}}}$$

quamobrem formula illa negative sumta $-\frac{A}{B}$ exprimet valorem hujus fractionis continuae

$$m-b+\cfrac{1}{2m-b+\cfrac{1}{3m-b+\cfrac{1}{4m-b+\text{etc.}}}}$$

quem igitur assignare liceret, si modo formulae integrales A et B expediri et a termino $x=0$ ad $x=\infty$ extendi possent. Verum istae formulae ita sunt comparatae, ut earum integratio nullo plane casu per quantitates cognitas exprimi queat, quod tamen non impedit, quo minus fractio $\frac{A}{B}$ valores satis cognitos involvere queat, etiamsi eos nullo adhuc modo assignare valeamus.

§. 247. Talium autem fractionum continuarum mihi quidem binae sequentes innotuere, quarum valores commode exhibere licet

$$n+\cfrac{1}{3n+\cfrac{1}{5n+\cfrac{1}{7n+\cfrac{1}{9n+\text{etc.}}}}}=\frac{e^{\frac{2}{n}}}{e^{\frac{2}{n}}-1},\ \text{et}$$

$$n-\cfrac{1}{3n-\cfrac{1}{5n-\cfrac{1}{7n-\cfrac{1}{9n-\text{etc.}}}}}=\cot.\tfrac{1}{n}.$$

Harum fractionum prior cum formulis postremi exempli collata praebet $m-b=n$, $2m-b=3n$, ideoque $m=2n$ et $b=n$, unde fit

$$A=\tfrac{1}{2n}\int\frac{\partial x}{x^{\frac{5}{2}}\,e^{\frac{1+xx}{2nx}}},\ \text{et}$$

$$B=\tfrac{1}{2n}\int\frac{\partial x}{x^{\frac{3}{2}}\,e^{\frac{1+xx}{2n}}},$$

unde jam discimus si hae duae formulae integrentur a termino $x=0$ usque ad terminum $x=\infty$, tum fore

$$\frac{A}{B}=\frac{1+e^{\frac{2}{n}}}{1-e^{\frac{2}{n}}},$$

quanquam nulla adhuc via analytica patet, hanc convenientiam demonstrandi.

SUPPLEMENTUM VI.

IN FINE SECTIONIS I. TOM. I.

DE

INTEGRATIONE FORMULARUM DIFFERENTIALIUM.

De formulis integralibus duplicatis. *Novi Commentarii Academiae Scientiar. Petropolitanae Tom. XIV. Pars I. Pag.* 72 — 103.

§. 1. Si corporis cujusque propositi vel soliditatem vel superficiem vel alias hujusmodi quantitates definire velimus, id per duplicem integrationem fieri solet; formula enim differentialis bis integranda tali forma $Z \partial x \partial y$ exprimitur, binas variabiles x et y continente, quarum altera sola in priori integratione ut variabilis spectatur, posterior vero integratio ad alteram jam ut variabilem spectatam instituitur. Hinc quantitas per duplicem istam integrationem resultans duplex signum integrale praefigendo indicari solet hoc modo $\int\int Z \partial x \partial y$, quippe qua duplicatione formula differentialis proposita $Z \partial x \partial y$ bis integrari debere est intelligenda. Hujusmodi igitur expressiones geminato signo summatorio affectas his formulas integrales duplicatas appello, quarum usus cum latissime pateat, in earum indolem hic diligentius inquirere, earumque proprietates et affectiones accuratius evolvere constitui.

§. 2. Primum igitur cum x et y sint duae quantitates variabiles a se invicem non pendentes, Z vero denotet earum functionem quamcunque, formulae integralis duplicatae $\int\int Z\partial x\partial y$ vis ita exponi potest, ut quaerenda sit functio finita binarum istarum variabilium x et y, quae ita bis differentiata, ut in altera differentiatione sola x in altera sola y pro variabili habeatur, ad formulam $Z\partial x\partial y$ deducat. Ita si fuerit $Z = a$, evidens fore $\int\int a\partial x\partial y = axy$; generalius vero erit $\int\int a\partial x\partial y = axy + X + Y$, denotante X functionem quamcunque ipsius x et Y ipsius y, quandoquidem hae duae quantitates per geminam illam differentiationem ex calculo tolluntur.

§. 3. In genere autem si V fuerit ejusmodi functio ipsarum x et y, quae bis differentiata ita ut modo est praeceptum, praebeat $Z\partial x\partial y$; erit quidem utique $V = \int\int Z\partial x\partial y$; verum duplex integratio insuper functiones arbitrarias X et Y, illam ipsius x, hanc ipsius y inducit, ut sit generalissime

$$\int\int Z\partial x\partial y = V + X + Y.$$

Ex quo statim perspicitur, hujusmodi formulas differentiales necessario affectas esse producto $\partial x\partial y$, neque propterea secundum hanc significationem tales formulas $\int\int Z\partial x^2$ vel $\int\int Z\partial y^2$ quicquam significare; siquidem per ipsam rei naturam excluduntur, dum in altera integratione sola x, in altera vero sola y ut variabilis tractatur.

§. 4. Constituta sic forma hujusmodi formularum integralium duplicatarum $\int\int Z\partial x\partial y$, ita ut x et y sint duae quantitates variabiles a se invicem non pendentes, et Z functio finita ex iis utcunque composita, haud difficile est duplicem integrationem, quam involvunt, instituere, quod quidem prout primo vel x vel y sola variabilis consideratur, duplici modo fieri potest. Sumta scilicet primo y pro variabili, altera x ut constans trac-

tatur, quaeriturque integrale $\int Z \partial y$, quod erit certa quaedam functio ipsarum x et y; qua inventa suscipiatur formula differentialis $\partial x \int Z \partial y$, in qua jam y ut constans solaque x ut variabilis tractetur, ejusque quaeratur integrale $\int \partial x \int Z \partial y$, qui erit valor quaesitus formulae integralis duplicatae propositae $\int\int Z \partial x \partial y$. Si in hac duplici integratione ordo variabilium x et y invertatur, valor quaesitus ita exprimetur $\int \partial y \int Z \partial x$, qui ab illo non discrepabit.

§. 5. Ob hunc consensum fit, ut talis forma $\int\int Z \partial x \partial y$ promiscue sive hoc modo $\int \partial x \int Z \partial y$ sive hoc $\int \partial y \int Z \partial x$ exhiberi possit; utrovis autem utamur, regulae vulgares integrationis sunt observandae, si modo notetur in ea integratione, in qua vel x vel y pro constante sumatur, constantem introductam ejusdem fore functionem quamcunque. Veluti si proponatur haec forma

$$\int\int \frac{\partial x \partial y}{xx+yy} = \int \partial x \int \frac{\partial y}{xx+yy}, \text{ ob}$$

$$\int \frac{\partial y}{xx+yy} = \frac{1}{x} \text{ Arc. tang. } \frac{y}{x} + \frac{\partial X}{\partial x},$$

denotante $\frac{\partial X}{\partial x}$ functionem quamcunque ipsius x, erit

$$\int\int \frac{\partial x \partial y}{xx+yy} = \int \frac{\partial x}{x} \text{ Arc. tang. } \frac{y}{x} + X,$$

ubi in integratione adhuc perficienda y pro constante habetur. Simili vero modo reperitur

$$\int\int \frac{\partial x \partial y}{xx+yy} = \int \frac{\partial y}{y} \text{ Arc. tang. } \frac{x}{y} + Y,$$

in qua integratione x constans assumitur, in quo quidem exemplo consensus binorum valorum inventorum non satis est perspicuus.

§. 6. Interim tamen veritas consensus per series facile ostenditur; cum enim sit

$$\text{Arc. tang. } \frac{x}{y} = \frac{\pi}{2} - \text{Arc. tang. } \frac{y}{x},$$

denotante $\frac{\pi}{2}$ angulum rectum, et

$$\text{Arc. tang.}\ \frac{y}{x} = \frac{y}{x} - \frac{y^3}{3x^3} + \frac{y^5}{5x^5} - \frac{y^7}{7x^7} + \frac{y^9}{9x^9} - \text{etc.}$$

erit

$$\int \frac{\partial x}{x}\ \text{Arc. tang.}\ \frac{y}{x} = \frac{-y}{x} + \frac{y^3}{9x^3} - \frac{y^5}{25x^5} + \frac{y^7}{49x^7} - \text{etc.} + \mathrm{f}:y \ \text{et}$$

$$\int \frac{\partial y}{y}\ \text{Arc. tang.}\ \frac{x}{y} = \frac{\pi}{2}\, ly - \frac{y}{x} + \frac{y^3}{9x^3} - \frac{y^5}{25x^5} + \frac{y^7}{49x^7} - \text{etc.} + \mathrm{f}:x$$

ex quarum utraque oritur

$$\int\int \frac{\partial x \partial y}{xx+yy} = \mathrm{X} + \mathrm{Y} - \frac{y}{x} + \frac{y^3}{9x^3} - \frac{y^5}{25x^5} + \frac{y^7}{49x^7} - \text{etc.}$$

Verum ubi ambae integrationes succedunt, convenientia sponte se offert: quod quidem pluribus exemplis ostendisse superfluum foret, cum ejus ratio ex natura differentialium et integralium perfecte sit demonstrata.

§. 7. Haec igitur tenenda sunt de istiusmodi formulis integralibus duplicatis, quando binae variabiles x et y nullo plane nexu inter se cohaerent, ita ut in altera integratione altera, in altera vero altera constans accipiatur. Verum tales formulae non confundendae sunt cum iis, quibus ut initio dixi, soliditas et superficies corporum quorumcunque exprimi solet. Etsi enim hae formulae etiam duplicem integrationem requirunt, et in priori altera binarum variabilium puta y sola ut variabilis tractatur altera x pro constante assumta, tamen priori integratione peracta, ea per omnes valores ipsius y extendi, sicque tandem loco y extremus valor, quem recipere potest, statui debet, qui plerumque ab x pendet, ita ut hoc valore post primam integrationem loco y constituto in posteriori integratione y tanquam functio quaedam ipsius x ingrediatur, neque propterea pro constanti haberi queat, qua conditione fit, ut altera integratio plurimum immutetur, etsi prior simili modo ut ante absolvatur.

§. 8. Quod discrimen quo clarius perspiciatur, exemplum attulisse juvabit. Quaeratur ergo soliditas sphaerae, cujus centrum sit C et radius C A $= a$, ac primo quidem portio ejus quadranti A C B insistens, cujus elementum est columella Y Z y z areolae Y $y = \partial x \partial y$ insistens, positis C P $= x$, et P Y $= y$, erit ejus altitudo Y Z $= \sqrt{(aa - xx - yy)}$; hinc soliditas columellae elementaris $= \partial x \partial y \sqrt{(aa - xx - yy)}$, quam bis integrari oportet. Maneat primo intervallum C P $= x$ constans, et integrale $\int \partial y \sqrt{(aa - xx - yy)}$ ita sumtum, ut evanescat posito $y = 0$, dabit portiunculam areolae P p Y q insistentem, quae ergo erit

Fig. 3.

$$= \tfrac{1}{2} y \sqrt{(aa - xx - yy)} + \tfrac{1}{2}(aa - xx) \text{ Arc. sin. } \frac{y}{\sqrt{(aa - xx)}}.$$

Jam hoc valore in altera integratione uti oportet, sed antequam is inducatur, per totam distantiam P M extendi debet, ut habeatur elementum soliditatis toti areolae P p M m insistens; puncto autem Y ad M usque promoto, fit $y = \sqrt{(aa - xx)}$, qui ergo valor loco y substitui debet, ita ut in sequente integratione quantitas y minime ut constans consideretur, haecque tractandi methodus plurimum a praecedente discrepet.

§. 9. Posito ergo $y = \sqrt{(aa - xx)}$, fit

$$\int \partial y \sqrt{(aa - xx - yy)} = \tfrac{\pi}{4}(aa - xx),$$

cum sit Arc. sin. $1 = \frac{\pi}{2}$; sicque integratio adhuc absolvenda erit

$$\int \partial x \int \partial y \sqrt{(aa - xx - yy)} = \tfrac{\pi}{4} \int (aa - xx) \partial x,$$

ubi quidem unica variabilis x inest, sed non ideo, quod jam hic y pro constanti habeatur, sed quia pro y certa quaedam functio ipsius x est substituta. Haec altera vero integratio ita instituta, ut evanescat posito $x = 0$, dabit soliditatem portionis sphaerae, quae areae C B M P insistit, quae idcirco erit $= \frac{\pi}{4}(aax - \frac{1}{3}x^3)$; unde sphaerae octans seu portio toti qua-

dranti A C B insistens prodibit, punctum P usque in A promovendo, ut fiat $x = a$. Tum ergo soliditas octantis sphaerae erit $= \frac{\pi}{6} a^3$, hincque totius sphaerae $= \frac{4\pi}{3} a^3$, uti constat. Ex quo exemplo intelligitur, talem soliditatis investigationem plurimum differre ab integratione duplicata formularum primo exposita.

§. 10. Quod si non totum octantem sphaerae, sed eam tantum ejus portionem quae areae rectangulari C E D F insistit investigare velimus, prior integratio ut ante instituenda est, sed ea peracta ipsi y valor P M debet tribui, qui quidem est constans, et propterea haec investigatio ad primum genus videtur accedere, verum tamen eo discrepat, quod integrale determinatum prodeat, cum ibi functiones indefinitae X et Y inveherentur. Posito ergo ut ante sphaerae radio C A $= a$, sit rectanguli C E F D latus C D $= e$ et C E $= f$: et solidum elementare areolae P p Y q insistens erit ut ante Fig. 4.

$$\tfrac{1}{2} y \sqrt{(aa - xx - yy)} + \tfrac{1}{2}(aa - xx) \text{ Arc. sin. } \frac{y}{\sqrt{(aa - xx)}},$$

quod usque ad M extensum, ubi fit $y = f$, erit

$$\tfrac{1}{2} f \sqrt{(aa - ff - xx)} + \tfrac{1}{2}(aa - xx) \text{ Arc. sin. } \frac{f}{\sqrt{(aa - xx)}},$$

unde solidum areae C P E M insistens sequenti integrali exprimetur

$$\tfrac{1}{2} f \int \partial x \sqrt{(aa - ff - xx)} + \tfrac{1}{2} \int (aa - xx) \partial x \text{ Arc. sin. } \frac{f}{\sqrt{(aa - xx)}},$$

si quidem ita definiatur, ut evanescat posito $x = 0$. Evolvamus ergo seorsim has binas formulas.

§. 11. Ac prima quidem statim praebet

$$\int \partial x \sqrt{(aa - ff - xx)} = \tfrac{1}{2} x \sqrt{(aa - ff - xx)} + \tfrac{1}{2}(aa - ff) \text{ Arc. sin. } \frac{x}{\sqrt{(aa - ff)}},$$

altera autem ob

$$\partial \,.\, \text{Arc. sin. } \frac{f}{\sqrt{(aa - xx)}} = \frac{f x \partial x}{(aa - xx)\sqrt{(aa - ff - xx)}},$$

ita transformatur

$$\int (aa-xx)\partial x \text{ Arc. sin. } \frac{f}{\sqrt{(aa-xx)}} = (aax-\tfrac{1}{3}x^3)\text{Arc. sin. } \frac{f}{\sqrt{(aa-xx)}}$$
$$-f\int \frac{(aa-\frac{1}{3}xx)xx\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}},$$

ad quam postremam partem integrandam, notetur esse

$$\text{Arc. sin. } \frac{fx}{\sqrt{(aa-ff)(aa-xx)}} = \int \frac{af\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}},$$

hujus ergo dabitur multiplum quoddam, quod illi formae adjectum praebeat talem formam

$$\int \frac{(aa-\frac{1}{3}xx)xx\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}} + m\text{ Arc. sin. } \frac{fx}{\sqrt{(aa-ff)(aa-xx)}}$$
$$= \int \frac{(aaxx-\frac{1}{3}x^4+maf)\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}},$$

ut $aaxx-\frac{1}{3}x^4+maf$ fiat per $aa-xx$ divisibile, id quod fit sumendo $m=-\frac{2a^3}{3f}$; hincque erit

$$\int \frac{(aa-\frac{1}{3}xx)xx\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}} = \frac{2a^3}{3f}\text{Arc. sin. } \frac{fx}{\sqrt{(aa-ff)(aa-xx)}}$$
$$-\tfrac{1}{3}\int \frac{(2aa-xx)\partial x}{\sqrt{(aa-ff-xx)}}.$$

§. 12. Cum igitur sit

$$\int \frac{(2aa-xx)\partial x}{\sqrt{(aa-ff-xx)}} = \tfrac{1}{2}(3aa+ff)\text{Arc. sin. } \frac{x}{\sqrt{(aa-ff)}} + \tfrac{1}{2}x\sqrt{(aa-ff-xx)},$$

$$\text{erit } \int \frac{(aa-\frac{1}{3}xx)xx\partial x}{(aa-xx)\sqrt{(aa-ff-xx)}} =$$

$$\frac{2a^3}{3f}\text{Arc. sin. } \frac{fx}{\sqrt{(aa-ff)(aa-xx)}} - \tfrac{1}{6}(3aa+ff)\text{Arc. sin. } \frac{x}{\sqrt{(aa-ff)}} - \tfrac{1}{6}x\sqrt{(aa-ff-xx)}$$

hincque $\int(aa-xx)\partial x$ Arc. sin. $\frac{fx}{\sqrt{(aa-xx)}} =$

$$(aax-\tfrac{1}{3}x^3)\text{ Arc. sin. }\frac{f}{\sqrt{(aa-xx)}}-\tfrac{2}{3}a^3\text{ Arc. sin. }\frac{fx}{\sqrt{(aa-ff)(aa-xx)}}$$
$$+\tfrac{1}{6}f(3aa+ff)\text{ Arc. sin. }\frac{x}{\sqrt{(aa-ff)}}+\tfrac{1}{6}fx\sqrt{(aa-ff-xx)}.$$

Quare posito $x = CD = e$, erit soliditas portionis sphaerae rectangulo C D E F insistentis

$$\tfrac{1}{4}ef\sqrt{(aa-ee-ff)}+\tfrac{1}{4}f(aa-ff)\text{ Arc. sin }\frac{e}{\sqrt{(aa-ff)}}+\tfrac{1}{6}e(3aa-ee)\times$$
$$\text{Arc. sin. }\frac{f}{\sqrt{(aa-ee)}}-\tfrac{1}{3}a^3\text{ Arc. sin. }\frac{ef}{\sqrt{(aa-ee)(aa-ff)}}+\tfrac{1}{12}f(3aa-ff)\times$$
$$\text{Arc. sin. }\frac{e}{\sqrt{(aa-ff}}+\tfrac{1}{12}ef\sqrt{(aa-ee-ff)},$$

quae expressio reducitur ad hanc

$$\tfrac{1}{3}ef\sqrt{(aa-ee-ff)}+\tfrac{1}{6}f(3aa-ff)\text{ Arc. sin. }\frac{e}{\sqrt{(aa-ff)}}+\tfrac{1}{6}e(3aa-ee)\times$$
$$\text{Arc. sin. }\frac{f}{\sqrt{(aa-ee)}}-\tfrac{1}{3}a^3\text{ Arc. sin. }\frac{e}{\sqrt{(aa-ee)(aa-ff)}}.$$

§. 13. Si ergo rectanguli terminus F usque ad peripheriam porrigatur, ut sit $ee+ff=aa$, primum membrum evanescit, et arcus circulares tria reliqua afficientes abeunt in angulum rectum seu $\frac{\pi}{2}$, eritque soliditas

$$\tfrac{\pi}{2}(\tfrac{1}{3}aae+\tfrac{1}{2}aaf-\tfrac{1}{6}e^3-\tfrac{1}{6}f^3-\tfrac{1}{3}a^3)$$

seu ob $f=\sqrt{(aa-ee)}$ soliditas erit

$$\tfrac{\pi}{12}[(2aa+ee)\sqrt{(aa-ee)}-2a^3+3aae-e^3]$$

quod solidum fit maximum, si $f=e=\frac{a}{\sqrt{2}}$, fitque id tum $=\frac{\pi a^3(5-2\sqrt{2})}{12\sqrt{2}}$, dum soliditas octantis sphaerae est $=\frac{\pi}{6}a^3$. Ita ut nostrum solidum sit ad octantem sphaerae ut $5-2\sqrt{2}$ ad $2\sqrt{2}$. Sin autem punctum F non ad peripheriam quadrantis pertingat, fueritque $f=e$ erit soliditas quaesita

$$=\tfrac{1}{3}ee\sqrt{(aa-2ee)}+\tfrac{1}{3}e(3aa-ee)\text{ Arc. sin. }\frac{e}{\sqrt{(aa-ee)}}$$
$$-\tfrac{1}{3}a^3\text{ Arc. sin. }\frac{ee}{aa-ee}.$$

Quare si fuerit

$$\text{Arc. sin.} \frac{e}{\sqrt{(aa-ee)}} : \text{Arc. sin.} \frac{ee}{(aa-ee)} = a^3 : e(3aa-ee)$$

solidum algebraice exprimetur.

Fig. 5. §. 14. Quo autem rem generalius complectamur, quaeramus solidum areae cuicunque G Q H R insistens, cujus elementum cum areolae $Yy = \partial x \partial y$ insistat, idque sit

$$= \partial x \partial y \sqrt{(aa-xx-yy)},$$

prima integratio sumto x constante praebet

$$\tfrac{1}{2}\partial x[y\sqrt{(aa-xx-yy)} + (aa-xx)\,\text{Arc. sin.} \frac{y}{\sqrt{(aa-xx)}}].$$

Sint jam ex natura curvae G Q H R distantiae extremae $PQ = q$ et $PR = r$, atque solidum elementare areolae Q R insistens erit

$$\tfrac{1}{2}\partial x \left\{ \begin{array}{l} +r\sqrt{(aa-xx-rr)} + (aa-xx)\,\text{Arc. sin.} \frac{r}{\sqrt{(aa-xx)}} \\ -q\sqrt{(aa-xx-qq)} - (aa-xx)\,\text{Arc. sin.} \frac{q}{\sqrt{(aa-xx)}} \end{array} \right\}.$$

Quare cum q et r possint esse functiones quaecunque ipsius x, evidens est quantum absit, quo minus quantitas y in sequente integratione pro constanti habeatur. Sequens autem integratio a valore $x = CE$ usque ad valorem $x = CF$ est extendenda.

Fig 6. §. 15. Si figura basis G Q H R a recta C A trajiciatur, ut quaeratur solidum basi C G H insistens, cujus natura exprimatur aequatione quacunque inter $CP = x$, $PR = r$, erit solidum

$$\tfrac{1}{2}\int \partial x[r\sqrt{(aa-xx-rr)} + (aa-xx)\,\text{Arc. sin.} \frac{r}{\sqrt{(aa-xx)}}]$$

ubi problema non inelegans se offert, *quo figura basis* C G H *quaeritur, ut solidum ei insistens algebraice exprimatur.* Statuatur in hunc finem $r = u\sqrt{(aa-xx)}$, ut solidum indefinitum areae C P R G insistens sit

$$\tfrac{1}{2}\int (aa-xx)\partial x[u\sqrt{(1-uu)} + \text{Arc. sin. } u]$$

quae expressio transformatur in hanc

$$\tfrac{1}{2}(a a x - \tfrac{1}{3} x^3)[u \sqrt{(1 - u u)} + \text{Arc. sin. } u]$$
$$- \int (a a x - \tfrac{1}{3} x^3) \partial u \sqrt{(1 - u u)}.$$

Fiat jam

$$\int (a a x - \tfrac{1}{3} x^3) \partial u \sqrt{(1 - u u)} = n a^3 \text{ Arc sin. } u + a^3 \mathrm{U},$$

existente U functione algebraica ipsius u, et cum sit soliditas

$$\tfrac{1}{2}(a a x - \tfrac{1}{3} x^3) u \sqrt{(1 - u u)} - a^3 \mathrm{U} + (\tfrac{1}{2} a a x - \tfrac{1}{6} x^3 - n a^3) \text{Arc. sin. } u,$$

ea erit algebraica casu $- x^3 + 3\, a\, a\, x = 6\, n\, a^3$: dummodo u evanescat posito $x = 0$, tum enim soliditas erit

$$= n a^3 u \sqrt{(1 - u u)} - a^3 \mathrm{U}.$$

§. 16. Ponamus $\partial \mathrm{U} = \mathrm{U}' \partial u$, ac prodibit haec inter x et u aequatio

$$a a x - \tfrac{1}{3} x^3 = \frac{n a^3}{1 - u u} + \frac{a^3 \mathrm{U}'}{\sqrt{(1 - u u)}}.$$

Fingatur

$$\mathrm{U} = m\, u \sqrt{(1 - u u)}, \text{ erit } \mathrm{U}' = \frac{m - 2 m u u}{\sqrt{(1 - u u)}},$$

et ut u evanescat posito $x = 0$, debet esse $m = - n$, ut fiat

$$a a x - \tfrac{1}{3} x^3 = \frac{2 n a^3 u u}{1 - u u}, \text{ seu } u = \sqrt{\frac{3 a a x - x^3}{6 n a^3 + 3 a a x - x^3}},$$

hincque

$$r = \sqrt{\frac{(a a - x x)(3 a a x - x^3)}{6 n a^3 + 3 a a x - x^3}}.$$

Jam ob

$$u \sqrt{(1 - u u)} = \frac{\sqrt{6 n a^3 (3 a a x - x^3)}}{6 n a^3 + 3 a a x - x^3},$$

fit soliditas illa

$$= \frac{2 n a^3 \sqrt{6 n a^3 (3 a a x - x^3)}}{6 n a^3 + 3 a a x - x^3}.$$

Si haec soliditas locum habere debeat, facto $x = a$, fit $n = \tfrac{1}{3}$,

$$r = \sqrt{\frac{(a a - x x)(3 a a x - x^3)}{2 a^3 + 3 a a x - x^3}} = \sqrt{\frac{x (a - x)(3 a a - x x)}{(a + x) 2 a - x)}},$$

ac posito $x = a$, erit soliditas $= {}^{1} a^{3}$, et curva pro basi inventa est linea quarti ordinis.

§. 17. Quae hic de soliditate portionis sphaericae datae basi insistentis sunt tradita, simili calculo ad quaevis alia corpora accommodari possunt, cum tantum in formula $Z \partial x \partial y$ quantitas Z alio modo per x et y determinetur, dum hic erat $Z = \sqrt{(aa - xx - yy)}$. Quin etiam si superficies corporis cujuscunque datae basi imminens definiri debeat, id integratione gemina similis formulae differentialis $Z \partial x \partial y$ eodem modo expedietur: ita si corpus sit sphaera, elementum superficiei areolae elementari basis $\partial x \partial y$ imminentis est $\frac{a \partial x \partial y}{\sqrt{(aa - xx - yy)}}$ ita ut sit $Z = \frac{a}{\sqrt{(aa - xx - yy)}}$, cujus gemina integratio pari modo pro ratione basis, cui imminens portio superficiei quaeritur, est instituenda. Atque in genere quantitates quaecunque aliae cujusvis corporis, quae certae basi respondeant, ope similium operationum determinabuntur.

§. 18. Quaecunque ergo Z fuerit functio ipsarum x et y, pro integrali duplicato $\int\int Z \partial x \partial y$ primo quaeritur integrale $\int Z \partial y$, quantitate x ut constante spectata, idque extendatur per totam quantitatem y, sicque extremi valores ipsius y in computum ingredientur, quae erunt functiones ipsius x, ex basis figura cognitae; sicque pro $\int Z \partial y$ orietur functio ipsius x, quae in ∂x ducta denuo more solito debet integrari. Idem tenendum est, si ordine inverso primo formula $\int Z \partial x$ integretur, spectato y ut constante, quod integrale dum per totum intervallum x extenditur, extremi valores ipsius x eidem y respondentes, qui erunt functiones ipsius y, invehentur, sicque $\int Z \partial x$ abibit in functionem ipsius y tantum, quae per ∂y multiplicata denuo ita integrari debet, ut integrale per totum

intervallum y extendatur. Utroque scilicet modo integratio per totam basin est extendenda, eademque praecepta sunt observanda, qualiscunque Z fuerit functio ipsarum x et y.

§. 19. Basi ergo data, determinatio integrationum perinde se habet, ac si quantitas Z esset constans, quaereretur que tantum integrale $\int\int \partial x \partial y$, quo area basis exprimitur. Quare ad praecepta, quae in determinatione horum integralium observari oportet, stabilienda, sufficiet posuisse $Z = 1$, ut integrale duplicatum $\int\int \partial x \partial y$ definiendum sit, sive autem sumatur x sive y, extremi valores utriusque determinabuntur per aequationem basis figuram exprimentem. Scilicet priori integratione peracta, ubi punctum Y ubicunque intra terminos extremos erat assumtum, tum hoc punctum in peripheriam basis transferatur, quo pacto x et y fient coordinatae basis, inter quas aequatio datur, ex qua deinceps sive y per x sive x per y determinabitur. Fig. 7.

§. 20. Quae quo clarius perspiciantur, sumamus basis figuram esse circulum centrum in G et radium GQ habentem, ponamusque $CF = f$, $FG = g$, et $GQ = c$, erit puncto Y in peripheriam hujus circuli translato

$$cc = (f - x)^2 + (g - y)^2.$$

Jam ad aream hujus circuli investigandam sit primo x constans, eritque $\int \partial y = y + C$, et quia y habet geminum valorem in nostra basi

$$y = g \pm \sqrt{[cc - (f - x)^2]},$$

haec integratio ita determinetur, ut integrale evanescat, dum ipsi y minor horum valorum $g - \sqrt{[cc - (f - x)^2]}$ tribuitur, ita ut sit

$$\int \partial y = y - g + \sqrt{[cc - (f - x)^2]}.$$

Nunc ergo y usque ad alterum terminum

$$y = g + \sqrt{[cc - (f - x)^2]}$$

extenso erit

$$\int \partial y = 2\sqrt{[cc - (f - x)^2]},$$

quod jam per ∂x multiplicatum et integratum praebet

$$\int \partial x \int \partial y = C - (f - x)\sqrt{[cc - (f - x)^2]} - cc \text{ Arc. sin. } \frac{f - x}{c},$$

quod ut evanescat posito

$$x = f - c, \text{ fit } C = cc \text{ Arc. sit. } 1 = \tfrac{\pi}{2} cc.$$

Porro statuatur $x = f + c$, et ob

$$cc \text{ Arc. sin. } \frac{f - x}{c} = -cc \text{ A sin. } 1 = -\tfrac{\pi}{2} cc,$$

erit area quaesita tota $= \frac{\pi}{2} cc + \frac{\pi}{2} cc = \pi cc$, uti constat.

§. 21. Si has determinationes accuratius perpendamus, videmus extremos valores ipsius x ita esse comparatos, ut alter sit maximus, siquidem basis tota quadam curva in se redeunte terminetur. Hi ergo ambo valores reperientur, si aequatio naturam basis exprimens differentietur, et $\partial x = 0$ ponatur. Quando autem basis non una quadam linea curva terminatur, sed portione quapiam veluti C G H continetur, cujus basis C H sit maxima, tum minor terminus ipsius x manifesto est $= 0$, major autem ipsi C H aequalis: eodemque casu termini applicatae P R abscissae C P $= x$ respondentis sunt alter $= 0$, alter vero $=$ P R. Quacunque ergo basi proposita, ejus figura ante probe est examinanda, ipsiusque termini quaquaversus explorandi, quam investigatio areae vel cujusvis alius formulae integralis duplicatae suscipi queat: definitis autem terminis quibus area continetur, inde determinationes integrationum sunt petendae.

Fig. 6.

§. 22. His de integrationum determinatione expositis, insignes maximeque notatu dignae affectiones hujusmodi formularum integralium duplicatarum perpendi merentur, quae in earum transformatione occurrunt. Scilicet quemadmodum coordinatae ejusdem curvae infinitis modis sumi possunt, ita hic loco binarum variabilium x et y, binae quaecunque aliae variabiles in computum introduci possunt, sive eae pariter sint coordinatae, sive aliae quantitates utcunque definitae. Ita talis transformatio in genere ita concipi potest, ut loco x et y functiones quaecunque aliarum duarum variabilium t et v substituantur, hisque in aequationem pro basi datam introductis, simili modo limites harum quantitatum t et v quibus figura basis terminatur, definiri poterunt. Utcunque autem hae substitutiones sumantur, tandem post duplicem integrationem semper eadem quantitas resultet, necesse est.

§. 23. Si loco x et y aliae quaecunque binae coordinatae orthogonales introducantur puta t et v, quod fit in genere ponendo

$$x = f + mt + v\sqrt{(1 - mm)} \text{ et}$$
$$y = g + t\sqrt{(1 - mm)} - mv,$$

manifestum est, elementum areae basis, quod ante erat $\partial x \partial y$, nunc per $\partial t \partial v$ exprimitur debere. Cum autem inde sit

$$\partial x = m\partial t + \partial v\sqrt{(1 - mm)} \text{ et}$$
$$\partial y = \partial t\sqrt{(1 - mm)} - m\partial v,$$

minime patet, quomodo loco $\partial x \partial y$ per has substitutiones oriri possit $\partial t \partial v$; dum potius prodiret

$$\partial x \partial y = m\partial t^2\sqrt{(1 - mm)} + (1 - 2mm)\partial t \partial v$$
$$- m\partial v^2\sqrt{(1 - mm)},$$

quae autem formula, utcunque ad geminam integrationem adap-

tatur, semper in maximos errores inducet. Multo minus ergo hinc colligere licet, si loco x et y aliae functiones ipsarum t et v substituantur, cujusmodi expressio loco $\partial x \partial y$ adhiberi debeat.

§. 24. Ac primo quidem observo nullam hic esse rationem, cur expressio loco $\partial x \partial y$ in calculum introducenda ei aequalis esse debeat; quod tum demum necesse esset, si binae integrationes eodem modo ut ante secundum binas variabiles instituerentur. Cum autem nunc aliae variabiles t et v adsint, atque altera integratio per variabilitatem ipsius t, altera ipsius v, sit administranda, quae operationes a praecedentibus plurimum differunt; formula jam loco $\partial x \partial y$ introducenda non ex aequalitate aestimari, sed potius ad scopum, qui est propositus, accommodari debet. Et quoniam jam binas integrationes secundum binas variabiles t et v distingui oportet, manifestum est formulam loco $\partial x \partial y$ adhibendam necessario producto $\partial t \partial v$ affectam esse, et hujusmodi formam $Z \partial t \partial v$ habere debere.

§. 25. Quo haec certius expediantur, maneat primo x, et loco y introducatur alia variabilis u, ita ut sit y functio quaecunque ipsarum x et u, et $\partial y = P \partial x + Q \partial u$. Si jam in priori integratione x constans sumatur, erit utique $\partial y = Q \partial u$, hinc $\int\int \partial x \partial y = \int \partial x \int Q \partial u$, ita ut nunc loco formulae $\partial x \partial y$ habeatur $Q \partial x \partial u$, cujus integrale duplicatum proinde etiam hoc modo exprimi poterit $\int \partial u \int Q \partial x$, ubi in priori integratione $\int Q \partial x$ quantitas u sumitur pro constante. Quodsi nunc simili modo u retineatur et loco x introducatur functio quaecunque ipsarum t et u, ut sit $\partial x = R \partial t + S \partial u$, in tractatione formulae $\int \partial u \int Q \partial x$ prior integratio $\int Q \partial x$, in qua u constans statuitur, abibit in hanc $\int Q R \partial t$; ita ut integrale du-

plicatum sit $\int \partial u \int QR \partial t$, seu promiscue $\int\int QP \partial t \partial u$; unde manifestum est ob has ambas substitutiones loco formulae $\partial x \partial y$ hanc $QR \partial t \partial u$ tractari debere.

§. 26. Introducamus nunc statim loco x et y has duas novas variabiles t et u, per quas illae ita determinentur, ut sit

$$\partial x = R \partial t + S \partial u \text{ et } \partial y = T \partial t + V \partial u,$$

unde valore ipsius ∂x in forma $\partial y = P \partial x + Q \partial u$ substituto, fit $\partial y = PR \partial t + (PS + Q) \partial u$, ita ut sit $PR = T$ et $P S + Q = V$, unde fit $P = \frac{T}{R}$ et $\frac{ST}{R} + Q = V$, sicque $QR = VR - ST$. Quare vi harum substitutionum loco $\partial x \partial y$ uti debemus formula $(VR - ST) \partial t \partial u$, quae bis integrata justis abhibitis determinationibus, aeque aream totius basis praebere debet, atque ipsa formula $\partial x \partial y$ bis integrata. Quod autem hic pro formula areae baseos $\int\int \partial x \partial y$ est ostensum, locum habet pro quacunque alia formula $\int\int Z \partial x \partial y$, quippe quae per easdem substitutiones transformatur in hanc $\int\int Z (VR - ST) \partial t \partial u$, dummodo in Z loco x et y assumti valores substituantur. Pari enim modo binas integrationes ex figura basis determinari oportet.

§. 27 Quod si ergo ponatur

$$\partial x = R \partial t + S \partial u \text{ et } \partial y = T \partial t + V \partial u,$$

loco $\partial x \partial y$ consequimur $(RV - ST) \partial t \partial u$, quae formula plurimum differt ab ea, cui productum $\partial x \partial y$ revera est aequale; etiamsi enim termini per ∂t^2 et ∂u^2 affecti, utpote ad duplicem integrationem inepti, rejiciantur tamen quod restat $(RV + ST)$ $\partial t \partial u$ ratione signi a vera formula discrepat. Verum hic non leve dubium exoritur quod cum coordinatae x et y pari passu ambulent, nostra formula potius differentiam $RV - ST$ quam

inversam $S\,T - R\,V$ complectatur; quod dubium eo magis augetur, quod si superius ratiocinium respectu x et y invertissemus eaedem substitutiones nos revera ad formulam $(S\,T - R\,V)\,\partial t\,\partial u$ perduxissent. Sed quia totum discrimen tantum in signo versatur, alteraque formula alterius est negativa, hine determinatio absoluta areae basis, quippe cujus quantitas absoluta quaeritur, nullam mutationem realem patitur.

§. 28. Haec autem magis fient perspicua, si modum
Fig. 7. quo supra (20) ad aream $E\,Q\,H\,R$ inveniendam usi sumus attentius consideremus. Primum scilicet ex integratione formulae $\int\int \partial x\,\partial y$ deduximus hanc aream $= \int \partial x\,(P\,R - P\,Q)$, ubi quidem $P\,Q$ a $P\,R$ subtraximus, quia manifesto erat $P\,R > P\,Q$, sed in ipso calculo nulla continetur ratio, quae praecipiat, ut potius $P\,Q$ a $P\,R$ quam vicissim $P\,R$ a $P\,Q$ subtrahamus, sicque non adversante calculo potuissemus aequo jure eandem aream per $\int \partial x\,(P\,Q - P\,R)$ exprimere, quo pacto ea negativa sed priori aequalis proditura fuisset. Ex quo perspicuum est signum $+$ vel $-$ non quantitatem areae, quae quaeritur, afficere, et calculum pari jure ad utrumque perducere posse. Quam ob causam superius dubium ita diluetur, ut dicamus aream quaesitam ita exprimi debere, ut sit

$$= \pm \int\int \partial t\,\partial u\,(R\,V - S\,T),$$

et ut area positive expressa prodeat, quovis casu eo signo utendum esse, quo $\pm\,(R\,V - S\,T)$ reddatur quantitas positiva.

§. 29. Hinc etiam dubia, quae forte oriri possent circa inventionem areae curvarum, quarum partes utrinque ad axem sunt dispositae, et quibus tyrones saepe non parum turbari solent, facile resolvuntur. Si enim curvae $Q\,A\,R$ ad

axem A P relatae area tota Q A R abscissae $AP = x$ respondens definiri debeat, ejusque partes A P Q et A P R seorsim considerentur, certum est si altera A P Q affirmative spectetur ut sit $= + Q$, alteram A P R negative concipi debere, ut sit $= - R$. Neque tamen hinc sequitur, aream totam Q A R fore $= Q - R$, quippe quae evanesceret, si ambae partes A P Q et A P R essent aequales; sed perinde ac si ambo puncta Q et R ad eandem axis partem sita essent, area perpetuo est $= \pm \int \partial x\,(PR - PQ)$, unde ob $\int PQ \,.\, \partial x = Q$ et $\int PR \,.\, \partial x = - R$, fit tota area $= \pm (Q + R)$, uti rei natura postulat. Fig. 8.

§. 30. Ope autem talium substitutionum, quibus loco binarum variabilium x et y binae quaecunque aliae introducantur t et u, saepenumero integrationes plurimum sublevari facilioresque reddi possunt, et quovis casu haud difficile est substitutiones maxime idoneas reperire. Veluti si area circuli E Q H R ad axem C P relati definiri debeat, ubi ob $CF = f$, $FG = g$, $GQ = c$, erat $cc = (f - x)^2 + (g - y)^2$, poni conveniet Fig. 7.

$$f - x = \frac{t}{\sqrt{(1+uu)}} \text{ et } g - y = \frac{tu}{\sqrt{(1-uu)}},$$

ut fiat $tt = cc$ et $t = c$. Tum vero ob

$$\partial x = \frac{-\partial t}{\sqrt{(1+uu)}} + \frac{tu\,\partial u}{(1+uu)^{\frac{3}{2}}}, \text{ et}$$

$$\partial y = \frac{-u\,\partial t}{\sqrt{(1+uu)}} - \frac{t\,\partial u}{(1+uu)^{\frac{3}{2}}},$$

loco $\partial x \partial y$ per §. 27. adipiscimur

$$\partial t \partial u \left(\frac{t}{(1+uu)^2} + \frac{tuu}{(1+uu)^2}\right) = \frac{t\,\partial t \partial u}{1+uu},$$

cujus duplex integrale ita exprimatur $\int \frac{\partial u}{1+uu} \int t \partial t$. Jam vero est $\int t \partial t = \frac{1}{2} t t = \frac{1}{2} c c$, et area tota erit $\frac{1}{2} c c \int \frac{\partial u}{1+uu}$, dum ipsi u omnes valores possibiles tribuuntur, quandoquidem u non amplius aequationem pro basi afficiebat.

§. 31 Quo hunc usum clarius explicemus, consideremus iterum sphaeram centrum C et radium $CA = a$ habentem, cujus portio basi circulari perpendiculariter insistens quaeri debeat. Quia radium C A per centrum cujus circuli G ducere licet, sit $FG = g = 0$, ut fiat $cc = (f+x)^2 + yy$, et solidum quaesitum

$$= \int\int \partial x \partial y \sqrt{(aa - xx - yy)};$$

statuatur jam

$$x = \frac{t}{\sqrt{(1+uu)}} \text{ et } y = \frac{tu}{\sqrt{(1+uu)}},$$

ut fiat $xx + yy = tt$, et

$$\sqrt{(aa - xx - yy)} = \sqrt{(aa - tt)},$$

et pro $\partial x \partial y$ prodeat $\frac{t \partial t \partial u}{1+uu}$, ita ut soliditas quaesita ita exprimatur $\int\int \frac{t \partial t \partial u \sqrt{(aa-tt)}}{1+uu}$, quae integrationes determinari debebunt ex aequatione hinc pro figura basis oriunda

$$cc = ff - \frac{2ft}{\sqrt{(1+uu)}} + tt,$$

unde fit

$$\text{vel } t = \frac{f \pm \sqrt{(cc + ccuu - ffuu)}}{\sqrt{(1+uu)}},$$

$$\text{vel } \sqrt{(1+uu)} = \frac{2ft}{ff - cc + tt}.$$

§. 32. Consideretur primo t ut constans, fietque integrale

$$= \int t \partial t \sqrt{(aa - tt)} \,.\, \text{Arc. tang.}\, u,$$

ubi constantem adjici non est necesse, quia evanescente u simul

y evanescit, quaeramus enim primo solidum semicirculo insistens. At integrali hoc primo extenso ad terminum extremum, ob Arc. tang. $u =$ Arc. cos. $\frac{1}{\sqrt{(1+uu)}}$, fit id

$$\int t\,\partial t \sqrt{(aa - tt)} \,.\, \text{Arc. cos.}\, \frac{ff - cc + tt}{2ft},$$

cujus integrationis limites sunt $t = f - c$ et $t = f + c$. Si non soliditatem hujus portionis sphaerae, sed ejus superficiem basi quasi imminentem definire voluissemus, perventuri fuissemus ad hanc formulam

$$\int \frac{at\,\partial t}{\sqrt{(aa - tt)}}\, \text{Arc. cos.}\, \frac{ff - cc + tt}{2ft},$$

at operae pretium non videtur ejus integrationem fusius prosequi.

§. 33. Methodus autem hujusmodi formulas integrales duplicatas tractandi haud parum illustrabitur, si eam ad problema illud quondam famosum Florentinum accommodemus, quo in superficie sphaerica portio geometrice assignabilis requirebatur, cujus superficies algebraice exprimi possit. Immineat talis sphaerae portio curvae GRH, cujus propterea figura est determinanda: in qua si ponatur CP $= x$, PR $= y$, superficies sphaerae imminens hac formula integrali duplicata exprimitur $\int\int \frac{a\,\partial x\,\partial y}{\sqrt{(aa - xx - yy)}}$. Fig. 6.
Jam nulla substitutione adhibita, si primo x pro constante habeatur, prodibit

$$\int a\,\partial x\, \text{Arc. sin.}\, \frac{y}{\sqrt{(aa - xx)}},$$

qua portio sphaerae aream indefinitam CPRG tegens exprimitur; et quaestio nunc huc redit, ut ejusmodi aequatio algebraica inter x et y assignetur, unde pro tota area CHRG portio superficiei sphaericae ei respondentis fiat algebraice assignabilis.

§. 34. Ponamus brevitatis gratia $\frac{y}{\sqrt{(aa-xx)}} = v$, ut sit $y = v\sqrt{(aa - xx)}$, ac posito $x = 0$ fiat $v = n$: quoniam superius integrale evanescere debet posito $x = 0$. Erit ergo superficies sphaerica aream indefinitam C P R G tegens

$$= ax \text{ Arc. sin. } v - a\int \frac{x\partial v}{\sqrt{(1-vv)}},$$

sumto hoc integrali ita ut evanescat posito $x = 0$. Statuatur nunc

$$\int \frac{x\partial v}{\sqrt{(1-vv)}} = f \text{ Arc. sin. } v - a\text{V},$$

denotante V functionem quamcunque algebraicam ipsius v, quae abeat in N posito $x = 0$, eritque superficies nostra

$$= ax \text{ Arc. sin. } v - af \text{ Arc. sin. } v + aa\text{V} + fa \text{ Arc. sin. } n - aa\text{N},$$

atque x per v ita determinabitur, ut sit

$$x = f - \frac{a\partial\text{V}\sqrt{(1-vv)}}{\partial v},$$

sit jam C H $= b$, ac ponatur $x = h$, quo casu fiat $v = m$ et $\text{V} = \text{M}$, et cum superficies proposita sit

$$ah \text{ Arc. sin. } m - af \text{ Arc. sin. } m + aa\text{M} + af \text{ Arc. sin. } n - aa\text{N},$$

ea algebraica esse nequit nisi sit

$$h \text{ Arc. sin. } m - f \text{ Arc. sin. } m + f \text{ Arc. sin. } n = 0.$$

§. 35. Hic igitur primo arcus quorum sinus sunt m et n inter se commensurabiles reddi debent, nisi forte sit $n = 0$, quo casu sufficit fieri $h = f$. Quod etsi facile infinitis modis praestari potest, tamen hoc problema multo facilius adhibendis substitutionibus ante expositis resolvetur. Ponatur ergo

$$x = \frac{t}{\sqrt{(1+uu)}} \text{ et } y = \frac{tu}{\sqrt{(1+uu)}},$$

ut fiat $xx + yy = tt$, et pro $\partial x\partial y$ prodeat $\frac{t\partial t\partial u}{1+uu}$, atque superficies portionis sphaericae hac formula integrali duplicata exprimetur $\int\int \frac{at\partial t\partial u}{(1+uu)\sqrt{(aa-tt)}}$. Sumatur primo u constans

erit ea

$$= \int \frac{a \partial u}{1+uu} [b - \sqrt{(aa - tt)}],$$

quae jam facile absolute integrabilis reddi potest: ponatur enim aequalis functioni algebraicae cuicunque ipsius u, quae sit $= V$, eritque

$$b - \sqrt{(aa - tt)} = \frac{\partial V (1+uu)}{a \partial u},$$

et portio superficiei sphaericae adeo indefinita erit $= V$, ubi pro V functionem algebraicam quamcunque ipsius u accipere licet.

§. 36. Simplicissimae solutiones deducentur ex hac hypothesi $V = \frac{a(\alpha + \beta u)}{\sqrt{(1+uu)}}$, unde fit

$$\frac{\partial V}{a \partial u} = - \frac{\alpha u + \beta}{(1+uu)^{\frac{3}{2}}},$$

hincque

$$b - \sqrt{(aa - tt)} = \frac{\beta - \alpha u}{\sqrt{(1+uu)}}.$$

Ponatur $b = 0$, et cum per substitutiones sit

$$u = \frac{y}{x} \text{ et } t = \sqrt{(xx + yy)},$$

erit pro curva quaesita

$$\sqrt{(xx+yy)(aa - xx - yy)} = \alpha y - \beta x,$$

et pro superficie

$$V = \frac{a(\alpha x + \beta y)}{\sqrt{(xx+yy)}}.$$

Hinc casus simplicissimus oritur, ponendo $\beta = 0$ et $\alpha = a$, unde prodit $aaxx - (xx+yy)^2 = 0$, seu $yy = ax - xx$; ita ut curva G R H sit circulus diametro A C descriptus, et $V = \frac{aax}{\sqrt{(xx+yy)}}$. Infiniti alii circuli diametrum $= a$ habentes ac per centrum sphaerae transeuntes reperiuntur, si sit

$$\beta = \sqrt{(aa - \alpha\alpha)},$$

unde fit

$$a x + y \surd (a a - \alpha \alpha) = x x + y y \text{ et}$$

$$V = \frac{a [\alpha x + y \surd (a a - \alpha \alpha)]}{\surd (x x + y y)} = a \surd (x x + y y):$$

ubi notandum est, quantitatem V pro natura rei constantem quandam assumere.

§. 37. Concipiatur ergo octans sphaerae super quadrante A C B extractus, cujus radius C A $= a$, qui simul sit diameter
Fig. 9. semicirculi C R A, in quo si ducatur corda quaecunque C R, et perpendiculum R P, ut sit C P $= x$ et P R $= y$, erit C R $= t$, et u erit tangens anguli A C R. Quoniam igitur posuimus $b = 0$, prius integrale, quo u erat constans, est $\surd (a a - t t)$, quod cum evanescat si $t = a$, evidens est, id non per cordam C R $= t$ sed per ejus complementum R S extendi. Hinc repetita integratio $\int \frac{a \partial u}{1 + u u} \surd (a a - t t)$ eam sphaericae superficiei portionem exprimit, quae trilineo R V A S imminet, quae ergo ob

$$\surd (a a - t t) = \frac{a u}{\surd (1 + u u)}, \text{ est } = \frac{-a a}{\surd (1 + u u)} + a a,$$

integrali scilicet ita sumto, ut evanescat cum angulo A C R. Quare ob $\frac{1}{\surd (1 + u u)} = \cos.$ A C R, ducto perpendiculo S T, erit illa superficies

$$= a (a - \text{C T}) = \text{C A . A T} = \text{A V}^2,$$

ducta corda A V. Consequenter portio superficiei sphaerae spatio C E R A S B inter quadrantem et semicirculo intercepto imminens, aequatur quadrato radii sphaerae.

§. 38. Contemplemur autem adhuc ejusmodi casum, quo prima integratio evanescat posito $t = 0$, seu sit $b = a$, ac ponatur $V = \frac{1}{2} a a u$, quae expressio simul superficiem quae-

sitam praebet. Erit ergo

$$a - \sqrt{(aa - tt)} = \tfrac{1}{2} a (1 + uu) \text{ et}$$ Fig 10.

$$\sqrt{(aa - tt)} = \tfrac{1}{2} a (1 - uu),$$

ita ut sit

$$t = \tfrac{1}{2} a \sqrt{(3 + 2uu - u^4)} \text{ seu}$$

$$t = \tfrac{1}{2} a \sqrt{(1 + uu)(3 - uu)},$$

ubi est $CR = t$, et u denotat tangentem anguli ACR. Ex hac aequatione patet, si sit $u = 0$, fore $t = \frac{a\sqrt{3}}{2}$; scilicet curva quaesita radio AC ita in E occurrit, ut sit $CE = CA . \frac{\sqrt{3}}{2}$, eique perpendiculariter insistit. Tum si angulus ACR augeatur ad semirectum ACF, ut fiat $u = 1$, erit $t = a$; hocque casu curva per ipsum punctum F transit, ibique quadrantem osculabitur; ac simul distantia t fit maxima. Dehinc curva introrsum reflectitur, et t evanescit si $u = \sqrt{3}$: hoc est, curva centro C ita immergitur, ut ejus tangens in C cum radio CA faciat angulum 60°.

§. 39. Tota ergo curva in quadrante descripta figuram habebit $ERFGC$, et ducta in ea ex C recta utcunque CR, angulique ECR tangens sit $= u$, tum portio superficiei sphaericae sectori ECR imminens algebraice poterit assignari, eritque ea $= \tfrac{1}{2} aau$. Quare si CR ad occursum cum tangente AT producatur, ob $AT = au$, ea portio praecise aequabitur triangulo CAT: et portio imminens sectori ECF erit $= \tfrac{1}{2} aa$, si autem angulus ECR major semirecto sumatur, ut sit $u > 1$, quia tum

$$\sqrt{(aa - tt)} = \sqrt{(aa - xx - yy)},$$

quae est elevatio superficiei sphaericae supra quadrantem, fit negativa, superficies in inferiori octante capi debet. Quodsi hujus curvae aequationem inter coordinatas $CP = x$ et $PR = y$

desideremus, ob

$$tt = xx + yy \text{ et } u = \frac{y}{x},$$

habebimus

$$4xx + 4yy = aa\left(3 + \frac{2yy}{xx} - \frac{y^4}{x^4}\right) = \frac{aa(xx+yy)(3xx-yy)}{x^4},$$

quae divisa per $xx + yy$ praebet

$$4x^4 = 3aaxx - aayy, \text{ seu } yy = 3xx - \frac{4x^4}{aa}.$$

§. 40. Hanc solutionem reddere possumus generaliorem ponendo $V = abu$, fietque

$$a - \sqrt{(aa - tt)} = b(1 + uu), \text{ hinc}$$
$$\sqrt{(aa - tt)} = a - b - buu, \text{ ergo}$$
$$tt = 2ab - bb + 2(a - b)buu - bbu^4$$
$$= (1 + uu)(2ab - bb - bbuu).$$

Qua ad coordinatas orthogonales translata, divisio per $xx + yy$ iterum succedet, fietque

$$x^4 = (2ab - bb)xx - bbyy \text{ seu}$$
$$y = \frac{x}{b}\sqrt{(2ab - bb - xx)},$$

ac portio superficiei sphaericae sectori E C R hujus curvae imminens erit $= \frac{aby}{x} = b \,.\, AT$: quae expressio locum habet, quamdiu $uu < \frac{a-b}{b}$; hoc est donec anguli E C R tangens fiat $= \sqrt{\frac{a-b}{b}}$, ubi fit $t = a$. Tum vero angulo E C R ultra aucto, perpendiculares super curva erectae ad hemisphaerium inferius protendi debent, quo casu superficies eo magis augetur. Si ergo sit $b = a$, quia $\sqrt{(aa - tt)}$ ubique fit quantitas negativa, quantitas $b \,.\, AT$ portionem superficiei sphaericae ad inferius hemisphaerium continuatae exprimit.

§. 41. Sit adhuc $b = a$, ac ponatur

$$V = \frac{a^2 (\alpha + \beta u)}{\sqrt{(1+uu)}} - \alpha a^2$$

ut superficies assignanda evanescat posito $u = 0$, eritque

$$a - \sqrt{(aa - tt)} = \frac{a(\beta - \alpha u)}{\sqrt{(1+uu)}} \text{ et}$$

$$\sqrt{(aa - tt)} = a - \frac{a(\beta - \alpha u)}{\sqrt{(1+uu)}},$$

ubi notandum est, si haec expressio fiat negativa, ibi in hemisphaerium inferius descendi. Ex his autem prodit

$$\frac{t}{aa} = \frac{2(\beta - \alpha u)}{\sqrt{(1+uu)}} - \frac{(\beta - \alpha u)^2}{1+uu}.$$

Quare evanescente angulo E C R, cujus tangens $= u$, erit

$$\frac{tt}{aa} = 2\beta - \beta\beta, \text{ at si } u = \frac{\beta}{\alpha}, \text{ evanescit } t.$$

Pro altera parte axis C A fit u negativum, ac posito $u = -v$ habetur superficies negative expressa

$$V = \frac{aa(\alpha - \beta v)}{\sqrt{(1+vv)}} - \alpha a^2,$$

et curva hac definietur aequatione

$$\frac{tt}{aa} = \frac{2(\beta + \alpha v)}{\sqrt{(1+vv)}} - \frac{(\beta + \alpha v)^2}{1+vv},$$

unde posito v infinito prodit $\frac{tt}{aa} = 2\alpha - \alpha\alpha$; ubi recta C R fit in curvam normalis, quod etiam evenit, ubi

$$v = \frac{\alpha}{\beta} \text{ et } \frac{tt}{aa} = 2\sqrt{(\alpha\alpha + \beta\beta)} - \alpha\alpha - \beta\beta.$$

Quare ne fiat t imaginarium, oportet sit $\sqrt{(\alpha\alpha + \beta\beta)} < 2$.

§. 42. Consideremus casum quo

$$\alpha = -\frac{1}{\sqrt{2}} \text{ et } \beta = \frac{1}{\sqrt{2}},$$

ut sit superficies

$$V = aa\left(\frac{1}{\sqrt{2}} - \frac{1+u}{\sqrt{2(1+uu)}}\right) \text{ et}$$

$$\frac{tt}{aa} = \frac{2(1+u)}{\sqrt{2(1+uu)}} - \frac{(1+u)^2}{2(1+uu)},$$

ubi patet si $u = -1$ fore $t = 0$; tum vero ut sequitur

si $u = 0$, $u = 1$, $u = 7$, $u = \infty$,

erit $t = a\sqrt{\frac{2\sqrt{2}-1}{2}}$, $t = a$, $t = a\sqrt{\frac{24}{25}}$, $t = a\sqrt{\frac{2\sqrt{2}-1}{2}}$,

ubi notandum, casibus $u = 1$ et $u = \infty$ rectam CR fore in curvam normalem. In hoc ergo quadrante curva nostra fere cum quadrante confunditur, cum ubique sit proxime $s = a$: cui portio superficiei sphaericae imminens erit $= aa\sqrt{2}$, quae deficit a superficie totius octantis, quae est $\frac{\pi}{2}aa$, parte satis parva $aa(\frac{\pi}{2}-\sqrt{2}) = 0{,}15658\,aa$. Ad alteram axis CA partem haec curva in centrum incidit, ubi tangens cum CA faciet angulum semirectum.

§. 43. Verum solutio §. 35. data multo magis amplificari poteat, cum enim superficies sphaerae assignanda hac formula exprimatur $\int\frac{a\partial u}{1+uu}\int\frac{t\partial t}{\sqrt{(aa-tt)}}$, et in integratione $\int\frac{t\partial t}{\sqrt{(aa-tt)}}$ quantitas u ut constans consideretur, integrale ita exhiberi poterit $U - \sqrt{(aa-tt)}$, denotante U functionem quamcunque ipsius u; quae formula, quoniam evanescit si

$$\sqrt{(aa-tt)} = U \text{ et } t = \sqrt{(aa-UU)},$$

ab hoc termino quantitas t ulterius protendi est concipienda. Denotet jam V aliam quamcunque functionem ipsius u, quae abeat in C posito $u = 0$, ac ponatur superficies

$$\int\frac{a\partial u}{1+uu}[U - \sqrt{(aa-tt)} = aV - aC,$$

eritque hinc

$$U - \sqrt{(aa-tt)} = \frac{\partial V(1+uu)}{\partial u},$$

ideoque

$$\sqrt{(aa-tt)} = U - \frac{\partial V(1+uu)}{\partial u},$$

unde alter terminus ipsius t definitur.

§. 44. Hinc igitur solutio problematis Florentini ita generalissime adornabitur. Constituto quadrante circuli A C B, cui octans sphaerae insistat, radio C A existente $= a$, ductoque radio quocunque C S, vocetur anguli A C S tangens $= u$; tum primo curva E Q G ita construatur, ut sit $CQ = \sqrt{(aa - UU)}$, et perpendiculum ex Q ad sphaericam usque superficiem erectum $QM = U$, denotante U functionem quamcunque algebraicam ipsius u. Si $u = 0$ abeat C Q in C E, et Q M in E I. Deinde alia describatur curva F R H, ut sit Fig. 11.

$$CR = \sqrt{[aa - (U - \frac{\partial V(1 + uu)}{\partial u})^2]},$$

et perpendiculum ex R ad sphaeram usque pertingens

$$RN = U - \frac{\partial V(1 + uu)}{\partial u},$$

denotante V aliam, quamcunque functionem algebraicam ipsius u, quae abeat in C si $u = 0$; quo casu simul C E in C F et R N in F K abeat. Jam his duabus curvis constructis, portio superficiei sphaericae areae E Q R F imminens et intra terminos I, K, M, N contenta, algebraice exprimetur, eritque ea $= a(V - C)$.

§. 45. Haec de natura formularum integralium duplicatarum commentandi occasionem praebuit problema aeque elegans atque utile in analysi, si quidem ejus solutionem evolvere liceret. Quaerebatur scilicet inter omnia corpora ejusdem soliditatis id, quod minima superficie contineretur: quod quidem ad ternas coordinatas orthogonales x, y et z relatum, posito $\partial z = p\,\partial x + q\,\partial y$ ita analytice exprimitur, ut inter omnes relationes harum trium variabilium, quae eandem quantitatem hujus formulae integralis duplicatae $\int\int z\,\partial x\,\partial y$ contineant, ea definiatur, cui minima quantitas hujus $\int\int \partial x\,\partial y \sqrt{(1 + pp + qq)}$ respondeat. Quod problema si per theoriam variationum aggre-

diamur, effici oportebit ut fiat

$$a\delta\int\int\partial x\partial y\sqrt{(1+pp+qq)}=\delta\int\int z\partial x\partial y,$$

ita ut totum negotium ad variationes hujusmodi formularum integralium duplicatarum indagandas reducatur.

§. 46. Quoniam utraque formula duplicem integrationem exigit, si in priori x pro constante habeatur, nostra aequatio ita repraesentabitur

$$a\delta\int\partial x\int\partial y\sqrt{(1+pp+qq)}=\delta\int\partial x\int z\partial y.$$

Verum hic probe animadvertendum est, postquam integralia

$$\int\partial y\sqrt{(1+pp+qq)} \text{ et } \int z\partial y$$

fuerint inventa, tum variabilem y non amplius indefinitam seu ab x non pendentem relinqui, quin potius pro y certam functionem ipsius x, quam figura corporis exigit, substitui oportere, ita ut in secunda integratione quantitas y non ut constans seu ab x non pendens spectari queat. Quia autem ob figuram corporis etiamnunc incognitam ista functio non constat, neutiquam apparet, quomodo variationes istiusmodi formularum duplicatarum determinari debeant.

§. 47. Ipsa vero hujus quaestionis natura alias praeterea determinationes requirere videtur, quarum ratio in solu-
Fig 12. tione haberi debeat. Nam quemadmodum si curva quaeritur, quae inter omnes alias eandem aream includentes brevissimo arcu contineatur, non solum basis A P sed etiam duo puncta B et M, per quae curva transeat, praescribi solent, ita etiam in nostro problemate non modo basis, cui corpus tanquam columna insistat pro cognita assumi debere videtur, sed etiam ipsi extremi termini superficiei quaesitae. Quodsi enim hae res non praescribantur omnes, ne quaestioni quidem certae locus relinquitur: nam etiamsi

basis praescriberetur, termini vero supremi superficiei arbitrio nostro relinquerentur, manifestum est, quo altior fuerit columna, eo magis soliditatem auctum iri eadem manente superficie suprema; quandoquidem superficies laterum non in computum ducitur. Multo minus autem problema sine basis praescriptione ullam vim retineret, quoniam basi coarctanda quantumvis magna soliditas cum minima superficie posset esse conjuncta.

SUPPLEMENTUM VII.

AD TOM. I. SECT. II. CAP. V.

DE

COMPARATIONE QUANTITATUM TRANSCENDENTIUM IN FORMA $\int \frac{p\,\partial x}{\sqrt{(A + 2Bx + Cxx)}}$ CONTENTARUM.

Plenior explicatio circa comparationem quantitatum in formula integrali $\int \frac{Z\,\partial z}{\sqrt{(1 + mzz + nz^4)}}$ contentarum, denotante Z functionem quamcunque rationalem ipsius zz. *Acta Academiae Scientiar. Petropolitanae. Tom. V. Pars II. Pag.* 3 — 22.

§. 1. Etsi hoc argumentum jam saepius tractavi atque Illustrissimus *La Grange* plures egregias observationes super hujusmodi formulis cum publico communicavit: id tamen neutiquam adhuc satis exploratum, multo minus exhaustum est censendum, sed plurima adhuc maxime abscondita involvere videtur, quae profundissimam indagationem requirunt atque insignia incrementa Analyseos pollicentur. Imprimis autem ipsae operationes analyticae, quae me primum ad hanc investigationem perduxerunt, ita sunt comparatae, ut non nisi per plures ambages totum negotium conficiant, unde merito etiamnunc methodus directa ad easdem comparationes perducens maxime est desideranda. Praeterea vero universa haec investigatio multo latius patet, quam eas formulas in-

tegrales, quas primo sum contemplatus, ubi pro littera Z tantum vel quantitatem constantem vel functionem integram ipsius zz hujus formae $F + Gzz + Hz^4 + Iz^6 + Kz^8$ etc. assumsi, quibus casibus ostendi, propositis duabus quibuscunque quantitatibus hujus generis, semper tertiam ejusdem generis inveniri posse, quae a summa illarum discrepet quantitate algebraica, quae quidem evanescat casu quo Z est tantum quantitas constans.

§. 2. Nunc autem observavi, easdem comparationes institui posse, si pro Z accipiatur functio quaecunque rationalis ipsius zz, quae scilicet habeat hujusmodi formam

$$\frac{F + Gzz + Hz^4 + Iz^6 + Kz^8 + etc.}{\mathfrak{F} + \mathfrak{G}zz + \mathfrak{H}z^4 + \mathfrak{I}z^6 + \mathfrak{K}z^8 + etc.},$$

ubi quidem hoc discrimen occurrit, quod differentia inter summam duarum hujusmodi formularum et tertiam formulam ejusdem generis inveniendam non amplius sit quantitas algebraica, verumtamen per logarithmos et arcus circulares semper exhiberi possit; ita ut nunc ista investigatio multo latius pateat, quam eam adhuc eram complexus. Atque hinc fortasse, si omnes operationes, quae ad hunc scopum manuducunt, debita attentione perpendantur, faciliorem viam apperire poterunt ad methodum directam perveniendi, totumque hoc argumentum maxime abstrusum feliciori successu perscrutandi.

§. 3. Quo autem haec omnia clarius perspici queant, denotet iste character $\Pi : z$ eam quantitatem transcendentem, quae ex integratione formulae propositae $\int \frac{Z\partial z}{\sqrt{(1 + mzz + nz^4)}}$ nascitur, dum integrale ita capi assumitur, ut evanescat posito $z = 0$; unde statim manifestum est, fore quoque $\Pi : 0 = 0$. Deinde cum Z involvat tantum pares potestates ipsius z, cujusmodi etiam in formula radicali insunt, evidens est, si loco z scribatur $-z$,

tum valorem quoque istius formulae integralis ideoque etiam characteris $\Pi : z$ in sui negativum abire, ita ut sit $\Pi : (-z) = -\Pi : z$. His praenotatis si proponantur duae quaecunque hujusmodi quantitates $\Pi : p$ et $\Pi : q$, semper invenire licet tertiam quantitatem ejusdem generis $\Pi : r$, quae a summa illarum formularum $\Pi : p + \Pi : q$ differat quantitate vel algebraica vel saltem per logarithmos et arcus circulares assignabili. Regula vero, qua ex datis litteris p et q tertia r elicitur, semper manet eadem, quaecunque functio per litteram Z designetur: semper enim erit

$$r = \frac{p\sqrt{(1 + mqq + nq^4)} + q\sqrt{(1 + mpp + np^4)}}{1 - nppqq}$$

Hinc autem pro sequentibus combinationibus observasse juvabit, fore

$$\sqrt{(1 + mrr + nr^4)} =$$

$$\frac{(mpq + [\sqrt{(1 + mpp + np^4)}] [\sqrt{(1 + mqq + nq^4)}] (1 + nppqq) + 2npq(pp + qq)}{(1 - nppqq)^2}.$$

§. 4. Non solum autem haec iuvestigatio adstringitur ad hujusmodi formulas $\Pi : p$ et $\Pi : q$ pro arbitrio accipiendas, sed adeo ad quotcunque formulas datas potest extendi, ita ut, quotcunque hujusmodi formulae fuerint propositae, scilicet

$$\Pi : f + \Pi : g + \Pi : h + \Pi : i + \Pi : k + \text{etc.}$$

semper nova hujusmodi formula $\Pi : r$ assignari possit, quae ab illarum summa discrepet quantitate vel algebraica vel saltem per logarithmos et arcus circulares assignabili. Quin etiam formulas illas, quas tanquam datas spectavimus, ita definire licebit, ut discrimen illud, sive algebraicum sive a logarithmis arcubusque circularibus pendens, prorsus evanescat, ita ut futurum sit

$$\Pi : r = \Pi : f + \Pi : g + \Pi : h + \Pi : i + \Pi : k + \text{etc.}$$

Atque haec fere sunt, ad quae hanc investigationem generalio-

rem, quam hic exponere constitui, mihi quidem extendere licuit; quamobrem singulas operationes, quae me huc perduxerunt, succincte sum propositurus.

Operatio I.

§. 5. Universam hanc investigationem inchoavi a consideratione hujus aequationis algebraicae

$$\alpha+\gamma(xx+yy)+2\delta xy+\zeta xxyy=0,$$

ex qua, cum sit quadratica, tam pro x quam pro y radicem extrahendo, colligitur vel

$$y=\frac{-\delta x+\sqrt{[-\alpha\gamma+(\delta\delta-\gamma\gamma-\alpha\zeta)xx-\gamma\zeta x^4]}}{\gamma+\zeta xx},\ \text{vel}$$

$$x=\frac{-\delta y+\sqrt{[-\alpha\gamma+(\delta\delta-\gamma\gamma-\alpha\zeta)yy-\gamma\zeta y^4]}}{\gamma+\zeta yy},$$

ita ut hinc fiat

$$\sqrt{[-\alpha\gamma+(\delta\delta-\gamma\gamma-\alpha\zeta)xx-\gamma\zeta x^4]}=\gamma y+\delta x+\zeta xxy,\ \text{et}$$

$$\sqrt{[-\alpha\gamma+(\delta\delta-\gamma\gamma-\alpha\zeta)yy-\gamma\zeta y^4]}=\gamma x+\delta y+\zeta xyy.$$

§. 6. Nunc litteras α, γ, δ, ζ, ita definio, ut ambae formulae radicales ad formam

$$\sqrt{(1+mxx+nx^4)}\ \text{et}\ \sqrt{(1+myy+ny^4)}$$

reducantur, quem in finem facio

$1^\circ.\ -\alpha\gamma=k,$

$2^\circ.\ \delta\delta-\gamma\gamma-\alpha\zeta=mk,$ et

$3^\circ.\ -\gamma\zeta=nk,$

ex quarum aequalitatum prima fit $\alpha=\frac{-k}{\gamma}$, ex tertia $\zeta=\frac{-nk}{\gamma}$, qui valores in secunda substituti praebent

$$\delta\delta=\gamma\gamma+\frac{nkk}{\gamma\gamma}+mk,$$

ideoque

$$\delta=\sqrt{\left(\gamma\gamma+\frac{nkk}{\gamma\gamma}+mk\right)}=\frac{1}{\gamma\gamma}\sqrt{(\gamma^4+m\gamma\gamma k+nkk)},$$

unde aequatio nostra nunc erit

$$-k+\gamma\gamma(xx+yy)+2xy\surd(\gamma^4+m\gamma\gamma k+nkk)-nkxxyy=0:$$

hinc igitur ambae nostrae formulae irrationales erunt

$$\surd k(1+mxx+nx^4)=\gamma y+\tfrac{1}{\gamma}x\surd(\gamma^4+m\gamma\gamma k+nkk)-\tfrac{nk}{\gamma}xxy,$$

$$\surd k(1+myy+ny^4)=\gamma x+\tfrac{1}{\gamma}y\surd(\gamma^4+m\gamma\gamma k+nkk)-\tfrac{nk}{\gamma}xyy.$$

§. 7. Cum nunc ambae quantitates x et y ita a se invicem pendeant, quemadmodum aequatio assumta declarat, litteras adhuc indefinitas γ et k ita definiamus, ut posito $x=0$ fiat $y=a$. Oportebit igitur esse $-k+\gamma\gamma aa=0$, ideoque $k=\gamma\gamma aa$, quo valore substituto aequatio nostra erit

$$0=\gamma\gamma(xx+yy-aa)+2\gamma\gamma xy\surd(1+maa+na^4)$$
$$-n\gamma\gamma aaxxyy,$$

hincque fiet per $\gamma\gamma$ dividendo

$$0=(xx+yy-aa)+2xy\surd(1+maa+na^4)-naaxxyy.$$

Tum vero ambae nostrae formulae radicales ita exprimentur

$$\surd(1+mxx+nx^4)=\tfrac{y}{a}+\tfrac{x}{a}\surd(1+maa+na^4)-naxxy,$$

$$\surd(1+myy+ny^4)=\tfrac{x}{a}+\tfrac{y}{a}\surd(1+maa+na^4)-naxyy.$$

§. 8. Quo has formulas tractatu faciliores reddamus, ponamus $\surd(1+maa-na^4)=\mathfrak{A}$, similique modo

$$\surd(1+mxx+nx^4)=\mathfrak{X} \text{ et}$$
$$\surd(1+myy+ny^4)=\mathfrak{Y},$$

et aequatio nostra erit

$$xx+yy-aa+2\mathfrak{A}xy-naaxxyy=0;$$

unde reperitur

$$y=-\frac{\mathfrak{A}x+a\mathfrak{X}}{1-naaxx}, \text{ et } x=-\frac{\mathfrak{A}y+a\mathfrak{Y}}{1-naayy},$$

unde patet si fuerit $y = 0$ fore $x = a$; tum vero erunt formulae radicales

$$\sqrt{(1 + mxx + nx^4)} = \mathfrak{X} = \frac{y}{a} + \frac{\mathfrak{A}x}{a} - naxxy,$$
$$\sqrt{(1 + myy + ny^4)} = \mathfrak{Y} = \frac{x}{a} + \frac{\mathfrak{A}y}{a} - naxyy.$$

§. 9. Quemadmodum autem tam y per x quam x per y exprimere licuit, ita etiam $\mathfrak{Y}$ per solum x et $\mathfrak{X}$ per solum y exprimere licebit. Calculo autem instituto reperietur fore

$$\mathfrak{X} = \frac{(-may + \mathfrak{A}\mathfrak{Y})(1 + naayy) - 2nay(aa + yy)}{(1 - naayy)^2},$$
$$\mathfrak{Y} = \frac{(-max + \mathfrak{A}\mathfrak{X})(1 + naaxx) - 2nax(aa + xx)}{(1 - naaxx)^2}.$$

§. 10. Praecipue autem circa nostram aequationem

$$xx + yy - aa + 2\mathfrak{A}xy - naaxxyy = 0$$

notari meretur, quod ternae quantitates xx, yy, aa perfecte inter se sint permutabiles. Si enim membrum irrationale ad alteram partem transferatur, ut sit

$$xx + yy - aa - naaxxyy = -2\mathfrak{A}xy,$$

et quadrata sumantur, restituendo pro $\mathfrak{A}^2$ valorem suum $1 + maa + na^4$, prodibit ista aequatio

$$\left.\begin{array}{llll} +x^4 - 2xxyy - 4maaxxyy & -2na^4xxyy + nna^4x^4y^4 \\ +y^4 - 2aaxx & -2naax^4yy \\ +a^4 - 2aayy & -2naaxxy^4 \end{array}\right\} = 0,$$

ubi permutabilitas litterarum a, x, y manifesto in oculos incurrit. In ipsis quidem formulis superioribus, ubi ipsa quantitas a ingreditur, permutabilitas non adeo est manifesta, sed prorsus elucebit, si loco a scribamus $-b$, itemque $\mathfrak{B}$ loco $\mathfrak{A}$; tum enim, quemadmodum erat

$$y = -\frac{x\mathfrak{B} - b\mathfrak{X}}{1 - nbbxx} \text{ et } x = -\frac{y\mathfrak{B} - b\mathfrak{Y}}{1 - nbbyy},$$

ita erit $b = -\frac{x\mathfrak{Y} - y\mathfrak{X}}{1 - nxxyy}$; similique modo pro formulis radicalibus seu litteris majusculis erit

$$\mathfrak{Y} = \frac{(mbx + \mathfrak{B}\mathfrak{X})(1 + nbbxx) + 2nbx(bb + xx)}{(1 - nbbxx)^2},$$
$$\mathfrak{X} = \frac{(mby + \mathfrak{B}\mathfrak{Y})(1 + nbbyy) + 2nby(bb + yy)}{(1 - nbbyy)^2},$$
$$\mathfrak{B} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2},$$

sicque perfecta permutabilitas perspicitur.

Operatio II.

§. 11. Differentiemus nunc nostram aequationem algebraicam assumtam, quae est

$$xx + yy - aa + 2\mathfrak{A}xy - naaxxyy = 0,$$

et aequatio differentialis erit

$$\partial x(x + \mathfrak{A}y - naaxyy) + \partial y(y + \mathfrak{A}x - naaxxy) = 0,$$

sive

$$\frac{\partial x}{y + \mathfrak{A}x - naaxxy} + \frac{\partial y}{x + \mathfrak{A}y - naaxyy} = 0.$$

Ex superioribus autem constat esse

$$y + \mathfrak{A}x - naaxxy = a\mathfrak{X} \text{ et}$$
$$x + \mathfrak{A}y - naaxyy = a\mathfrak{Y},$$

unde aequatio differentialis hanc induet formam

$$\frac{\partial x}{a\mathfrak{X}} + \frac{\partial y}{a\mathfrak{Y}} = 0, \text{ sive}$$
$$\frac{\partial x}{\sqrt{(1 + mxx + nx^4)}} + \frac{\partial y}{\sqrt{(1 + myy + ny^4)}} = 0.$$

§. 12. Inventa igitur hac aequatione differentiali, denotet iste character $\Gamma : x$ integrale $\int \frac{\partial x}{\mathfrak{X}}$, et character $\Gamma : y$ integrale $\int \frac{\partial y}{\mathfrak{Y}}$, utroque integrali ita sumto, ut evanescat posito vel $x = 0$ vel $y = 0$, atque aequationem illam differentialem integrando fiet $\Gamma : x + \Gamma : y = C$. Cum autem sumto $x = 0$

fiat etiam $\Gamma : x = 0$ et $y = a$, erit constans illa $C = \Gamma : a$, ita ut habeamus hanc aequationem $\Gamma : x + \Gamma : y = \Gamma : a$.

§. 13. Quoniam hic nulla amplius variabilitatis ratio tenetur, patet, sumtis binis litteris x et y pro lubitu, litteram a ita semper definiri posse, ut fiat

$$\Gamma : a = \Gamma : x + \Gamma : y.$$

Si enim in §. 10. loco b scribatur $-a$, sumi debet

$$a = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy},$$

quae comparatio jam casum constituit specialem investigationis generalis, quam suscepimus. Si enim loco x et y scribamus p et q, at r loco a, tum vero $\mathfrak{P}$, $\mathfrak{Q}$ et $\mathfrak{R}$ loco $\mathfrak{X}$, $\mathfrak{Y}$ et $\mathfrak{A}$, atque si, sumtis pro lubitu quantitatibus p, q, capiatur $r = \frac{p\mathfrak{Q} + q\mathfrak{P}}{1 - nppqq}$, tum utique erit $\Gamma : r = \Gamma : p + \Gamma : q$, ita ut hoc casu discrimen illud inter $\Gamma : r$ et summam $\Gamma : p + \Gamma : q$ plane evanescat. Sicque jam evolvimus casum, quo in nostra forma generali $\int \frac{Z \partial z}{\sqrt{(1 + mzz + nz^4)}}$ pro Z sumitur quantitas constans.

Operatio III.

§. 14. Quo nunc propius ad nostrum institutum accedamus, sint X et Y tales functiones ipsarum x et y, qualem volumus esse Z ipsius z, et quoniam modo invenimus

$$\frac{\partial x}{\sqrt{(1 + mxx + nx^4)}} + \frac{\partial y}{\sqrt{(1 + myy + ny^4)}} = 0,$$

ponamus esse

$$\frac{X \partial x}{\sqrt{(1 + mxx + nx^4)}} + \frac{Y \partial y}{\sqrt{(1 + myy + ny^4)}} = \partial V,$$

ita ut, si X et Y essent quantitates constantes, foret $\partial V = 0$. Hinc ergo si loco $\frac{\partial y}{\sqrt{(1 + myy + ny^4)}}$ scribamus $\frac{-\partial x}{\sqrt{(1 + mxx + nx^4)}}$, fiet

$$\partial V = \frac{(X-Y)\partial x}{\sqrt{(1+mxx+nx^4)}},$$

vel etiam

$$\partial V = \frac{(Y-X)\partial y}{\sqrt{(1+myy+ny^4)}}.$$

At si loco radicalium suos valores rationales scribamus, erit

$$\partial V = \frac{a(X-Y)\partial x}{y+\mathfrak{A}x-naaxxy}, \text{ vel}$$

$$\partial V = \frac{a(Y-X)\partial y}{x+\mathfrak{A}y-naaxyy}.$$

§. 15. Cum autem nulla sit ratio, cur istud differentiale ∂V potius per ∂x quam per ∂y exprimamus; consultum erit novam quantitatem in calculum introducere, quae aeque referatur ad x et ad y. Hunc in finem faciamus productum $xy = u$, ac statuamus

$$\frac{\partial x}{x+\mathfrak{A}x-naaxxy} = -\frac{\partial y}{x+\mathfrak{A}y-naaxyy} = s\,\partial u.$$

Hinc igitur erit

$$\partial x = s\,\partial u\,(y+\mathfrak{A}x-naaxxy) \text{ et}$$

$$\partial y = -s\,\partial u\,(x+\mathfrak{A}y-naaxyy),$$

ex quibus colligimus

$$y\,\partial x + x\,\partial y = s\,\partial u\,(yy-xx) = \partial u,$$

sicque habebimus $s = \frac{1}{yy-xx}$, ita ut habeamus

$$\frac{\partial x}{y+\mathfrak{A}x-naaxxy} = -\frac{\partial y}{x+\mathfrak{A}y-naaxyy} = \frac{\partial u}{yy-xx}.$$

Hoc igitur valore substituto nanciscimur

$$\partial V = \frac{a(X-Y)\partial u}{yy-xx} = \frac{-a\,\partial u\,(X-Y)}{xx-yy}.$$

§. 16. Cum autem X et Y sint functiones rationales pares ipsarum x et y, in quibus tantum insunt potestates pares harum litterarum; facile intelligitur, formulam $X-Y$ semper esse divisibilem per $xx-yy$, et quotum praeter pro-

ductum $xy = u$ insuper involvere summam quadratorum $xx + yy$; quamobrem statuamus $xx + yy = t$, et cum aequatio nostra fundamentalis fiat

$$t - aa + 2\mathfrak{A}u - naauu = 0,$$

ex ea fit

$$t = aa - 2\mathfrak{A}u + naauu,$$

ita ut t aequetur functioni rationali ipsius u. Quod si ergo hic valor ubique loco t scribatur, differentiale nostrum quaesitum ∂V per solam variabilem u exprimetur, ita ut posito $\partial V = U\partial u$ semper sit U functio rationalis ipsius u, quae ergo si fuerit integra, tum V aequabitur functioni algebraicae ipsius u; sin autem sit functio fracta, tum integrale $V = \int U\partial u$ semper per logarithmos et arcus circulares exhiberi poterit. Hoc ergo integrale si ita capiatur, ut evanescat posito $u = xy = 0$, id etiam evanescet posito $x = 0$ vel $y = 0$. Atque hinc integrando impetrabimus

$$\int \frac{X\partial x}{\sqrt{(1+mxx+nx^4)}} + \int \frac{Y\partial y}{\sqrt{(1+myy+ny^4)}} = C + V = C + \int U\partial u.$$

§. 17. Quod si igitur characteres $\Pi : x$ et $\Pi : y$ denotent valores horum integralium, ita ut utrumque evanescat sumto vel $x = 0$ vel $y = 0$, quoniam facto $x = 0$ per hypothesin fit $y = a$, manifestum est constantem hanc fore $\Pi : a$, sicque aequatio finita resultabit ista

$$\Pi : x + \Pi : y = \Pi : a + \int U\partial u.$$

§. 18. Accuratius autem in valores hujus fractionis U pro quovis casu inquiramus. Ac primo quidem, si sumatur

$$Z = \alpha + \beta zz + \gamma z^4 + \delta z^6 + \text{etc.}$$

erit simili modo

$$X = \alpha + \beta xx + \gamma x^4 + \delta x^6 + \text{etc. et}$$
$$Y = \alpha + \beta yy + \gamma y^4 + \delta y^6 + \text{etc.},$$

quare cum invenerimus

$$\partial V = U\,\partial u = -\frac{a\,\partial u(X-Y)}{xx-yy}, \text{ erit}$$

$$U = -\frac{a(X-Y)}{xx-yy}, \text{ ideoque}$$

$$U = -\frac{a[\beta(xx-yy)+\gamma(x^4-y^4)+\delta(x^6-y^6)+\text{etc.}]}{xx-yy},$$

unde fit

$$U = -a\beta - a\gamma(xx+yy) - a\delta(x^4+xxyy+y^4) - \text{etc.}$$

Cum igitur sit $xx+yy=t$ et $xy=u$, erit

$$U = -a\beta - a\gamma t - a\delta(tt-uu) - \text{etc.},$$

unde cum sit

$$t = aa - 2\mathfrak{A}u + naauu,$$

calculo subducto altiores potestates omittendo, fiet

$$\begin{aligned}\int U\,\partial u = -a\beta u - a\gamma\left(aau - \mathfrak{A}uu + \tfrac{1}{3}naau^3\right)\\ -a\delta\left(a^4uu - 2aa\mathfrak{A}uu + \tfrac{2}{3}na^4u^3 - n\mathfrak{A}a^2u^4 + \tfrac{1}{5}n^2a^4u^5\right)\\ +\tfrac{4}{3}\mathfrak{A}^2u^5\\ +\tfrac{1}{3}u^3.\end{aligned}$$

Atque hinc intelligitur, si functio Z ad altiores potestates exsurgat, quomodo valor integralis ipsius $\int U\,\partial u$ inde inveniri queat.

§. 19. Sin autem Z fuerit functio fracta, scilicet

$$Z = \frac{\alpha+\beta zz+\gamma z^4}{\zeta+\eta zz+\theta z^4},$$

hincque

$$X = \frac{\alpha+\beta xx+\gamma x^4}{\zeta+\eta xx+\theta x^4} \text{ et}$$

$$Y = \frac{\alpha+\beta yy+\gamma y^4}{\zeta+\eta yy+\theta y^4}, \text{ erit}$$

$$X - Y = \frac{(\beta\zeta-\alpha\eta)(xx-yy)+(\gamma\zeta-\alpha\theta)(x^4-y^4)+(\gamma\eta-\beta\theta)x^2y^2(x^2-y^2)}{\zeta\zeta+\zeta\eta(xx+yy)+\zeta\theta(x^4+y^4)+\eta^2x^2y^2+\eta\theta x^2y^2(xx+yy)+\theta\theta x^4y^4}.$$

Hinc igitur introductis litteris t et u erit

$$\frac{X-Y}{xx-yy} = \frac{\beta\zeta-\alpha\eta+(\gamma\zeta-\alpha\theta)t+(\gamma\eta-\beta\theta)uu}{\zeta\zeta+\zeta\eta t+\zeta\theta(tt-2uu)+\eta\eta uu+\eta\theta tuu+\theta\theta u^4},$$

quam ob rem cum sit

$$U = -\frac{a(X-Y)}{xx-yy}, \text{ ob}$$

$$t = aa - 2\mathfrak{A}u + naauu,$$

manifestum est, integrale formulae $\int U \partial u$ nisi fuerit algebraicum, semper, concessis logarithmis et arcubus circularibus, exhiberi posse. Sicque per has tres operationes omnia praestitimus, quibus opus est ad omnia problemata huc spectantia solvenda

Problema I.

§. 20. *Si* $\Pi : x$ *et* $\Pi : y$ *denotent valores formularum integralium*

$$\int \frac{X\partial x}{\sqrt{(1+mxx+nx^4)}} \text{ et } \int \frac{Y\partial y}{\sqrt{(1+myy+ny^4)}},$$

ubi X *et* Y *sint functiones pares similes ipsarum* x *et* y, *atque dentur binae hujusmodi formulae* $\Pi : x$ *et* $\Pi : y$; *invenire tertiam formulam ejusdem generis* $\Pi : z$, *ut sit*

$$\Pi : z = \Pi : x + \Pi : y + W,$$

ita ut W *sit functio vel algebraica vel per logarithmos et arcus circulares assignabilis.*

Solutio.

Cum dentur binae quantitates x et y, ex iis formentur radicales

$$\mathfrak{X} = \sqrt{(1 + mxx + nx^4)} \text{ et}$$

$$\mathfrak{Y} = \sqrt{(1 + nyy + ny^4)},$$

ex quibus definiatur quantitas z, eodem modo quo supra litteram a per x et y definire docuimus, ita ut sit $z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy}$, ejusque valor irrationalis

$$\mathfrak{Z} = \sqrt{(1 + mzz + nz^4)} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(xx+yy)}{(1-nxxyy)^2},$$

tum in superioribus formulis ubique loco a et $\mathfrak{A}$ scribamus z et $\mathfrak{Z}$, et capiatur $= -\frac{z(X-Y)}{xx-yy}$, quam quantitatem vidimus semper reduci posse ad functionem ipsius u, existente $u = xy$, ac ponatur $V = \int U \partial u$, in qua integratione quantitates z et $\mathfrak{Z}$ pro constantibus sunt habendae, ita ut littera V spectari possit tanquam functio ipsius $u = xy$, quandoquidem etiam z et $\mathfrak{Z}$ per x et y determinantur. Probe autem teneatur, in ista formula integrali solam quantitatem u ut variabilem esse tractandam. Hac igitur quantitate V inventa erit

$$\Pi : x + \Pi : y = \Pi : z + V,$$

unde cum debeat esse

$$\Pi : z = \Pi : x + \Pi : y + W,$$

patet esse $W = -V$, ideoque quantitatem vel algebraicam, vel per logarithmos et arcus circulares assignabilem.

Corollarium 1.

§. 21. Totum ergo negotium hic redit ad integrationem formulae $U \partial u$, existente

$$u = xy \text{ et } U = -\frac{z(X-Y)}{xx-yy},$$

quam supra vidimus semper per u exprimi posse, siquidem in hac integratione litterae z et $\mathfrak{Z}$ ut quantitates constantes tractentur.

Corollarium 2.

§. 22. Cum igitur pro data indole binarum functionum X et Y haec integratio nulla laboret difficultate, ipsumque integrale per u, hoc est per xy exprimatur, cujus valorem ex datis quantitatibus x et y semper exhibere liceat, loco quantitatis V scribemus in posterum characterem $\Phi : xy$, unde pro quibusque aliis litteris loco x et y assumtis intelligitur significatus characterum $\Phi : bq$, $\Phi : ab$ etc.

Corollarium 3.

§. 23. Hoc igitur charactere recepto, si pro datis quantitatibus x et y capiamus $z = \frac{x\mathfrak{Y} - y\mathfrak{X}}{1 - nxxyy}$, unde fit

$$\mathfrak{Z} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2}, \text{ erit}$$

$$\Pi : z = \Pi : x + \Pi : y - \Phi : xy.$$

Problema 2.

§. 24. *Servatis omnibus characteribus, quos hactenus explicavimus, si dentur ternae formulae,* $\Pi : p, \Pi : q, \Pi : r$, *invenire quartam ejusdem generis* $\Pi : z$, *ut fiat*

$$\Pi : z = \Pi p + \Pi : q + \Pi : r + W,$$

ita ut W *sit quantitas algebraica, vel per logarithmos arcusve circulares assignabilis.*

Solutio.

Ex datis binis quantitatibus p et q, ideoque etiam $\mathfrak{P}$ et $\mathfrak{Q}$ inde oriundis, capiatur $x = \frac{p\mathfrak{Q} + q\mathfrak{P}}{1 - nppqq}$, simulque

$$\mathfrak{X} = \frac{(mpq + \mathfrak{P}\mathfrak{Q})(1 + nppqq) + 2npq(pp + qq)}{(1 - nppqq)^2}.$$

Tum vero etiam colligatur valor characteris $\Phi : pq$, eritque per praecedentia

$$\Pi : x = \Pi : p + \Pi : q - \Phi : pq, \text{ sive}$$

$$\Pi : p + \Pi : q = \Pi : x + \Phi : pq,$$

quo valore substituto erit

$$\Pi : z = \Pi : x + \Pi : r + \Phi : pq + W.$$

Ex praecedente autem problemate, si loco y hic scribamus r et capiamus $z = \frac{x\mathfrak{R} + r\mathfrak{X}}{1 - nrrxx}$, unde fit

$$\mathfrak{Z} = \frac{(mrx + \mathfrak{R}\mathfrak{X})(1 + nrrxx) + 2nrx(rr + xx)}{(1 - nrrxx)^2}, \text{ erit}$$

$$\Pi : z = \Pi : x + \Pi : r - \Phi : rx,$$

qua forma cum praecedente collata colligitur

$$W = -\Phi : pq - \Phi : rx,$$

ita ut sit

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r - \Phi : pq - \Phi : rx.$$

Problema 3.

§. 25. *Propositis hujusmodi formulis* $\Pi : p$, $\Pi : q$, $\Pi : r$, $\Pi : s$, *invenire quintam ejusdem generis* $\Pi : z$, *ut fiat*

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r + \Pi : s + W,$$

ita ut W sit quantitas algebraica, vel per logarithmos arcusve circulares assignabilis.

Solutio.

Ex datis binis p et q quaeratur x, ut sit $x = \frac{p\mathfrak{Q} + q\mathfrak{P}}{1 - nppqq}$, item

$$\mathfrak{X} = \frac{(mpq + \mathfrak{P}\mathfrak{Q})(1 + nppqq) + 2npq(pp + qq)}{(1 - nppqq)^2},$$

eritque

$$\Pi : x = \Pi : p + \Pi : q - \Phi : pq.$$

Simili modo ex binis datis r et s quaeratur y, ut sit $y = \frac{r\mathfrak{S} + s\mathfrak{R}}{1 - nrrss}$, eritque

$$\mathfrak{Y} = \frac{(mrs + \mathfrak{R}\mathfrak{S})(1 + nrrss) + 2nrs(rr + ss)}{(1 - nrrss)^2},$$

tum vero

$$\Pi : y = \Pi : r + \Pi : s - \Phi : rs.$$

Nunc denique ex inventis x et y sumatur $z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy}$, et

$$\mathfrak{Z} = \frac{(mxy + \mathfrak{Y}\mathfrak{X})(1 + nxxyy) + 2nxy(xx + yy)}{(1 - nxxyy)^2},$$

eritque

$$\Pi : z = \Pi : x + \Pi : y - \Phi : xy.$$

Quod si ergo loco $\Pi : x$ et $\Pi : y$ valores modo inventi substituantur, fiet

$$\Pi : z = \Pi : p + \Pi : q + \Pi : r + \Pi : s - \Phi : pq - \Phi : rs - \Phi : xy.$$

Corollarium 1.

§. 26. Hinc jam abunde intelligitur, si proponantur quotcunque hujusmodi formulae, quemadmodum novam ejusdem generis $\Pi : z$ investigari oporteat, quae ab illis junctim sumtis discrepet quantitate algebraica, vel per logarithmos arcusve circulares assignabili.

Corollarium 2.

§. 27. Quod si omnes illae formulae fuerint inter se aequales earumque numerus $= \lambda$, semper nova formula $\Pi : z$ inveniri poterit, ut sit $\Pi : z = \lambda \, \Pi : p + W$, existente W quantitate vel algebraica, vel per logarithmos arcusve circulares assignabili. Quin etiam, duabus hujusmodi formulis $\Pi : p$ et $\Pi : q$ propositis, inveniri poterit $\Pi : z$ ut sit

$$\Pi : z = \lambda : \Pi : p + \mu \, \Pi : q + W.$$

Scholion.

§. 28. Hoc igitur modo non solum principia et fundamenta, quibus hoc argumentum innititur, succincte ac dilucide mihi quidem exposuisse videor: sed hoc argumentum etiam multo latius amplificavi quam adhuc est factum. Semper autem maxime est optandum, ut via magis directa detegatur, quae ad easdem investigationes perducat. Nemo enim certe dubitabit, quin hinc maxima in universam Analysin incrementa essent redundatura.

Applicatio
ad quantitates transcendentes
in forma $\int \frac{\partial z(\alpha+\beta zz)}{\sqrt{(1+mzz+nz^4)}} = \Pi : z$ contentas.

§. 29. Cum igitur hic sit $Z = \alpha + \beta zz$, propositis duabus formulis hujus generis $\Pi : x$ et $\Pi : y$, sumtoque

$$z = \frac{x\mathfrak{Y} + y\mathfrak{X}}{1 - nxxyy}, \text{ hincque}$$

$$\mathfrak{Z} = \frac{(mxy + \mathfrak{X}\mathfrak{Y})(1 + nxxyy) + 2nxy(x^2 + y^2)}{(1 - nxxyy)^2},$$

ex §. 18. ubi $u = xy$ et $a = z$, erit

$$\Pi : z = \Pi : x + \Pi : y + \beta xyz,$$

ita ut character ante adhibitus $\phi : y$ hoc casu accipiat valorem βxyz. Hujus igitur regulae ope propositis duabus hujusmodi formulis $\Pi : x$ et $\Pi : y$, tertia $\Pi : z$ semper reperiri potest, quae a summa illarum differat quantitate algebraica βxyz.

§. 30. Ponamus igitur, quotcunque hujusmodi formulas transcendentes proponi

$$\Pi : a, \; \Pi : b, \; \Pi : c, \; \Pi : d, \; \Pi : e, \; \Pi : f, \; \Pi : g, \text{ etc.}$$

et ex singulis quantitatibus a, b, c, d, e, colligi valores irrationales litteris germanicis insignitas

$$\mathfrak{A} = \sqrt{(1 + maa + na^4)}; \; \mathfrak{B} = \sqrt{(1 + mbb + nb^4)};$$
$$\mathfrak{C} = \sqrt{(1 + mcc + nc^4)}; \; \mathfrak{D} = \sqrt{(1 + mdd + nd^4)};$$
$$\text{etc.} \qquad \qquad \text{etc}$$

semper nova formula ejusdem generis exhiberi poterit, quae a summa earum discrepet quantitate algebraica, quantuscunque etiam fuerit earum formularum datarum numerus. Operationes autem ad hunc finem perducentes commodissime sequenti modo instituentur.

§. 31. Primo scilicet ex binis datarum a et b quaeratur p, ut sit

$$p = \frac{a\mathfrak{B} + b\mathfrak{A}}{1 - naabb} \text{ et } \mathfrak{P} = \frac{(mab + \mathfrak{A}\mathfrak{B})(1 + naabb) + 2nab(aa + bb)}{(1 - naabb)^2}.$$

Deinde ex hac quantitate p, cum datarum tertia c juncta, definiatur q, ut sit

$$q = \frac{p\mathfrak{C} + c\mathfrak{P}}{1 - nccpp} \text{ et } \mathfrak{Q} = \frac{(mcp + \mathfrak{C}\mathfrak{P})(1 + ccpp) + 2ncp(cc + pp)}{(1 - nccpp)^2}.$$

Tertio ex hac quantitate q cum quarta datarum d juncta, quaeratur r, ut sit

$$r = \frac{q\mathfrak{D} + d\mathfrak{Q}}{1 - nddqq} \text{ et } \mathfrak{R} = \frac{(mdq + \mathfrak{D}\mathfrak{Q})(1 + nddqq) + 2ndq(dd + qq)}{(1 - nddqq)^2}.$$

Quarto ex ista quantitate r cum quinta datarum e definiatur s, ut sit

$$s = \frac{r\mathfrak{E} + e\mathfrak{R}}{1 - neerr} \text{ et } \mathfrak{S} = \frac{(mer - \mathfrak{E}\mathfrak{R})(1 + neerr) + 2ner(ee + rr)}{(1 - neerr)^2}.$$

Haeque operationes continuentur, donec omnes quantitates datae in computum fuerint ductae.

§. 32. His autem omnibus valoribus inventis, sequentes comparationes desideratae ordine ita se habebunt

I. $\Pi : p = \Pi : a + \Pi : b + \beta abp$

II. $\Pi : q = \Pi : a + \Pi : b + \Pi : c + \beta abp + \beta cpq$

III. $\Pi : r = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \beta abp + \beta cpq + \beta dqr$

IV. $\Pi : s = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \Pi : e + \beta abp + \beta cpq + \beta dqr + \beta ers$

V. $\Pi : t = \Pi : a + \Pi : b + \Pi : c + \Pi : d + \Pi : e + \Pi : f + \beta abp + \beta cpq + \beta dqr + \beta ers + \beta fst.$

etc. etc.

§. 33. Cum igitur ista formula transcendens

$$\Pi : z = \int \frac{\partial z(\alpha + \beta z z)}{\sqrt{(1 + m z z + z^4)}},$$

in se contineat arcus omnium sectionum conicarum a vertice sumtos, harum formularum ope, quotcunque proponantur arcus in quavis sectione conica, qui omnes a vertice sint sumti, semper novus in eadem sectione conica arcus pariter a vertice abscindi poterit, qui a summa illorum arcuum datorum discrepet quantitate algebraice assignabili. Quin etiam nihil impedit, quo minus aliqui inter arcus datos capiantur negativi, quandoquidem jam annotavimus esse $\Pi : (-z) = -\Pi : z$, ita ut nostra determinatio etiam accommodari possit ad arcus inter terminos quoscunque interceptos. Hocque modo tractatio, quam nuper circa comparationem talium arcuum dedi, multo generalior reddi poterit.

§. 34. Caeterum, quemadmodum hoc casu, quo sumsimus $Z = \alpha + \beta z z$, character supra usurpatus $\phi : x y$ abiit in $\beta x y z$, dum scilicet ex binis quantitatibus x et y, secundum praecepta data tertia z determinatur: ita etiam, quaecunque alia functio loco Z adhibeatur, quoniam posuimus

$$\phi : x y = a \int \frac{(X - Y) \partial u}{x x - y y}, \text{ existente } u = x y,$$

integratione absoluta, functio inde resultans tantum quantitatem u cum litteris a et $\mathfrak{A}$ continebit, quandoquidem littera t ita exprimebatur $t = a a - 2 \mathfrak{A} u + n a a u u$, cum invento integrali ubique loco u scribatur $x\ y$, at vero loco a et $\mathfrak{A}$ litterae z et $\mathfrak{Z}$; atque hoc modo obtinebitur valor characteris $\phi : x y$ pro quovis casu proposito, quae functio, nisi fuerit algebraica, semper per logarithmos et arcus circulares exhiberi poterit; siquidem, uti assumsimus, littera Z fuerit functio rationalis par ipsius z.

SUPPLEMENTUM VIII.

AD TOM. I. SECT. II. CAP. VI.

DE COMPARATIONE QUANTITATUM TRANSCENDENTIUM IN FORMA $\int \frac{P \partial z}{\sqrt{(A + 2Bz + Czz + 2Dz^3 + Ez^4)}}$ CONTENTARUM.

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1). Dilucidationes super methodo elegantissima, qua illustris *de la Grange* usus est, in integranda aequatione differentiali $\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}}$. *Acta Acad. Imp. Sc. Tom. II. P. I. Pag.* 20 — 57.

§. 1. Postquam diu et multum in perscrutanda aequatione differentiali $\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}}$ desudassem, atque imprimis in methodum directam, quae via facili ac plana ad ejus integrale perduceret, nequicquam inquisivissem; penitus obstupui, cum mihi nunciaretur, in volumine quarto *Miscellaneorum Taurinensium* ab illustri *de la Grange* talem methodum esse expositam, cujus ope pro casu, quo

$$X = A + Bx + Cxx + Dx^3 + Ex^4 \text{ et}$$
$$Y = A + By + Cyy + Dy^3 + Ey^4,$$

propositae aequationis differentialis hoc integrale algebraicum atque adeo completum felicissimo successu elicuit

$$\frac{\sqrt{X} + \sqrt{Y}}{x - y} = \sqrt{[\Delta + D(x + y) + E(x + y)^2]}$$

ubi Δ denotat quantitatem constantem arbitrariam per integrationem ingressam.

§. 2. Istud autem egregium inventum eo magis sum admiratus, quod equidem semper putaveram, talem methodum in investigando idoneo factore, quo aequatio proposita integrabilis redderetur, quaeri oportere, cum vulgo omnis methodus integrandi vel in separatione variabilium, vel in idoneo multiplicatore contineri videatur, etiamsi certis casibus quoque ipsa differentiatio ad integrale perducere queat, quemadmodum tam a me ipso quam ab aliis per plurima exempla est ostensum. Ad hanc autem tertiam viam illa ipsa methodus *Grangiana* rite referri posse videtur.

§. 3. Quanquam autem facile est inventis aliquid addere, tamen in re tam ardua plurimum intererit, hanc methodum ab illustri *de la Grange* adhibitam accuratius perpendisse atque ad usum analyticum magis accommodasse; siquidem totum negotium multo facilius ac simplicius expediri posse videtur. Quamobrem, quae de hoc argumento, quod merito maximi momenti est censendum, sum meditatus, hic data opera fusius sum expositurus.

§. 4. Quoniam autem hoc integrale ab illustri *de la Grange* inventum, ab iis formis quas ipse olim dederam, plurimum discrepat, ac simplicitate non mediocriter antecellit, ante omnia visum est scitari, quomodo aequationi differentiali satisfaciat. Hunc in finem pono brevitatis gratia $\sqrt{X} + \sqrt{Y} = V$, ut habeam

$$\frac{V}{x-y} = \sqrt{[\Delta + D(x+y) + E(x+y)^2]},$$

quam aequationem ita differentiare oportet, ut constans arbitraria Δ ex differentiali excedat. Sumtis igitur quadratis erit

$$\frac{VV}{(x-y)^2} = \Delta + D(x+y) + E(x+y)^2,$$

quae differentiata dat

$$\frac{2V\partial V}{(x-y)^2} - \frac{2VV(\partial x - \partial y)}{(x-y)^3} - D(\partial x + \partial y) - 2E(x+y)(\partial x + \partial y) = 0.$$

§. 5. Quo nunc calculus planior reddatur, seorsim partes vel per ∂x vel per ∂y affectas investigemus. Pro elemento igitur ∂x, si y ut constans spectetur, erit $\partial V = \frac{X'\partial x}{2\sqrt{X}}$, unde singulae partes ita se habebunt

$$\partial x \left[\frac{VX'}{(x-y)^2\sqrt{X}} - \frac{2VV}{(x-y)^3} - D - 2E(x+y)\right]$$

ubi notetur esse $V = \sqrt{X} + \sqrt{Y}$, hincque

$$VV\sqrt{X} = (X+Y)\sqrt{X} + 2X\sqrt{Y}:$$

unde hic duplicis generis termini occurrunt, dum vel per $\sqrt{X}$ vel per $\sqrt{Y}$ sunt affecti. Duo autem termini adsunt $\sqrt{Y}$ affecti, qui sunt

$$-\frac{4X\sqrt{Y}}{(x-y)^3} + \frac{X'\sqrt{Y}}{(x-y)^2},$$

qui ergo junctim sumti dabunt

$$\frac{\sqrt{Y}}{(x-y)^3}[X'(x-y) - 4X],$$

quae forma ob

$$X = A + Bx + Cxx + Dx^3 + Ex^4, \text{ hincque}$$
$$X' = B + 2Cx + 3Dxx + 4Ex^3, \text{ dabit}$$
$$X'(x-y) - 4X = -4A - B(3x+y)$$
$$-2C(xx+xy) - D(x^3 + 3xxy) - 4Ex^3y.$$

Termini autem per $\sqrt{X}$ affecti sunt

$$\frac{\sqrt{X}}{(x-y)^3}[X'(x-y) - 2(X+Y) - D(x-y)^3 - 2E(x+y)(x-y)^3].$$

Cum igitur sit

$$X + Y = 2A + B(x+y) + C(x^2+y^2)$$
$$+ D(x^3+y^3) + E(x^4+y^4),$$

facta substitutione iste postremus factor erit

$$-4\,A - B\,(x+3\,y) - 2\,C\,(x\,y+y\,y)$$
$$-D\,(3\,x\,y\,y+y^3) - 4\,E\,x\,y^3,$$

quae forma a praecedente hoc tantum discrepat, quod litterae x et y sunt permutatae.

§. 6. Quod si ergo brevitatis gratia ponamus

$$M = 4\,A + B\,(3\,x+y) + 2\,C\,(x\,x+x\,y)$$
$$+ D\,(x^3+3\,x\,x\,y) + 4\,E\,x^3\,y,$$
$$N = 4\,A + B\,(x+3\,y) + 2\,C\,(y\,y+x\,y)$$
$$+ D\,(y^3+3\,x\,y\,y) + 4\,E\,x\,y^3,$$

hinc pars elemento ∂x affecta ita erit expressa

$$-\frac{\partial x}{(x-y)^3\sqrt{X}}(M\sqrt{Y}+N\sqrt{X}).$$

§. 7. Simili modo ob $\partial V = \frac{Y'\partial Y}{2\sqrt{Y}}$, partes elemento ∂y affectae erunt

$$\frac{\partial y}{\sqrt{Y}}\left[\frac{VY'}{(x-y)^2} + \frac{2VV\sqrt{Y}}{(x-y)^3} - D\sqrt{Y} - 2\,E\,(x+y)\sqrt{Y}\right].$$

Haec jam forma ob

$$V = \sqrt{X} + \sqrt{Y} \text{ et } V\,V\sqrt{Y} = (X+Y)\sqrt{Y} + 2\,Y\sqrt{X},$$

continebit sequentes terminos per $\sqrt{X}$ affectos

$$\frac{\sqrt{X}}{(x-y)^3}\,[Y'\,(x-y) + 4\,Y],$$

quae forma ex priore praecedentis calculi oritur, si litterae x et y permutentur, simulque signa; unde patet hanc expressionem praebere valorem $+$ N. Reliqui autem termini per $\sqrt{X}$ affecti erunt

$$\frac{\sqrt{Y}}{(x-y)^3}\,[Y'(x-y) + 2\,(X+Y) - D\,(x-y)^3 - 2\,E\,(x+y)(x-y)^3].$$

Haec forma iterum ex permutatione litterarum et signorum ex

forma praecedentis calculi oritur, quae ergo cum esset $-$ N, haec erit $+$ M. Hoc igitur modo partes elementum ∂y continentes erunt

$$\frac{+\partial y}{(x-y)^2 \sqrt{Y}}(N\sqrt{X}+M\sqrt{Y}).$$

§. 8. Conjungendis igitur his membris, aequatio differentialis ex forma *Grangiana* orta erit

$$\left(\frac{\partial y}{\sqrt{Y}}-\frac{\partial x}{\sqrt{Y}}\right)\left[\frac{N\sqrt{X}+M\sqrt{Y}}{(x-y)^2}\right]=0,$$

quae per factorem communem divisa praebet ipsam aequationem differentialem propositam $\frac{\partial x}{\sqrt{X}}=\frac{\partial y}{\sqrt{Y}}$; unde simul patet aequationem integralem exhibitam recte se habere, atque adeo valorem litterae Δ arbitrio nostro penitus relinqui.

§. 9. Antequam autem methodum *Grangianam* ad ipsam aequationem differentialem $\frac{\partial x}{\sqrt{X}}=\frac{\partial y}{\sqrt{Y}}$ in omni extensione acceptam applicemus, a casu simpliciore inchoëmus, quo aequatio adeo rationalis proponitur haec

$$\frac{\partial x}{a+2bx+cxx}=\frac{\partial y}{a+2by+cyy}.$$

Analysis
pro integratione aequationis differentialis

$$\frac{\partial x}{a+2bx+cxx}=\frac{\partial y}{a+2by+cyy}.$$

§. 10. Ponamus brevitatis gratia $a+2bx+cxx=X$ et $a+2by+cyy=Y$, ut fieri debeat $\frac{\partial x}{X}=\frac{\partial y}{Y}$, quae formulae cum inter se debeant esse aequales, utraque per idem elementum ∂t designetur, ita ut nanciscamur has duas formulas $\frac{\partial x}{\partial t}=X$ et $\frac{\partial y}{\partial t}=Y$. Quod si ergo jam statuamus

$$x-y=q, \text{ erit } \frac{\partial q}{\partial t}=X-Y=2bq+cq(x+y),$$

unde per q dividendo erit $\frac{\partial q}{q\partial t}=2b+c(x+y)$.

§. 11. Nunc primas formulas differentiemus, sumto elemento ∂t constante, et facto

$$\partial X = X' \partial x \text{ et } \partial Y = Y' \partial y$$

orientur hae duae aequationes

$$\frac{\partial \partial x}{\partial x \partial t} = X' \text{ et } \frac{\partial \partial y}{\partial y \partial t} = Y',$$

quae invicem additae praebent

$$\frac{\partial \partial x}{\partial x \partial t} + \frac{\partial \partial y}{\partial y \partial t} = X' + Y'.$$

Quare cum sit

$$X' = 2b + 2cx \text{ et } Y' = 2b + 2cy, \text{ erit}$$

$$\frac{1}{\partial t}\left(\frac{\partial \partial x}{\partial x} + \frac{\partial \partial y}{\partial y}\right) = 4b + 2c(x + y).$$

§. 12. Quoniam igitur hic postremus valor duplo major est praecedente $\frac{\partial q}{q \partial t}$, hoc modo deducti sumus ad hanc aequationem

$$\frac{\partial \partial x}{\partial x} + \frac{\partial \partial y}{\partial y} = \frac{2 \partial q}{q},$$

quae integrata dat $l \partial x + l \partial y = 2 l q +$ constans, hincque in numeris erit

$$\partial x \partial y = C q q \partial t^2, \text{ ita ut sit } C = \frac{\partial x \partial y}{q q \partial t^2}.$$

Quare cum sit $\frac{\partial x}{\partial t} = X$ et $\frac{\partial y}{\partial t} = Y$, aequatio integralis erit $\frac{XY}{(x-y)^2} = C$, quae ergo non solum est algebraica, sed etiam completa.

§. 13. Si igitur proposita fuerit haec aequatio differentialis

$$\frac{\partial x}{a + 2bx + cxx} = \frac{\partial y}{a + 2by + cyy},$$

ejus integrale completum ita erit expressum

$$\frac{(a + 2bx + cxx)(a + 2by + cyy)}{(x - y)^2} = C,$$

quae, utrinque addendo $bb - ac$, induet hanc formam

$$\frac{aa + 2ab(x+y) + 2acxy + bb(x+y)^2 + 2bcxy(x+y) + ccxxyy}{(x-y)^2} = \Delta\Delta,$$

sicque, extracta radice, integrale hanc formam habebit

$$\frac{a + b(x+y) + cxy}{x-y} = \Delta,$$

quae sine dubio est simplicissima, quandoquidem tam y per x quam x per y facillime exprimi potest, cum sit

$$y = \frac{(\Delta - b)x - a}{\Delta + b + cx} \text{ et } x = \frac{a + (\Delta + b)y}{\Delta - b - cy}.$$

§. 14. Calculum, quo hic usi sumus, perpendenti facile patebit, in his formis X et Y, non ultra quadrata progredi licere. Si enim ipsi X insuper tribuamus terminum $d\,x^3$ et ipsi Y terminum $d\,y^3$, pro priore forma prodit

$$\frac{X - Y}{x - y} = 2\,b + c\,(x+y) + d\,(xx + xy + yy) = \frac{\partial q}{q\,\partial t};$$

pro altera autem forma est

$$X' + Y' = 4\,b + 2\,c\,(x+y) + 3\,d(xx + yy) = \frac{\partial\partial x}{\partial x\,\partial t} + \frac{\partial\partial y}{\partial y\,\partial t}.$$

Quare si hinc duplum praecedentis auferamus, colligitur

$$\frac{\partial\partial x}{\partial x\partial t} + \frac{\partial\partial y}{\partial y\partial t} - \frac{2\partial q}{q\partial t} = d\,(x - y)^2,$$

quam aequationem non amplius integrare licet.

§. 15. Facile autem ostendi potest, talem aequationem differentialem, in qua ultra quadratum proceditur, nullo amplius modo algebraice integrari posse. Si enim tantum hic casus proponeretur $\frac{\partial x}{1 + x^3} = \frac{\partial y}{1 + x^3}$, notum est, utrinque integrale partim logarithmos partim arcus circulares involvere, ideoque quantitates transcendentes diversos, quae nullo modo inter se comparari possunt. Hujusmodi scilicet comparationes iis tantum casibus locum habere possunt, quando utrinque unius generis tantum quantitates transcendentes occurrunt.

Analysis
pro integratione aequationis

$$\frac{\partial x}{a+2bx+cxx}+\frac{\partial y}{a+2by+cyy}=0.$$

§. 16. Quod si hic ut ante ponamus

$$\frac{\partial x}{a+2bx+cxx}=\partial t,$$

statui debebit

$$\frac{\partial y}{a+2by+cyy}=-\partial t,$$

at vero si calculum simili modo quo ante instituere velimus, nihil plane proficimus. Postquam autem omnes difficultates probe perpendissem, tandem in artificium incidi, quo hunc casum expedire licuit, ita ut hinc non contemnendum incrementum methodo *Grangianae* attulisse mihi videar.

§. 17. Quoniam igitur has duas habeo aequationes

$$\frac{\partial x}{\partial t}=X \text{ et } \frac{\partial y}{\partial t}=-Y,$$

hinc formo istam novam aequationem

$$\frac{y\partial x+x\partial y}{\partial t}=yX-xY.$$

Jam facio $xy=u$, ut habeam

$$\frac{\partial u}{\partial t}=a(y-x)+cxy(x-y),$$

unde posito

$$x-y=q \text{ erit } \frac{\partial u}{\partial t}=q(cu-a),$$

quae aequatio per $cu-a$ divisa ductaque in c praebet

$$\frac{c\partial u}{(cu-a)\partial t}=cq,$$

hocque modo nacti sumus differentiale logarithmicum.

§. 18. Dein vero aequationes principales ut ante differentiemus, et obtinebimus

$$\frac{\partial\partial x}{\partial t\,\partial x} = X' \text{ et } \frac{\partial\partial y}{\partial t\,\partial y} = -Y',$$

quae invicem additae dant

$$\frac{1}{\partial t}\cdot\left(\frac{\partial\partial x}{\partial x} + \frac{\partial\partial y}{\partial y}\right) = X' - Y' = 2cq;$$

quare si hinc duplum praecedentis aequationis subtrahamus, remanebit

$$\frac{1}{\partial t}\cdot\left(\frac{\partial\partial x}{\partial x} + \frac{\partial\partial y}{\partial y} - \frac{2c\partial u}{cu-a}\right) = 0,$$

unde per ∂t multiplicando et integrando nanciscimur

$$l\partial x + l\partial y - 2l(cu-a) = lC, \text{ ideoque}$$

$$\frac{\partial x\partial y}{(cu-a)^2} = C\partial t^2.$$

Cum igitur sit

$$\partial x = X\partial t \text{ et } \partial y = -Y\partial t,$$

aequatio integralis nostra erit $-\frac{XY}{(cu-a)^2} = C$.

§. 19. Per hanc ergo analysin deducti sumus ad hanc aequationem integralem aequationis propositae

$$\frac{(a+2bx+cxx)(a+2by+cyy)}{(a-cxy)^2} = C.$$

Quae aequatio, si utrinque unitas subtrahatur, reducitur ad hanc formam

$$\frac{2ab(x+y)+ac(x+y)^2+4bbxy+2bcxy(x+y)}{(a-cxy)^2} = C.$$

§. 20. Illustremus hanc integrationem exemplo, ponendo $a=1$, $b=0$ et $c=1$, ita ut proposita sit haec aequatio differentialis $\frac{\partial x}{1+xx} + \frac{\partial y}{1+yy} = 0$, cujus integrale novimus esse

$$\text{Arc. tang. } x + \text{Arc. tang. } y = \text{Arc. tang. } \frac{x+y}{1-xy} = C,$$

sicque novimus esse $\frac{x+y}{1-xy} = C$. At vero nostra postrema formula dat pro hoc casu

$$\frac{(x+y)^2}{(1-xy)^2} = C, \text{ ideoque } \frac{x+y}{1-xy} = C,$$

quod egregie convenit.

§. 21. Consideremus etiam casum, quo $a = 1$, $b = \frac{1}{2}$ et $c = 1$, ita ut proponatur haec aequatio

$$\frac{\partial x}{1+x+xx} + \frac{\partial y}{1+y+yy} = 0,$$

cujus integrale est

$$\frac{2}{\sqrt{3}} \text{ Arc. tang. } \frac{x\sqrt{3}}{2+x} + \frac{2}{\sqrt{3}} \text{ Arc. tang. } \frac{y\sqrt{3}}{2+y} = C,$$

unde sequitur fore

$$\text{Arc. tang. } \frac{2(x+y+xy)\sqrt{3}}{4+2(x+y)-2xy} = C,$$

ideoque etiam $\frac{x+y+xy}{2+x+y-xy} = C$. At vero forma integralis inventa pro hoc casu dabit

$$\frac{x+y+(x+y)^2+xy+xy(x+y)}{(1-xy)^2} = C,$$

quae in factores resoluta dat

$$\frac{(1+x+y)(x+y+xy)}{(1-xy)^2} = C.$$

Prior vero aequatio

$$\frac{x+y+xy}{2+x+y-xy} = C \text{ inversa praebet } \frac{2+x+y-xy}{x+y+xy} = C,$$

et unitate subtracta $\frac{1-xy}{x+y+xy} = C$, atque haec in praecedentem ducta dat $\frac{1+x+y}{1-xy} = C$.

§. 22. Videamus igitur, utrum haec posteriores aequationes inter se conveniant, et quia constantes utrinque inter se discrepare possunt, ambas aequationes ita referamus:

$$\frac{1-xy}{x+y+xy} = \alpha \text{ et } \frac{1+x+y}{1-xy} = \beta;$$

unde cum sit $\frac{1}{a} = \frac{x+y+xy}{1-xy}$, evidens est fore $\beta - \frac{1}{a} = 1$, ex quo pulcherrimus consensus inter ambas formulas elucet. Ex his exemplis intelligitur aequationem generalem supra inventam hoc modo per factores repraesentari posse

$$\frac{[2b+c(x+y)]\,[a(x+y)+2bxy]}{(a-cxy)^2}.$$

Caeterum consideratio harum formularum haud injucundas speculationes suppeditare poterit.

§. 23. Sequenti autem modo forma illa integralis inventa

$$\frac{[2b+c(x+y)]\,[a(x+y)+2bxy]}{(a-cxy)^2} = C,$$

statim ad formam simplicissimam reduci potest; si enim ejus factores statuamus

$$\frac{2b+c(x+y)}{a-cxy} = P \text{ et } \frac{a(x+y)+2bxy}{a-cxy} = Q,$$

ut esse debeat $PQ = C$, exit

$$aP - cQ = \frac{2ab-2bcxy}{a-cxy} = 2b, \text{ unde fit } Q = \frac{aP-2b}{c},$$

sicque quantitati constanti aequari debet haec forma $\frac{aPP-2bP}{c}$; ex quo patet, etiam ipsam quantitatem P constanti aequari debere, ita ut jam aequatio nostra integralis sit

$$\frac{2b+c(x+y)}{a-cxy} = C, \text{ vel etiam } \frac{a(x+y)+2bxy}{a-cxy} = C.$$

Alia solutio facillima ejusdem aequationis

$$\frac{\partial x}{a+2bx+cxx} + \frac{\partial y}{a+2by+cyy} = 0.$$

§. 24. Postrema reductione probe perpensa, comperui, statim ab initio ad formam integralis simplicissimam perveniri posse, atque adeo non necesse esse ad differentialia secunda ascendere. Si enim ut ante ponamus $x+y=p$, $x-y=q$ et $xy=u$, ex formulis

$$\frac{\partial x}{\partial t} = X \text{ et } \frac{\partial y}{\partial t} = -Y$$

statim deducimus

$$\frac{\partial p}{\partial t} = X - Y = 2bq + cpq, \text{ unde fit } \frac{\partial p}{2b + cp} = q\,\partial t.$$

§. 25. Porro vero erit

$$\frac{y\,\partial x + x\,\partial y}{\partial t} = \frac{\partial u}{\partial t} = yX - xY = -aq + cqu,$$

unde fit $\frac{\partial u}{cu - a} = q\,\partial t$, quamobrem hinc statim colligimus hanc aequationem $\frac{\partial u}{2b + cp} = \frac{\partial u}{cu - a}$, cujus integratio praebet

$$l(2b + cp) = l(cu - a) + lC;$$

unde deducitur haec aequatio algebraica $\frac{2b + cp}{cu - a} = C$, quae, restitutis litteris x et y, dat $\frac{2b + c(x + y)}{cxy - a} = C$, quae est forma simplicissima aequationis integralis desideratae. Hic imprimis notatu dignum occurrit, quod casum primum hac ratione resolvere non licet.

§. 26. Ex forma autem integrali inventa facile aliae derivantur veluti, si addamus $\frac{2b}{a}$, orietur haec forma

$$\frac{a(x+y)2 + bxy}{cxy - a} = C,$$

quae per praecedentem divisa denuo novam formam suppeditat, scilicet

$$\frac{2b + c(x + y)}{a(x + y) + 2bxy} = C,$$

quae formae quomodo satisfaciant operae pretium erit ostendisse. Et quidem postrema forma differentiata, erit

$$\frac{-2ab(\partial x + \partial y) - 4bb(y\,\partial x + x\,\partial y) - 2bc(yy\,\partial x + xx\,\partial y)}{[a(x + y) + 2bxy]^2}$$

quae in ordinem redacta praebet

$$\partial x(2ab + 4bby + 2bcyy) + \partial y(2ab + 4bbx + 2bcxx) = 0.$$

Haec per $2b$ divisa et separata dat

$$\frac{\partial x}{a+2bx+cxx}+\frac{\partial y}{a+2by+cyy}=0,$$

quae est ipsa proposita.

Analysis

pro integratione aequationis

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}}=\frac{\partial y}{\sqrt{(A+By+Cyy)}}.$$

§. 27. Introducto novo elemento ∂t, deinceps pro constanti habendo, oriuntur hae duae aequationes

$$\frac{\partial x}{\partial t}=\sqrt{X} \text{ et } \frac{\partial y}{\partial t}=\sqrt{Y},$$

ubi litteris X et Y valores initio assignatos tribuamus. Videbimus autem, pro methodo, qua hic utemur, terminos litteris D et E affectos omitti debere. Sumtis ergo quadratis erit

$$\frac{\partial x^2}{\partial t^2}=X \text{ et } \frac{\partial y^2}{\partial t^2}=Y.$$

§. 28. Nunc istas formulas differentiemus, positoque, ut fieri solet, $\partial X=X'\partial x$ et $\partial Y=Y'\partial y$, nanciscemur has aequationes

$$\frac{2\partial\partial x}{\partial t^2}=X' \text{ et } \frac{2\partial\partial y}{\partial t^2}=Y',$$

ac posito $x+y=p$, fiet $\frac{2\partial\partial p}{\partial t^2}=X'+Y'$. Cum jam sit

$$X'=B+2Cx+3Dxx+4Ex^3 \text{ et}$$
$$Y'=B+2Cy+3Dyy+4Ey^3, \text{ erit}$$
$$X'+Y'=2B+2Cp+3D(xx+yy)$$
$$+4E(x^3+y^3)=\frac{2\partial\partial p}{\partial t^2},$$

quae aequatio manifesto integrationem admittet, si fuerit et $D=0$ et $E=0$, quemadmodum assumsimus. Multiplicando igitur per

∂p et integrando nanciscimur

$$\frac{\partial p^2}{\partial t^2} = \Delta + 2\,\mathrm{B}p + \mathrm{C}pp,$$

et radicem extrahendo

$$\frac{\partial p}{\partial t} = \sqrt{(\Delta + 2\,\mathrm{B}p + \mathrm{C}pp)}.$$

Cum igitur sit $\frac{\partial p}{\partial t} = \sqrt{\mathrm{X}} + \sqrt{\mathrm{Y}}$, aequatio integralis, quam sumus adepti erit

$$\sqrt{\mathrm{X}} + \sqrt{\mathrm{Y}} = \sqrt{[\Delta + 2\,\mathrm{B}(x+y) + \mathrm{C}(x+y)^2]},$$

quae adeo est algebraica; ubi notetur esse

$$\mathrm{X} = \mathrm{A} + \mathrm{B}x + \mathrm{C}xx \text{ et } \mathrm{Y} = \mathrm{A} + \mathrm{B}y + \mathrm{C}yy.$$

§. 29. Sumamus igitur quadrata, et nostra aequatio integralis erit

$$2\,\mathrm{A} + \mathrm{B}(x+y) + \mathrm{C}(x^2+y^2) + 2\sqrt{\mathrm{XY}}$$
$$= \Delta + 2\,\mathrm{B}(x+y) + \mathrm{C}(x+y)^2, \text{ sive}$$
$$2\,\mathrm{A} - \mathrm{B}(x+y) - 2\,\mathrm{C}xy + 2\sqrt{\mathrm{XY}} = \Delta,$$

quae penitus ab irrationalitate liberata, posito $\Delta - 2\,\mathrm{A} = \Gamma$, praebebit

$$\begin{aligned}4\,\mathrm{XY} = {} & 4\,\mathrm{AA} + 4\,\mathrm{AB}(x+y) + 4\,\mathrm{AC}(xx+yy) \\ & + 4\,\mathrm{BB}xy + 4\,\mathrm{BC}xy(x+y) + 4\,\mathrm{CC}xxyy \\ = {} & \Gamma^2 + 2\,\Gamma\mathrm{B}(x+y) + 4\,\Gamma\mathrm{C}xy + \mathrm{BB}(x+y)^2 \\ & + 4\,\mathrm{BC}xy(x+y) + 4\,\mathrm{CC}xxyy \text{ sive}\end{aligned}$$

$$\begin{aligned}(4\,\mathrm{AA} - \Gamma^2) + 2\,\mathrm{B}(2\,\mathrm{A} - \Gamma)(x+y) + 4(\mathrm{BB} - \Gamma\mathrm{C})xy \\ + 4\,\mathrm{AC}(xx+yy) - \mathrm{B}^2(x+y)^2 = 0.\end{aligned}$$

§. 30. Quod si jam hanc aequationem rationalem cum formula *canonica*, qua olim sum usus ad hujusmodi integrationes expediendas, comparemus, quae erat

$$\alpha + 2\,\beta(x+y) + \gamma(xx+yy) + 2\,\delta xy = 0,$$

dum scilicet loco $(x+y)^2$ scribamus $(xx+yy)+2xy$, reperiemus fore

$$\alpha = 4AA - \Gamma^2,\ \beta = B(2A-\Gamma),\ \gamma = 4AC - B^2,$$
$$\delta = BB - 2\Gamma C.$$

§. 31. Alio vero insuper modo eandem aequationem differentialem propositam integrare poterimus, introducendo literam $q = x - y$; tum enim habebimus

$$\frac{2\,\partial\partial q}{\partial t} = X' - Y'.$$

At vero erit

$$X' - Y' = 2Cq + 3Dq(x+y),$$

ubi iterum patet, statui debere tam $D=0$ quam $E=0$, ut integratio, multiplicando per ∂q, succedat. Hoc autem notato erit integrale $\frac{\partial q^2}{\partial t^2} = \text{Const.} + Cqq$, ideoque

$$\frac{\partial q}{\partial t} = \sqrt{(\Delta + Cqq)}.$$

§. 32. Cum igitur sit $\frac{\partial q}{\partial t} = \sqrt{X} - \sqrt{Y}$, hoc integrale ita erit expressum

$$\sqrt{X} - \sqrt{Y} = \sqrt{(\Delta + Cqq)}$$

quae aequatio sumtis quadratis abit in hanc

$$2A + B(x+y) + C(xx+yy) - 2\sqrt{XY}$$
$$= \Delta + C(x-y)^2, \text{ sive}$$
$$2A + B(x+y) + 2Cxy - 2\sqrt{XY} = \Delta,$$

unde fit

$$2\sqrt{XY} = 2A - \Delta + B(x+y) + 2Cxy,$$

ubi si ponatur $2A - \Delta = -\Gamma$, aequatio ab ante inventa prorsus non discrepat.

§. 33. Quod si autem proposita fuisset aequatio

$$\frac{\partial x}{\sqrt{(A + Bx + Cxx)}} + \frac{\partial y}{\sqrt{(A + By + Cyy)}} = 0,$$

integralia ante inventa ad hunc casum referentur, si modo loco $\sqrt{Y}$ scribatur $-\sqrt{Y}$; unde patet pro hoc casu haberi hanc aequationem

$$\sqrt{X} - \sqrt{Y} = \sqrt{[\Delta + 2B(x+y) + C(x+y)^2]},$$

vel etiam

$$\sqrt{X} + \sqrt{Y} = \sqrt{[\Delta + C(x-y)^2]}.$$

§. 34. Hic singularis casus occurrit, quando formulae $A + Bx + Cxx$ sunt quadrata. Sit enim

$$X = (a + bx)^2 \text{ et } Y = (a + by)^2$$

ideoque

$$A = a^2,\ B = 2ab,\ C = bb,$$

tum enim prior forma integralis erit

$$b(x-y) = \sqrt{[\Delta + 4ab(x+y) + bb(x+y)^2]}$$

sumtisque quadratis

$$-4bbxy = \Delta + 4ab(x+y),$$

ideoque

$$\Delta = a(x+y) + bxy,$$

cujus aequationis differentiale est

$$a(\partial x + \partial y) + b(x\partial y + y\partial x) = 0$$

ideoque

$$\partial x(a + by) + \partial y(a + bx) = 0.$$

Sin autem altera formula utamur, erit

$$2a + b(x+y) = \sqrt{[\Delta + bb(x-y)^2]},$$

unde quadratis sumtis, positoque $\Delta - 4aa = \Gamma$, prodit ut ante $\Gamma = a(x+y) + bxy$.

Analysis
pro integranda aequatione

$$\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}}$$

existente

$$X = A + Bx + Cxx + Dx^3 + Ex^4 \text{ et}$$
$$Y = A + By + Cyy + Dy^3 + Ey^4.$$

§. 35. Introducto iterum elemento ∂t, ut sit

$$\frac{\partial x}{\partial t} = \sqrt{X} \text{ et } \frac{\partial y}{\partial t} = \sqrt{Y},$$

ideoque sumtis quadratis

$$\frac{\partial x^2}{\partial t^2} = X \text{ et } \frac{\partial y^2}{\partial t^2} = Y,$$

statuamus

$$x + y = p \text{ et } x - y = q,$$

et quia hinc prodit

$$\partial x^2 - \partial y^2 = \partial p\, \partial q, \text{ erit}$$
$$\frac{d p\, \partial q}{\partial t^2} = X - Y = B(x - y) + C(x^2 - y^2)$$
$$+ D(x^3 - y^3) + E(x^4 - y^4).$$

§. 36. Quoniam igitur est

$$x = \frac{p+q}{2} \text{ et } y = \frac{p-q}{2},$$

his valoribus introductis reperietur

$$X - Y = Bq + Cpq + \tfrac{1}{4} Dq(3pp + qq)$$
$$+ \tfrac{1}{2} Epq(pp + qq),$$

unde per q dividendo oritur

$$\frac{\partial p \partial q}{q \partial t^2} = B + Cp + \tfrac{1}{4} D(3pp + qq)$$
$$+ \tfrac{1}{2} Ep(pp + qq).$$

§. 37. Nunc etiam formulas quadratas primo exhibitas differentiemus, et statuendo ut ante

$$\partial X = X' \partial x \text{ et } \partial Y = Y' \partial y, \text{ habebimus}$$

$$\frac{2 \partial \partial x}{\partial t^2} = X' \text{ et } \frac{2 \partial \partial y}{\partial t^2} = Y',$$

hincque addendo

$$\frac{2 \partial \partial p}{\partial t^2} = X' + Y'.$$

Cum vero sit

$$X' = B + 2 C x + 3 D x x + 4 E x^3 \text{ et}$$

$$Y' = B + 2 C y + 3 D y y + 4 E y^3, \text{ erit}$$

$$X' + Y' = 2 B + 2 C p + \tfrac{3}{2} D (p p + q q) + E p (p p + 3 q q),$$

ita ut substituto hoc valore fiat

$$\frac{\partial \partial p}{\partial t^2} = B + C p + \tfrac{3}{4} D (p p + q q) + \tfrac{1}{2} E p (p p + 3 q q),$$

a qua aequatione si priorem $\frac{\partial p \partial q}{q \partial t^2}$ subtrahamus, remanebit sequens

$$\frac{\partial \partial p}{\partial t^2} - \frac{\partial p \partial q}{q \partial t^2} = \tfrac{1}{2} D q q + E p q q.$$

§. 38. Haec jam aequatio per $q q$ divisa producit istam

$$\frac{1}{\partial t^2} \cdot \left(\frac{\partial \partial p}{q q} - \frac{\partial p \partial q}{q^3} \right) = \tfrac{1}{2} D + E p,$$

quae ducta in $2 \partial p$ manifesto fit integrabilis: prodit enim

$$\frac{\partial p^2}{q q \partial t^2} = \Delta + D p + E p p,$$

ex qua radice extracta colligitur

$$\frac{\partial p}{q \partial t} = \sqrt{(\Delta + D p + E p p)}.$$

Cum igitur posuerimus

$$p = x + y \text{ et } q = x - y, \text{ erit } \frac{\partial p}{\partial t} = \sqrt{X} + \sqrt{Y}$$

unde resultat haec aequatio integralis algebraica

$$\frac{\sqrt{X} + \sqrt{Y}}{x - y} = \sqrt{[\Delta + D (x + y) + E (x + y)^2]}$$

quae est ipsa forma ab illustri *de la Grange* inventa.

§. 39. Evolvamus ulterius hanc formam, ac sumtis quadratis erit

$$\frac{X + Y + 2\sqrt{XY}}{(x-y)^2} = \Delta + D(x+y) + E(x+y)^2.$$

Est vero

$$X + Y = 2A + B(x+y) + C(xx+yy)$$
$$+ D(x^3+y^3) + E(x^4+y^4),$$

unde si auferamus

$$[D(x+y) + E(x+y)^2](x-y)^2$$

remanebit

$$2A + B(x+y) + C(x^2+y^2) + Dxy(x+y)$$
$$+ 2Exxyy,$$

quo substituto aequatio integralis erit

$$\frac{2A + B(x+y) + C(x^2+y^2) + Dxy(x+y) + 2Ex^2y^2 + 2\sqrt{XY}}{(x-y)^2} = \Delta.$$

§. 40. Haec aequatio aliquanto concinnior reddi potest subtrahendo utrinque C et statuendo $\Delta - C = \Gamma$: habebitur enim hoc facto

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy + 2\sqrt{XY}}{(x-y)^2} = \Gamma,$$

unde deducimus

$$2\sqrt{XY} = \Gamma(x-y)^2 - 2A - B(x+y) - 2Cxy$$
$$- Dxy(x+y) - 2Exxyy,$$

sive ponendo

$$2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy = V,$$

aequatio nostra erit

$$2\sqrt{XY} = \Gamma(x-y)^2 - V,$$

quae sumtis quadratis abit in hanc:

$$4XY = \Gamma^2(x-y)^4 - 2\Gamma V(x-y)^2 + VV, \text{ sive}$$
$$4XY - VV = \Gamma^2(x-y)^4 - 2\Gamma V(x-y)^2.$$

§. 41. Facta nunc substitutione erit

$$\begin{aligned}4XY = {} & 4A^2 + 4AB(x+y) + 4AC(xx+yy) \\ & + 4AD(x^3+y^3) + 4AE(x^4+y^4) + 4BBxy \\ & + 4BCxy(x+y) + 4BDxy(xx+yy) \\ & + 4BExy(x^3+y^3) + 4CCxxyy \\ & + 4CDxxyy(x+y) + 4CExxyy(xx+yy) \\ & + 4DDx^3y^3 + 4DEx^3y^3(x+y) \\ & + 4EEx^4y^4.\end{aligned}$$

At vero porro colligitur fore

$$\begin{aligned}VV = {} & 4AA + 4AB(x+y) + 8ACxy \\ & + 4ADxy(x+y) + 8AExxyy + BB(x+y)^2 \\ & + 4BCxy(x+y) + 2BDxy(x+y)^2 \\ & + 4BE(x+y)xxyy + 4CCxxyy \\ & + 4CD(x+y)xxyy + 8CEx^3y^3 \\ & + DDxxyy(x+y)^2 + 4DEx^3y^3(x+y) \\ & + 4EEx^4y^4\end{aligned}$$

§. 42. Quod si jam posteriorem formulam a priore subtrahamus et singulos terminos ordine analogos disponamus, reperiemus

$$\begin{aligned}4XY - VV = {} & 4AC(x-y)^2 + 4AD(x+y)(x-y)^2 \\ & + 4AE(x+y)^2(x-y)^2 - B^2(x-y)^2 \\ & + 2BDxy(x-y)^2 + 4BExy(x+y)(x-y)^2 \\ & + 4CExxyy(x-y)^2 - DDxxyy(x-y)^2,\end{aligned}$$

quae expressio factorem habet communem $(x-y)^2$, per quem ergo si dividamus perveniemus ad hanc aequationem concinniorem

$$\begin{aligned}
&4AC + 4AD(x+y) + 4AE(x+y)^2 - BB \\
&\quad + 2BDxy + 4BExy(x+y) + (4CE - DD)xxyy \\
&= \Gamma\Gamma(x-y)^2 - 4\Gamma A - 2\Gamma B(x+y) - 4\Gamma Cxy \\
&\quad - 2\Gamma Dxy(x+y) - 4\Gamma Exxyy.
\end{aligned}$$

§. 43. Transferamus nunc omnes terminos ad partem sinistram, et loco $(x+y)^2$ scribamus $(xx+yy)+2xy$, tum vero $(xx+yy)-2xy$ loco $(x-y)^2$, quo facto talis oritur aequatio meae canonicae respondens

$$0 = \left\{\begin{array}{lll}
4AC + 4AD(x+y) + 4AE(x^2+y^2) & + 2BDxy + 4BExy(x+y) & + 4CExxyy \\
-BB + 2\Gamma C(x+y) - \Gamma\Gamma(x^2+y^2) & + 8AExy + 2\Gamma Cxy(x+y) & - DDxxyy \\
+4\Gamma A & + 2\Gamma^2 xy & + 4\Gamma Exxyy \\
 & + 4\Gamma Cxy &
\end{array}\right.$$

§. 44. Hinc ergo pro aequatione canonica litterae graecae α, β, γ, δ, ε, ζ, per latinas A, B, C, D, E, una cum constante Γ sequenti modo determinantur

$$\begin{aligned}
\alpha &= 4AC + 4\Gamma A - BB \\
\beta &= 2AD + \Gamma B \\
\gamma &= 4AE - \Gamma\Gamma \\
\delta &= BD + 4AE + \Gamma\Gamma + 2\Gamma C \\
\varepsilon &= 2BE + \Gamma C \\
\zeta &= 4CE + 4\Gamma E - DD,
\end{aligned}$$

ita ut aequatio canonica, qua olim sum usus, sit

$$\begin{aligned}
&\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy \\
&\quad + 2\varepsilon xy(x+y) + \zeta xxyy = 0.
\end{aligned}$$

§. 45. Haec autem aequatio integralis ad rationalitatem perducta latius patet quam aequatio proposita differentialis $\frac{\partial x}{\sqrt{X}} - \frac{\partial y}{\sqrt{Y}} = 0$:

simul enim complectitur integrale hujus

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\partial Y} = 0.$$

Scilicet haec aequatio complectitur duos factores, quorum alteruter alterutri satisfacit. Ex genesi autem patet hanc aequationem esse productum ex his factoribus

$$\Delta (x - y)^2 - V + 2 \sqrt{XY}, \text{ et}$$
$$\Delta (x - y)^2 - V - 2 \sqrt{XY}.$$

§. 46. Supra jam observavimus, ejusdem aequationis differentialis integrale hoc quoque modo exhiberi posse

$$\frac{M\sqrt{Y} + N\sqrt{X}}{(x-y)^3} = C \text{ (vide §. 8. et praec.)}$$

existente

$$M = 4A + B(3x + y) + 2Cx(x + y)$$
$$+ Dxx(x + 3y) + 4Ex^3 y,$$
$$N = 4A + B(3y + x) + 2Cy(x + y)$$
$$+ Dyy(y + 3x) + 4Exy^3,$$

ubi notasse juvabit esse

$$M + N = 8A + 4B(x + y) + 2C(x + y)^2$$
$$+ D(x + y)^3 + 4Exy(xx + yy),$$
$$M - N = 2B(x - y) + 2C(x + y)(x - y)$$
$$+ D(x - y)(x^2 + 4xy + y^2)$$
$$+ 4Exy(x + y)(x - y).$$

Interim tamen haud facile intelligitur, quomodo haec forma cum ante inventa consentiat, dum tamen de consensu certi esse possumus.

§. 47. Ex iis, quae hactenus sunt allata, satis liquet, eandem aequationem integralem innumeris modis exhiberi posse,

prout constans arbitraria alio atque alio modo repraesentatur; unde plurimum intererit certam legem stabilire, secundum quam quovis casu constantem illam arbitrariam exprimere velimus. Hunc in finem ista regula observetur: ut perpetuo integralia ita capiantur, ut posito $y = 0$ fiat $x = k$, hincque secundum legem compositionis $X = K$, existente

$$K = A + Bk + Ckk + Dk^3 + Ek^4.$$

Hac enim lege observata omnia integralia, utcunque diversa videantur, ad perfectum consensum perduci poterunt. Hoc igitur modo quae hactenus invenimus sequentibus theorematibus complectamur.

Theorema 1.

§. 48. Si haec aequatio differentialis

$$\frac{\partial x}{a + bx + cxx} - \frac{\partial y}{a + by + cyy} = 0,$$

ita integretur, ut posito $y = 0$ fiat $x = k$, integrale ita se habebit

$$\frac{2a + b(x + y) + 2cxy}{x - y} = \frac{2a + bk}{k}.$$

Theorema 2.

§. 49. Si haec aequatio differentialis

$$\frac{\partial x}{a + bx + cxx} + \frac{\partial y}{a + by + cyy} = 0$$

ita integretur, ut posito $y = 0$ fiat $x = k$, integrale supra triplici modo est inventum; erit enim

$$\text{I. } \frac{b + c(x + y)}{cxy - a} = -\frac{b + ck}{a},$$

$$\text{II. } \frac{a(x + y) + bxy}{cxy - a} = -k,$$

$$\text{III. } \frac{b + c(x + y)}{a(x + y) + bxy} = \frac{b + ck}{ak}.$$

Theorema 3.

§. 50. Si haec aequatio differentialis

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}}-\frac{\partial y}{\sqrt{(A+By+Cyy)}}=0$$

ita integretur, ut posito $y=0$ fiat $x=k$, integrale erit

$$-B(x+y)-2Cxy+2\sqrt{(A+Bx+Cxx)\times(A+By+Cyy)}=$$
$$-Bk+2\sqrt{A(A+Bk+Ckk)}, \text{ sive}$$
$$B(k-x-y)-2Cxy=2\sqrt{A(A+Bk+Ckk)}$$
$$-2\sqrt{(A+Bx+Cxx)(A+By+Cyy)}.$$

Corollarium.

§. 51. Hinc ergo patet, si aequatio differentialis proposita fuerit ista

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}}+\frac{\partial y}{\sqrt{(A+By+Cyy)}}=0,$$

eaque integretur ita, ut posito $y=0$ fiat $x=k$, integrale fore

$$B(k-x-y)-2Cxy=2\sqrt{(A+Bx+Cxx)\times(A+By+Cyy)}$$
$$-2\sqrt{A(A+Bk+Ckk)}.$$

Theorema 4.

§. 52. Si posito brevitatis gratia

$$X=A+Bx+Cxx+Dx^3+Ex^4$$
$$Y=A+By+Cyy+Dy^3+Ey^4$$
$$K=A+Bk+Ckk+Dk^3+Ek^4,$$

haec proponetur aequatio differentialis $\frac{\partial x}{\sqrt{X}}-\frac{\partial y}{\sqrt{Y}}=0$, quae ita integrari debeat, ut posito $y=0$ fiat $x=k$, ejus integrale ita erit expressum

$$\frac{2A+B(x+y)+2Cxy+Dxy(x+y)+2Exxyy+2\sqrt{XY}}{(x+y)^2}=\frac{2A+Bk+2\sqrt{AK}}{kk}$$

Sin autem aequatio proposita fuerit

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0,$$

ejus integrale erit

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy - 2\sqrt{XY}}{(x-y)^2} = \frac{2A + Bk - 2\sqrt{AK}}{kk}.$$

Corollarium 1.

§. 53. Quod si hic ponamus $D = 0$ et $E = 0$, casus oritur theorematis tertii, pro aequatione

$$\frac{\partial x}{\sqrt{(A + Bx + Cxx)}} - \frac{\partial y}{\sqrt{(A + By + Cyy)}} = 0,$$

cujus ergo integrale erit

$$\frac{2A + B(x+y) + 2Cxy + 2\sqrt{(A + Bx + Cxx)(A + By + Cyy)}}{(x-y)^2} = \frac{2A + Bk + 2\sqrt{A(A + Bk + Ckk)}}{kk},$$

quae forma si cum superiori comparetur, formulae irrationales eliminari poterunt. Quoniam enim ex priore est

$$2\sqrt{XY} = 2\sqrt{[A(A + Bk + Ckk)]} - B(k - x - y) + 2Cxy,$$

erit hoc integrale postremum

$$\frac{2A + B(2x + 2y - k) + 4Cxy + 2\sqrt{[A(A + Bk + Ckk)]}}{(x-y)^2} = \frac{2A + Bk + 2\sqrt{A(A + Bk + Ckk)}}{kk},$$

unde statim deduci potest aequatio canonica

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy = 0.$$

Corollarium 2.

§. 54. Ponamus nunc esse $A = 0$ et $B = 0$, ut sit

$$X = xx(C + Dx + Exx) \text{ et}$$
$$Y = yy(C + Dy + Eyy) \text{ et}$$
$$K = kk(C + Dk + Ekk),$$

aequatio differentialis integranda fiet

$$\frac{\partial x}{x\sqrt{(C+Dx+Exx)}}-\frac{\partial y}{y\sqrt{(C+Dy+Eyy)}}=0,$$

cujus ergo integrale erit

$$\frac{xy[2C+D(x+y)+2Exy]+2xy\sqrt{(C+Dx+Exx)(C+Dy+Eyy)}}{(x-y)^2}=\Delta,$$

atque hic constantem Δ per k definire non licebit: positio enim $y=0$ incongruum jam involvit. Interim tamen et haec integratio maxime est memoratu digna.

Corollarium 3.

§. 55. Quod si autem in hac postrema integratione loco x et y scribamus $\frac{1}{x}$ et $\frac{1}{y}$, primo aequatio differentialis erit

$$\frac{\partial y}{\sqrt{(Cyy+Dy+E)}}-\frac{\partial x}{\sqrt{(Cxx+Dx+E)}}=0;$$

tum vero integrale sequentem induet formam

$$\frac{2Cxy+D(x+y)+2E+2\sqrt{(Cxx+Dx+E)(Cyy+Dy+E)}}{(y-x)^2}=\Delta$$
$$=\frac{Dk+2E+2\sqrt{E(Ckk+Dk+E)}}{kk}.$$

Si igitur hic loco literarum E, D, C, scribamus A, B, C, prodibit aequatio differentialis supra tractata

$$\frac{\partial x}{\sqrt{(A+Bx+Cxx)}}-\frac{\partial y}{\sqrt{(A+By+Cyy)}}=0$$

cujus ergo integrale erit

$$\frac{2A+B(x+y)+2Cxy+2\sqrt{(A+Bx+Cxx)(A+By+Cyy)}}{(x-y)^2}=$$
$$\frac{Bk+2A+2\sqrt{A(A+Bk+Ckk)}}{kk},$$

quae egregie convenit cum ea in coroll. 1. data.

Corollarium 4.

§. 56. Contemplemur nunc etiam casum, quo formula $A+Bx+Cxx+Dx^3+Ex^4$ fit quadratum, quod sit $(a+bx+cxx)^2$, ita ut jam habeamus

$$A = aa,\ B = 2ab,\ C = bb + 2ac,\ D = 2bc,\ E = cc,$$

tum vero

$$\sqrt{X} = a + bx + cxx,\ \sqrt{Y} = a + by + cyy,$$
$$\sqrt{K} = a + bk + ckk,$$

atque aequatio differentialis pro priore casu erit

$$\frac{\partial x}{a + bx + cxx} - \frac{\partial y}{a + by + cyy} = 0,$$

cujus ergo integrale erit

$$\left\{\begin{array}{l} 2aa + 2ab\,(x+y) + 2\,(bb + 2ac)\,xy + 2bcxy\,(x+y) \\ \quad + 2ccxxyy + 2\,(a + bx + cxx)\,(a + by + cyy) \end{array}\right\} : (x - y)^2 = \Delta,$$

quae reducitur ad

$$\frac{aa + ab\,(x+y) + (bb + ac)\,xy + bcxy\,(x+y) + ccxxyy}{(x-y)^2} = \frac{aa + abk}{kk}.$$

Quod si jam utrinque addamus $\frac{1}{4}bb$, prodibit

$$\frac{[a + \frac{1}{2}b\,(x + y) + cxy]^2}{(x - y)^2} = \frac{(a + \frac{1}{2}bk)^2}{k^2},$$

unde extracta radice obtinetur forma integralis in theoremate primo assignata.

§. 57. Sin autem hoc modo alterum casum aequationis

$$\frac{\partial x}{a + bx + cxx} + \frac{\partial y}{a + by + cyy} = 0,$$

evolvere velimus, pervenimus ad hanc aequationem

$$\frac{2aa + 2ab(x+y) + 2(bb + 2ac)xy + 2bcxy(x+y) + 2ccxxyy}{(x-y)^2} - \frac{2(a + bx + cxx)(a + by + cyy)}{(x-y)^2} = \Delta,$$

quae evoluta praebet $\Delta = -2ac$, haecque aequatio manifesto est absurda, et nihil circa integrale quaesitum declarat, cujus rationem maximi momenti erit perscrutari.

Insigne Paradoxon.

§. 58. Cum hujus aequationis differentialis

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0,$$

integrale in genere inventum sit

$$\frac{2A + B(x+y) + yCxy + Dxy(x+y) + 2Exxyy - 2\sqrt{XY}}{(x-y)^2} = \Delta,$$

casu autem, quo statuitur

$$\sqrt{X} = a + bx + cxx \text{ et}$$
$$\sqrt{Y} = a + by + cyy,$$

aequatio absurda inde oriatur, quaeritur enodatio hujus insignis difficultatis ac praecipue modus, hinc verum integralis valorem investigandi.

Enodatio Paradoxi.

§. 59. Quemadmodum scilicet in analysi ejusmodi formulae occurrere solent, quae certis casibus indeterminatae atque adeo nihil plane significare videntur: ita hic simile quid usu venit, sed longe alio modo, cum neque ad fractionem, cujus numerator et denominator simul evanescunt, neque ad differentiam inter duo infinita perveniatur, quod exemplum eo magis est notatu dignum, quod non memini, similem casum mihi unquam se obtulisse. Istud singulare phaenomenon se nimirum exerit, quando ambae formulae X et Y evadunt quadrata, ad quod ergo resolvendum ad simile artificium recurri oportet, quo formulae X et Y non ipsis quadratis aequales sed ab iis infinite parum discrepare assumuntur.

§. 60. Statuamus igitur

$$X = (a + bx + cxx)^2 + \alpha \text{ et}$$
$$Y = (a + by + cyy)^2 + \alpha,$$

ita ut pro litteris majusculis A, B, C, D, E, fiat $A = aa + \alpha$, $B = 2ab$, $C = 2ac + bb$, $D = 2bc$, $E = cc$, ubi α denotat quantitatem infinire parvam, deinceps nihilo aequalem ponendam. Hinc ergo si brevitatis gratia ponamus

$$a + bx + cxx = R \text{ et } a + by + cyy = S, \text{ erit}$$
$$\sqrt{X} = R + \frac{\alpha}{2R} \text{ et } \sqrt{Y} = S + \frac{\alpha}{2S}.$$

§. 61. Hunc igitur consideremus formam integralis primo inventam, quae erat

$$\frac{\sqrt{X} - \sqrt{Y}}{x - y} = \sqrt{[\Delta + D(x+y) + E(x+y)^2]},$$

pro qua igitur habebimus

$$\sqrt{X} - \sqrt{Y} = R - S - \frac{\alpha(R-S)}{2RS}.$$

Quia vero hic erit

$$R - S = b(x-y) + c(xx - yy), \text{ fiet}$$
$$\frac{R-S}{x-y} = b + c(x+y).$$

At posito brevitatis gratia

$$x + y = p \text{ erit } \frac{R-S}{x-y} = b + cp,$$

unde aequatio nostra erit

$$b + cp - \frac{\alpha(b+cp)}{2RS} = \sqrt{(\Delta + 2bcp + ccpp)}.$$

§. 62. Sumantur nunc utrinque quadrata, et aequatio nostra sequentem induet formam $bb - \frac{\alpha}{RS}(b+cp)^2 = \Delta$. Altiores scilicet potestates ipsius α hic ubique praetermittuntur. Hic ergo ratio nostri paradoxi manifesto in oculos incidit, quia posito $\alpha = 0$ oritur $bb = \Delta$; unde, ut Δ maneat constans arbitraria, evidens est, differentiam inter bb et Δ etiam infinite parvam statui debere; quamobrem ponamus $\Delta = bb - \alpha\Gamma$, ac obtinebitur ista aequatio penitus determinata $\frac{(b+cp)^2}{RS} = \Gamma$, sive

$$[b+c(x+y)]^2 = \Gamma(a+bx+cxx)(a+by+cyy),$$

quae forma non multum discrepat a formula supra inventa.

§. 63. Haec quidem forma magis est complicata quam solutiones §. §. 24. et seqq. inventae: Sequenti autem artificio ad formam simplicissimam redigi poterit. Cum haec fractio $\frac{RS}{(b+cp)^2}$ debeat esse quantitas constans, sit ea $= F$, ut esse debeat $F(cp+b)^2 = RS$, et quemadmodum hic posuimus $x+y=p$, ponamus porro $xy=u$, fietque

$$RS = aa + abp + ac(pp-2u) + bbu + bcpu + ccuu,$$

atque aequatio jam secundum potestates ipsius p disposita erit

$$\begin{aligned} F(cp+b)^2 = acpp + abp &+ aa \\ +bcpu &+ bbu \\ &- 2acu \\ &+ ccuu, \end{aligned}$$

ubi primo utrinque dividamus, quatenus fieri potest, per $cp+b$, ac reperietur

$$F(cp+b) = ap + bu + \frac{(a-cu)^2}{cp+b}.$$

Dividamus nunc porro per $cp+b$, quatenus fieri potest, ac fiet

$$F = \frac{a}{c} - \frac{b}{c}\cdot\frac{(a-cu)}{(cp+b)} + \frac{(a-cu)^2}{(cp+b)^2}.$$

§. 64. Hac forma inventa, si statuamus

$\frac{a-cu}{cp+b} = V$, erit $F = \frac{a}{c} - \frac{b}{c}\cdot V + VV$.

Cum igitur ista expressio aequari debeat quantitati constanti, evidens est, ipsam quantitatem V constantem esse debere, ita ut jam nostrum integrale reductum sit ad hanc formam

$$\frac{a-cu}{cp+b} = \frac{a-cxy}{c(x+y)+b} = \text{Const.}$$

Subtrahamus utrinque $\frac{a}{b}$, fietque

$$\frac{cxy+a(x+y)}{b+c(x+y)} = \text{Const.}$$

quae forma per priorem divisa producit hanc

$$\frac{a(x+y)+cxy}{cxy-a} = \text{Const.}$$

quae formae conveniunt cum supra exhibitis.

Theorema 5.

§. 65. Si in genere haec ratio designandi adhibeatur, ut sit $Z = A + Bz + Czz + Dz^3 + Ez^4$, atque valor hujus formulae integralis $\int \frac{\partial z}{\sqrt{Z}}$, ita sumtus ut evanescat posito $z = 0$, designetur hoc charactere $\Pi : z$; tum, ut fiat $\Pi : k = \Pi : x \pm \Pi : y$, necesse est ut inter quantitates k, x, y, ista relatio subsistat

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy \mp 2\sqrt{XY}}{(x-y)^2} = \frac{2A + Bk \mp 2\sqrt{AK}}{kk},$$

cujus ratio ex superioribus est manifesta. Cum enim k denotet quantitatem constantem, erit

$$\partial . \Pi : x \pm \partial . \Pi : y = 0, \text{ sive } \frac{\partial x}{\sqrt{X}} \pm \frac{\partial y}{\sqrt{Y}} = 0,$$

cujus integrale modo ante vidimus ita exprimi

$$\frac{2A + B(x+y) + 2Cxy + Dxy(x+y) + 2Exxyy \mp 2\sqrt{XY}}{(x-y)^2} = \Delta.$$

Quare cum esse debeat $\Pi : x \pm \Pi : y = \Pi : k$, manifestum est posito $y = 0$, fieri debere $\Pi : x = \Pi : k$ ideoque $x = k$ unde constans indefinita Δ eodem prorsus modo definitur, uti est exhibita.

Corollarium 1.

§. 66. Hinc si formula $\Pi : z$ exprimat arcum cujuspiam lineae curvae abscissae sive applicatae Z respondentem, in

hac curva omnes arcus eodem modo inter se comparare licebit, quo arcus circulares inter se comparantur; quandoquidem, propositis duobus arcubus $\Pi : x$ et $\Pi : y$, tertius arcus $\Pi : k$ semper exhiberi poterit vel summae vel differentiae eorum arcuum aequalis.

Corollarium 2.

§. 67. Ita si in hac forma $\Pi : k = \Pi : x + \Pi : y$ statuatur $y = x$, prodibit $\Pi\, k = 2\, \Pi : x$; sicque arcus reperitur duplo alterius aequalis. At vero si in nostra forma faciamus $y = x$, tam numerator quam denominator in nihilum abeunt. Ut autem ejus verum valorem eruamus, utamur aequatione primum (§. 38.) inventa

$$\frac{\sqrt{X} - \sqrt{Y}}{x - y} = \sqrt{[\Delta + D(x+y) + E(x+y)^2]},$$

et jam in membro sinistro spectetur y ut constans; ipsi x vero valorem tribuamus infinite parum discrepantem, sive, quod eodem redit, loco numeratoris et denominatoris eorum differentialia substituantur, sumta sola x variabili, hocque modo pro casu $y = x$ membrum sinistrum evadit $\frac{X'}{2\sqrt{X}}$, ubi est

$$X' = B + 2\,C\,x + 3\,D\,x\,x + 4\,E\,x^3.$$

Nunc ergo sumtis quadratis habebitur

$$\frac{X'X'}{4X} = \Delta + 2\,D\,x + 4\,E\,x\,x,$$

existente Δ ut ante $= \frac{2A + Bk - 2\sqrt{AK}}{kk}$.

Corollarium 3.

§. 68. Verum sine his ambagibus duplicatio arcus ex altera forma $\Pi : k = \Pi : x - \Pi\, y$ deduci potest, ponendo $y = k$, siquidem hinc fit $\Pi : x = 2\,\Pi : k$, pro quo ergo casu relatio inter x et k hac aequatione exprimetur

$$\frac{2A+B(k+x)+2Ckx+Dkx(k+x)+2Ekkxx+2\surd KX}{(x-k)^2}$$
$$=\frac{2A+Bk+2\surd AK}{kk}.$$

Facile autem patet, quomodo hic ad triplicationem, quadruplicationem et quamlibet multiplicationem arcuum progredi debeat, quod argumentum olim fusius sum tractatus.

Theorema 6.

§. 69. Si in formis supra inventis ponatur tam $B=0$ quam $D=0$, ut sit

$$X=A+Cxx+Ex^4$$
$$Y=A+Cyy+Ey^4 \text{ et}$$
$$K=A+Ckk+Ek^4;$$

tum si ista aequatio $\frac{\partial x}{\surd X}\pm\frac{\partial y}{\surd Y}=0$ ita integretur, ut posito $y=0$ fiat $x=k$, tum aequatio integralis erit

$$\frac{A+Cxy+Exxyy\mp\surd XY}{(x-y)^2}=\frac{A\mp\surd AK}{kk}.$$

Corollarium 1.

§. 70. Hic notari meretur, istum casum adhuc alio modo ex forma generali deduci posse, si scilicet sumatur $A=0$ et $E=0$, tum enim prodit ista aequatio differentialis

$$\frac{\partial x}{\surd(Bx+Cxx+Dx^3)}\pm\frac{\partial y}{\surd(By+Cyy+Dy^3)}=0,$$

cujus ergo integrale erit

$$\frac{2B(x+y)+2Cxy+Dxy(x+y)\mp2\surd(Bx+Cxx+Dx^3)(By+Cyy+Dy^3)}{(x-y)^2}$$
$$=\frac{Bk}{kk}=\frac{B}{k},$$

ubi valor constantis admodum simplex evasit. Nunc in his formulis loco x et y scribamus xx et yy, at vero loco litterarum B et D scribamus A et E, fietque aequatio differentialis

$$\frac{\partial x}{\sqrt{(A+Cxx+Dx^4)}} \pm \frac{\partial y}{\sqrt{(A+Cyy+Dy^4)}} = 0,$$

cujus ergo integrale etiam hoc modo exprimetur

$$\frac{A(xx+yy)+2Cxxyy+Dxxyy(xx+yy)\mp 2xy\sqrt{XY}}{(xx-yy)^2} = \frac{A}{kk}.$$

Corollarium 2.

§. 71. Ecce ergo hac ratione pervenimus ad alium integralis formam non minus notabilem priore, atque adeo nunc ex earum combinatione formula radicalis $\sqrt{XY}$ eliminari poterit, quandoquidem ex posteriore fit

$$\mp 2\sqrt{XY} = \frac{A(xx-yy)^2}{kkxy} - \frac{A(xx+yy)}{xy} - 2Cxy$$
$$- Exy(xx+yy),$$

qui valor in priore substitutus conducit ad hanc aequationem rationalem

$$2A + 2Cxy + 2Exxyy$$
$$+ \frac{A(xx-yy)^2}{kkxy} - \frac{A(xx+yy)}{xy} - 2Cxy - Exy(xx+yy)$$
$$= \frac{2A(x-y)^2}{kk} \mp \frac{2(x-y)^2\sqrt{AK}}{kk},$$

quae porro reducta et per $(x-y)^2$ divisa revocatur ad hanc formam

$$\frac{2A \mp 2\sqrt{AK}}{kk} = \frac{A(x+y)^2}{kkxy} - Exy - \frac{A}{xy},$$

sive ad hanc

$$\frac{A}{kk}\cdot(xx+yy-kk) - Exxyy \pm \frac{2xy\sqrt{AK}}{kk} = 0,$$

quae egregie convenit cum aequatione canonica, qua olim sum usus: scilicet

$$0 = \alpha + \gamma(xx+yy) + 2\delta xy + \zeta xxyy,$$

si quidem est

$$\alpha = -A,\ \gamma = +\frac{A}{kk},\ 2\delta = \pm\frac{2\sqrt{AK}}{kk},\ \zeta = -E.$$

Corollarium 3.

§. 72. Methodo posteriore, qua hic usi sumus ad hanc aequationem integrandam, aequatio multo generalior tractari poterit, ubi in formulis radicalibus potestates usque ad sextam dimensionem assurgunt. Namque si tantum statuamus $A = 0$, ut sit aequatio

$$\frac{\partial x}{\sqrt{x(B+Cx+Dxx+Ex^3)}} + \frac{\partial y}{\sqrt{y(B+Cy+Dyy+Ey^3)}} = 0,$$

cujus integrale est

$$\frac{B(x+y)+2Cxy+Dxy(x+y)+2Exxyy}{(x-y)^2} \mp \frac{2\sqrt{xy(B+Cx+Dxx+Ex^3)(B+Cy+Dyy+Ey^3)}}{(x-y)^2} = \frac{B}{kk}.$$

Quod si jam hic loco x et y scribamus xx et yy, aequatio differentialis fiet

$$\frac{\partial x}{\sqrt{(B+Cxx+Dx^4+Ex^6)}} + \frac{\partial y}{\sqrt{(B+Cyy+Dy^4+Ey^6)}} = 0,$$

ejus ergo integrale erit

$$\frac{B(xx+yy)+2Cxxyy+Dxxyy(xx+yy)+2Ex^4y^4}{(xx-yy)^2} \mp \frac{2xy\sqrt{(B+Cxx+Dx^4+Ex^6)(B+Cyy+Dy^4+Ey^6)}}{(xx-yy)^2} = \frac{B}{kk}.$$

Nunc autem ostendamus, quomodo ope methodi illustris *de la Grange* idem integrale impetrari queat.

Analysis

pro integratione aequationis differentialis

$$\frac{\partial x}{\sqrt{X}} + \frac{\partial y}{\sqrt{Y}} = 0,$$

existente

$$X = B + Cxx + Dx^4 + Ex^6 \text{ et}$$
$$Y = B + Cyy + Dy^4 + Ey^6.$$

§. 73. Posito igitur
$$\frac{\partial x}{\sqrt{X}} = \partial t \text{ erit } \frac{\partial y}{\sqrt{Y}} = \mp \partial t,$$
hincque sumtis quadratis
$$\frac{\partial x^2}{\partial t^2} = X \text{ et } \frac{\partial y^2}{\partial t^2} = Y,$$
hinc formentur hae aequationes
$$\frac{xx\partial x^2}{\partial t^2} = xxX \text{ et } \frac{yy\partial y^2}{\partial t^2} = yy$$
Jam introducantur duae novae variabiles p et q, ut sit
$$xx + yy = 2p \text{ et } xx - yy = 2q,$$
ex quo fit $x\partial x + y\partial y = \partial p$, et $x\partial x - y\partial y = \partial q$, hincque $xx\partial x^2 - yy\partial y^2 = \partial p\partial q$; quamobrem habebimus
$$\frac{\partial p\partial q}{\partial t^2} = xxX - yyY,$$
quae aequatio dividatur per $xx - yy = 2q$, prodibitque
$$\frac{\partial p\partial q}{2q\partial t^2} = \frac{xxX - yyY}{xx - yy},$$
quae forma, valoribus pro X et Y substitutis, dabit
$$\frac{\partial p\partial q}{2q\partial t^2} = B + 2Cp + D(3pp + qq) + 4E(p^3 + pqq).$$

§. 74. Nunc porro aequationes $\frac{\partial x^2}{\partial t^2}$ et $\frac{\partial y^2}{\partial t^2}$ differentiatae dabunt
$$\frac{2\partial\partial x}{\partial t^2} = X' \text{ et } \frac{2\partial\partial y}{\partial t^2} = Y'.$$
Ex priore fit $\frac{2x\partial\partial x}{\partial t^2} = xX'$, cui addatur $\frac{2\partial x^2}{\partial t^2} = 2X$, ut prodeat
$$\frac{2(x\partial\partial x + \partial x^2)}{\partial t^2} = \frac{2\partial . x\partial x}{\partial t^2} = xX' + 2X.$$
Simili modo erit $\frac{2\partial . y\partial y}{\partial t^2} = yY' + 2Y$, quae duae aequationes invicem additae dabunt
$$\frac{2\partial . \partial p}{\partial t^2} = \frac{2\partial\partial p}{\partial t^2} = xX' + yY' + 2(X + Y).$$
Substitutis autem valoribus et facta substitutione respectu litterarum

p et q, reperitur

$$2X + 2Y = 4B + 4Cp + 4D(pp + qq) + 4Ep(pp + 3qq).$$

Deinde ob

$$xX' = 2Cxx + 4Dx^4 + 6Ex^6 \text{ et}$$

$$yY' = 2Cyy + 4Dy^4 + 6Ey^6 \text{ erit}$$

$$xX' + yY' = 4Cp + 8D(pp + qq) + 12Ep(pp + 3qq),$$

ex quibus conjunctis fit

$$\frac{2\partial\partial p}{\partial t^2} = 4B + 8Cp + 12D(pp + qq) + 16Ep(pp + 3qq).$$

§. 75. Ab hac formula subtrahatur supra inventa $\frac{\partial p \partial q}{2q\partial t^2}$ quater sumta, ac remanebit

$$\frac{2\partial\partial p}{\partial t^2} - \frac{2\partial p \partial q}{q\partial t^2} = 8Dqq + 32Epqq.$$

Nunc utrinque multiplicetur per $\frac{\partial p}{qq}$, et prodibit

$$\frac{1}{\partial t^2} \cdot \left(\frac{2\partial p \partial\partial p}{qq} - \frac{2\partial p^2 \partial q}{q^3}\right) = 8D\partial p + 32Ep\partial p,$$

cujus integrale sponte se offert ita expressum

$$\frac{\partial p^2}{qq\partial t^2} = 4\Delta + 8Dp + 16Epp,$$

ideoque extracta radice

$$\frac{\partial p}{q\partial t} = 2\sqrt{(\Delta + 2Dp + 4Epp)}.$$

§. 76. Cum nunc sit

$$\frac{\partial p}{\partial t} = x\sqrt{X} \mp y\sqrt{Y} \text{ et } 2q = xx - yy,$$

facta substitutione orietur haec aequatio

$$\frac{x\sqrt{X} \mp y\sqrt{Y}}{xx - yy} = \sqrt{[\Delta + D(xx + yy) + E(xx + yy)^2]},$$

quae sumtis quadratis reducetur ad istam formam

$$\frac{xxX + yyY \mp 2xy\sqrt{XY}}{(xx - yy)^2} = \Delta + D(xx + yy) + E(xx + yy)^2.$$

Est vero

$$xxX + yyY = B(xx + yy) + C(x^4 + y^4) + D(x^6 + y^6) + E(x^8 + y^8),$$

hincque pervenietur ad hanc aequationem

$$\frac{B(xx+yy) + C(x^4+y^4) + Dxxyy(xx+yy) + 2Ex^4y^4 \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \Delta.$$

§. 77. Sumamus nunc ut supra constantem Δ ita, ut posito

$y = 0$ fiat $x = k$ et $X = K = B + Ckk + Dk^4 + Ek^6$,

et aequatio integralis induet hanc formam

$$\frac{B(xx+yy) + C(x^4+y^4) + Dxxyy(xx+yy) + 2Ex^4y^4 \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \frac{B + Ckk}{kk},$$

quae aliquanto simplicior evadit, si utrinque subtrahamus C: erit enim

$$\frac{B(xx+yy) + 2Cxxyy + Dxxyy(xx+yy) + 2Ex^4y^4 \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \frac{B}{kk},$$

quae egregie convenit cum integrali supra §. 72. exhibito.

§. 78. Hic casus notatu dignus se offert, dum $B = 0$, tum autem aequatio differentialis ita se habebit

$$\frac{\partial x}{x\sqrt{(C + Dxx + Ex^4)}} \pm \frac{\partial y}{y\sqrt{(C + Dyy + Ey^4)}} = 0,$$

cujus ergo integrale per constantem Δ expressum erit

$$\frac{C(x^4+y^4) + Dxxyy(xx+yy) + 2Ex^4y^4 \mp 2xy\sqrt{XY}}{(xx-yy)^2} = \Delta.$$

Hoc autem casu integratio non ita determinari potest, ut posito $y = 0$ fiat $x = k$, quia integrale posterioris membri hoc casu manifesto abit in infinitum; quamobrem alio modo integrationem determinari conveniet, veluti ut posito $y = b$ fiat $x = a$, tum autem erit ista constans

$$\Delta = \frac{C(a^4 + b^4) + Da^2b^2(aa + bb) + 2Ea^4b^4 \mp 2ab\sqrt{AB}}{(aa + bb)^2},$$ existente

$$A = C + Daa + Ea^4 \text{ et } B = C + Dbb + Eb^4.$$

Conclusio.

§. 79. Qui processum analyseos hic usitatae comparare voluerit cum methodo, qua illustris *de la Grange* usus est in *Miscellan. Taur. Tom. IV.* facile perspiciet, eam multo facilius ad scopum desideratum perducere atque multo commodius ad quosvis casus applicari posse. Introduxerat autem Vir Ill. in calculum formulam $\frac{\partial t}{T}$, cujus loco hic simplici elemento ∂t sumus usi, ac deinceps quantitatem T tanquam functionem litterarum p et q spectavit, quae positio satis difficiles calculos postulavit, dum nostra methodo longe concinnius easdem integrationes investigare licuit. Quanquam autem nullum est dubium, quin ista analyseos species insigne incrementum polliceatur, tamen nondum patet, quemadmodum ad alias integrationes ea accommodari queat, praeter hos ipsos casus, quos hic tractavimus et quos olim ex aequatione canonica derivaveram.

2.) Methodus succinctior comparationes quantitatum transcendentium in forma $\int\frac{P\partial z}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}$ contentarum inveniendi.

M. S. Academiae exhib. die 3 *Nov.* 1777.

In Capite VI. Sect. II. Institutionum mearum Calculi Integralis Tom. I. insignes tradidi comparationes inter quantitates maxime transcendentes, ad quam deductus eram methodo penitus indirecta. Postquam igitur non ita pridem illustris *de la Grange* methodum maxime ingeniosam excogitasset easdem comparationes inveniendi, totum hoc argumentum multo succinctius et elegantius tractari poterit, quam mihi quidem tum temporis licebat, unde sequentia Supplementa Geometris haud displicebunt.

Hypothesis 1.

§. 80. Denotet hic perpetuo character $\Pi : z$ valorem formulae integralis $\int\frac{\partial z}{\sqrt{(\alpha+\beta z+\gamma zz+\delta z^3+\varepsilon z^4)}}$, ita sumtae ut evanescat posito $z = 0$. Ponatur autem brevitatis gratia $\alpha +$
Fig. 13. $\beta z + \gamma zz + \delta z^3 + \varepsilon z^4 = Z$, ita ut sit $\Pi : z = \int\frac{\partial z}{\sqrt{Z}}$. Tum vero concipiatur super axe oz exstructa ejusmodi curva O Z, cujus singuli arcus O Z abscissis $oz = z$ respondentes exprimantur per formulam $\Pi : z = \int\frac{\partial z}{\sqrt{Z}}$; atque haec curva ista insigni proprietate erit praedita, ut sumto in ea pro lubitu arcu quocunque F G, a quovis alio puncto X semper arcus X Y illi arcui F G aequalis geometrice abscindi possit, cujus demonstrationem solutio sequentis problematis suppeditabit.

Problema 1.

Si in curva modo descripta proponatur arcus quicunque $F\ G$, innumerabiles alios arcus $X\ Y$ in eadem curva geometrice assignare, qui singuli eidem arcui $F\ G$ sint aequales.

Solutio.

§. 81. Ductis ex punctis F et G ad axem $o\,z$ applicatis $F\,f$ et $G\,g$, vocentur abscissae $o\,f = f$ et $o\,g = g$, eruntque arcus $O\,F = \Pi : f$ et $O\,G = \Pi : g$, unde longitudo arcus propositi $F\ G$ erit $= \Pi : g - \Pi : f$. Simili modo pro quovis arcu quaesito $X\ Y$ vocentur abscissae $o\,x = x$ et $o\,y = y$, eruntque arcus $O\,X = \Pi : x$ et $O\,Y = \Pi : y$, ideoque arcus $X\ Y = \Pi : y - \Pi : x$, qui cum aequalis esse debeat arcui $F\ G$, habebitur ista aequatio $\Pi : y - \Pi : x = \Pi : g - \Pi : f$, cui satisfieri oportet.

§. 82. Quoniam puncta F et G considerantur ut fixa, dum puncta X et Y per totam curvam variari possunt, differentiatio nobis praebebit hanc aequationem $\partial . \Pi : y - \partial . \Pi : x = 0$. Quare cum sit per hypothesin

$$\Pi : x = \int \frac{\partial x}{\sqrt{X}} \text{ et } \Pi : y = \int \frac{\partial y}{\sqrt{Y}},$$

existente

$$X = \alpha + \beta x + \gamma x x + \delta x^3 + \varepsilon x^4 \text{ et}$$
$$Y = \alpha + \beta y + \gamma y y + \delta y^3 + \varepsilon y^4,$$

solutio problematis perducta est ad hanc aequationem differentialem $\frac{\partial y}{\sqrt{Y}} - \frac{\partial x}{\sqrt{X}} = 0$.

§. 83. Hic jam methodum ill. *de la Grangé* in subsidium vocantes statuamus $\frac{\partial x}{\sqrt{X}} = \partial t$, eritque $\frac{\partial y}{\sqrt{Y}} = \partial t$. Hic scili-

cet novum elementum ∂t in calculum introducimus, quod in sequentibus differentiationibus ut constans tractetur; tum igitur habebimus

$$\frac{\partial x}{\partial t} = \sqrt{X} \text{ et } \frac{\partial y}{\partial t} = \sqrt{Y}.$$

Quod si ergo porro statuamus $y + x = p$ et $y - x = q$, habebimus hinc

$$\frac{\partial p}{\partial t} = \sqrt{Y} + \sqrt{X} \text{ et } \frac{\partial q}{\partial t} = \sqrt{Y} - \sqrt{X},$$

quarum formularum productum praebet

$$\frac{\partial p \partial q}{\partial t^2} = Y - X.$$

Valoribus ergo loco Y et X substitutis erit

$$\frac{2 p\, \partial q}{\partial t^2} = \beta (y - x) + \gamma (y^2 - x^2) + \delta (y^3 - x^3) + \varepsilon (y^4 - x^4).$$

Quare cum sit

$$y = \frac{p+q}{2} \text{ et } x = \frac{p-q}{2} \text{ erit}$$

$$y - x = q,\ y^2 - x^2 = pq,\ y^3 - x^3 = \tfrac{1}{4} q (3pp + qq) \text{ et}$$
$$y^4 - x^4 = \tfrac{1}{2} pq (pp + qq),$$

quibus substitutis factaque divisione per q habebitur

$$\frac{\partial p \partial q}{q \partial t^2} = \beta + \gamma p + \tfrac{1}{4}\delta (3pp + qq) + \tfrac{1}{2}\varepsilon p (pp + qq),$$

cujus aequationis plurimus erit usus in sequenti calculo.

§. 84. Jam sumtis quadratis primae aequationes dabunt

$$\frac{\partial x^2}{\partial t^2} = X \text{ et } \frac{\partial y^2}{\partial t^2} = Y,$$

quae denuo differentientur, quem in finem ponamus brevitatis gratia

$$\partial X = X' \partial x \text{ et } \partial Y = Y' \partial y;$$

atque hinc nanciscemur

$$\frac{2\partial\partial x}{\partial t^2} = X' \text{ et } \frac{2\partial\partial y}{\partial t^2} = Y',$$

quibus additis erit

$$\frac{2\partial\partial p}{\partial t^2} = X' + Y'.$$

Cum igitur sit

$$X' = \beta + 2\gamma x + 3\delta x x + 4\varepsilon x^3 \text{ et}$$
$$Y' = \beta + 2\gamma y + 3\delta y y + 4\varepsilon y^3, \text{ erit}$$
$$\frac{2\partial\partial p}{\partial t^2} = 2\beta + 2\gamma(x+y) + 3\delta(x^2+y^2) + 4\varepsilon(x^3+y^3).$$

Introducendo igitur litteras p et q ut ante, fiet

$$x + y = p,\quad x^2 + y^2 = \tfrac{1}{2}(pp + qq),$$
$$x^3 + y^3 = \tfrac{1}{4}p(pp + 3qq),$$

sicque ista aequatio hanc induet formam

$$\frac{2\partial\partial p}{\partial t^2} = 2\beta + 2\gamma p + \tfrac{3}{2}\delta(pp+qq) + \varepsilon p(pp + 3qq).$$

§. 85. Ab hac jam postrema aequatione subtrahatur praecedens bis sumta, ac remanebit

$$\frac{2\partial\partial p}{\partial t^2} - \frac{2\partial p\partial q}{q\partial t^2} = \delta q q + 2\varepsilon p q q.$$

Hinc per qq dividendo habebimus

$$\frac{1}{\partial t^2}\left(\frac{2\partial\partial p}{qq} - \frac{2\partial p\partial q}{q^3}\right) = \delta + 2\varepsilon p,$$

cujus utrumque membrum manifesto integrationem admittit, si ducatur in elementum ∂p. Hoc enim facto aequatio integralis erit

$$\frac{\partial p^2}{qq\partial t^2} = C + \delta p + \varepsilon pp.$$

§. 86. Initio autem vidimus esse $\frac{\partial p}{\partial t} = \sqrt{X} + \sqrt{Y}$, hincque statim pervenimus ad aequationem integralem algebraicam hanc

$$\frac{(\sqrt{X}+\sqrt{Y})^2}{qq} = C + \delta p + \varepsilon pp.$$

Quare cum sit $p = x + y$ et $q = y - x$, haec aequatio evoluta fiet

$$\frac{X + Y + 2\sqrt{XY}}{(y - x)^2} = C + \delta(x + y) + \varepsilon(x + y)^2,$$

ubi constantem per integrationem ingressam secundum indolem problematis ita definiri oportet, ut dum punctum X incidit in punctum F, punctum Y in ipsum punctum G cadat, sive ut facto $x = f$ fiat $y = g$.

§. 87. Cum jam sit

$$X + Y = 2\alpha + \beta(x + y) + \gamma(x^2 + y^2) + \delta(x^3 + y^3) + \varepsilon(x^4 + y^4),$$

si terminos $\delta(x + y) + \varepsilon(x + y)^2$ in alteram partem transferimus, perveniemus ad hanc aequationem

$$\frac{2\alpha + \beta(x + y) + \gamma(x^2 + y^2) + \delta xy(x + y) + 2\varepsilon xxyy + 2\sqrt{XY}}{(y - x)^2} = C.$$

Subtrahamus autem insuper utrinque γ, et loco $C - \gamma$ scribamus Δ, hocque modo nostra aequatio reducetur ad hanc formam satis concinnam

$$\frac{2\alpha + \beta(x + y) + 2\gamma xy + \delta xy(x + y) + 2\varepsilon xxyy + 2\sqrt{XY}}{(y - x)^2} = \Delta.$$

§. 88. Quia nunc Δ ita determinari debet, ut sumto $x = f$ fiat $y = g$, si secundum analogiam statuamus

$$\alpha + \beta f + \gamma ff + \delta f^3 + \varepsilon f^4 = F \text{ et}$$
$$\alpha + \beta g + \gamma gg + \delta g^3 + \varepsilon g^4 = G,$$

erit ista constans Δ ita expressa

$$\Delta = \frac{2\alpha + \beta(f + g) + 2\gamma fg + \delta fg(f + g) + 2\varepsilon ffgg + 2\sqrt{FG}}{(g - f)^2}.$$

Hac igitur aequatione inventa, si ipsi x pro lubitu tribuatur valor quicunque, inde elici poterit valor ipsius y, ita ut alter terminus X arcus quaesiti X Y pro arbitrio assumi possit. Verum

facile patet, istam determinationem in calculos perquam molestos praecipitare, quandoquidem aequatio inventa quadratis sumendis ab irrationalitate $\sqrt{XY}$ liberari deberet. Sequenti autem modo ista investigatio sublevari poterit.

§. 89. Quoniam ista formula

$$2\alpha + \beta(x+y) + 2\gamma xy + \delta xy(x+y) + 2\varepsilon xxyy$$

essentialiter in calculum ingreditur, ejus loco brevitatis gratia scribamus hunc characterem $[x, y]$, cujus ergo valor erit cognitus, etiam si loco x et y aliae litterae accipiantur. Hoc igitur modo aequatio inventa ita referri poterit

$$\frac{[x, y] + 2\sqrt{XY}}{(y-x)^2} = \frac{[f, g] + 2\sqrt{FG}}{(g-f)^2},$$

quae ergo aequatio exprimit relationem inter bina ordinata x et y, ut problemati satisfiat, hoc est, ut fiat

$$\Pi : y - \Pi : x = \Pi : g - \Pi : f.$$

Quare cum hic etiam sequatur

$$\Pi : y - \Pi : g = \Pi : x - \Pi : f,$$

aequatio hinc ista exsurget

$$\frac{[g, y] + 2\sqrt{GY}}{(y-g)^2} = \frac{[f, x] + 2\sqrt{FX}}{(x-f)^2}.$$

§. 90. Ex hac jam aequatione cum priore conjuncta facile eliminari poterit formula radicalis $\sqrt{Y}$, sicque aequatio habebitur tantum litteram y tanquam incognitam involvens, unde ejus valor haud difficulter definiri potest. Calculum autem hunc instituenti patebit, tantum ad aequationem quadraticam perveniri, ita ut bini valores pro puncto Y reperiantur, quemadmodum rei natura postulat, dum sumto puncto X alterum punctum Y tam dextrorsum quam sinistrorsum cadere poterit. Hinc autem calculo fusius non immoramur, quandoquidem hic potissimum est propo-

situm, totam hujus problematis solutionem per methodum directam a priori repetere.

Hypothesis 2.

§. 91. Constituta super axe $o\,z$ curva O Z in priori
Fig 14. hypothesi descripta, concipiatur super eodem axe alia curva insuper descripta $\mathfrak{O}\,\mathfrak{Z}$, ita comparata, ut abscissae $o\,z = z$ respondeat arcus $\mathfrak{O}\,\mathfrak{Z} = \phi : z$, ita ut sit

$$\phi : z = \int \frac{\partial z\,(\mathfrak{A} + \mathfrak{B}\,z + \mathfrak{C}\,z\,z + \mathfrak{D}\,z^3 + etc.)}{\sqrt{Z}},$$

integrali hoc pariter ita sumto ut evanescat posito $z = 0$, existente ut ante

$$Z = \alpha + \beta z + \gamma\,z\,z + \delta\,z^3 + \varepsilon\,z^4.$$

Pro numeratore autem ponamus brevitatis gratia

$$\mathfrak{A} + \mathfrak{B}\,z + \mathfrak{C}\,z\,z + \mathfrak{D}\,z^3 + \text{etc.} = \mathfrak{Z},$$

ita ut sit $\phi : z = \int \frac{\mathfrak{Z}\,\partial z}{\sqrt{Z}}$.

§. 92. Ista jam curva hac ratione descripta hac insigni proprietate erit praedita, ut, si in priore curva rescissi fuerint arcus F G et X Y inter se aequales, productis iisdem applicatis in nova curva, arcuum hoc modo rescissorum $\mathfrak{F}\,\mathfrak{G}$ et $\mathfrak{X}\,\mathfrak{Y}$ differentia vel algebraice vel saltem per logarithmos et arcus circulares assignari possit, cujus rei veritatem solutio sequentis problematis demonstrabit.

Problema 2.

Si in curva secundum primam hypothesin descripta abscissi fuerint duo arcus aequales FG et XY, iisque in curva modo descripta respondeant arcus $\mathfrak{F}\,\mathfrak{G}$ *et* $\mathfrak{X}\,\mathfrak{Y}$, *quibus scilicet eaedem abscissae in axe conveniant, differentiam inter hos binos arcus investigare.*

Solutio.

§. 93. Quia igitur hic quaeritur differentia inter arcus $\mathfrak{F}\mathfrak{G}$ et $\mathfrak{X}\mathfrak{Y}$, ponatur ea $= V$, quae ergo spectari poterit tanquam certa functio ipsarum x et y, si quidem puncta $\mathfrak{F}$ et $\mathfrak{G}$ tanquam fixa consideramus. Cum igitur sit arcus

$\mathfrak{F}\mathfrak{G} = \Phi : g - \Phi : f$ et arcus

$\mathfrak{X}\mathfrak{Y} = \Phi : y - \Phi : x$,

habebimus

$$\Phi : y - \Phi : x = \Phi : g - \Phi : f + V,$$

unde differentiando habebimus

$$\frac{\mathfrak{Y}\partial y}{\sqrt{Y}} - \frac{\mathfrak{X}\partial x}{\sqrt{X}} = \partial V,$$

quia litteras f et g pro constantibus habemus.

§. 94. Ponamus nunc ut supra factam est

$$\frac{\partial x}{\sqrt{X}} = \frac{\partial y}{\sqrt{Y}} = \partial t,$$

et haec aequatio induet istam formam

$$(\mathfrak{Y} - \mathfrak{X})\partial t = \partial V.$$

Verum in solutione primi problematis deducti fuimus ad hanc aequationem finalem

$$\frac{\partial p^2}{qq\partial t^2} = C + \delta p + \varepsilon pp,$$

unde fit

$$\frac{\partial p}{\partial t} = \sqrt{(C + \delta p + \varepsilon pp)} = \sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)},$$

atque hinc colligimus

$$\partial t = \frac{\partial p}{q\sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)}},$$

ubi est $p = x + y$ et $q = y - x$. Hoc ergo valore inducto aequatio differentialis resolvenda est

$$\partial V = \frac{(\mathfrak{Y} - \mathfrak{X})\partial p}{q\sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)}},$$

ubi est

$$\mathfrak{X} = \mathfrak{A} + \mathfrak{B}x + \mathfrak{C}xx + \mathfrak{D}x^3 + \text{etc.}$$

similique modo

$$\mathfrak{Y} = \mathfrak{A} + \mathfrak{B}y + \mathfrak{C}yy + \mathfrak{D}y^3 + \text{etc.},$$

quousque libuerit continuando.

§. 95. Quod si jam hos valores substituamus, habebimus

$$\mathfrak{Y} - \mathfrak{X} = \mathfrak{B}(y-x) + \mathfrak{C}(y^2 - x^2) + \mathfrak{D}(y^3 - x^3) + \mathfrak{E}(y^4 - x^4) + \text{etc.}$$

unde si loco x et y introducamus quantitates p et q, ob $x = \frac{p-q}{2}$ et $y = \frac{p+q}{2}$, orientur sequentes valores

$$y - x = q,\ y^2 - x^2 = pq,\ y^3 - x^3 = \tfrac{1}{4}q(3pp+qq),$$
$$y^4 - x^4 = \tfrac{1}{2}pq(pp+qq),\ y^5 - x^5 = \tfrac{1}{16}q(5p^4 + 10ppqq + q^4).$$

§. 96. Quantitas ergo V per sequentes formulas integrales secundum numerum litterarum $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$, etc. determinatur

$$V = \mathfrak{B}\int \frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} + \mathfrak{C}\int \frac{p\,\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$$
$$+ \tfrac{1}{4}\mathfrak{D}\int \frac{(3pp+qq)\,\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} + \tfrac{1}{2}\mathfrak{E}\int \frac{p(pp+qq)\,\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$$
$$+ \tfrac{1}{16}\mathfrak{F}\int \frac{(5p^4+10ppqq+q^4)\,\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}} + \text{etc.}$$

Quarum formularum duae priores jam absolute exhiberi possunt, sive algebraice, quod evenit si $\varepsilon = 0$, sive per logarithmos, si valor ipsius ε fuerit positivus, sive per arcus circulares, si valor ipsius ε fuerint negativus. Reliquae vero formulae exigunt relationem inter p et q, quam deinceps investigabimus. Hic tantum notetur, potestates solas pares ipsius q in has formulas ingredi.

§. 97. Hic autem littera Δ eundem valorem constantem designat, quem supra jam definivimus, qui erat

$$\Delta = \frac{2\alpha + \beta(f+g) + 2\gamma fg + \delta fg(f+g) + 2\varepsilon ffgg + 2\sqrt{FG}}{(g-f)^2}.$$

Praeterea vero cum esse debeat

$$\Phi : y - \Phi : x = \Phi : g - \Phi : f + V,$$

evidens est, casu quo $x = f$ et $y = g$ fieri debere $V = 0$; quamobrem formulae illae integrales pro V inventae ita capi debebunt, ut posito $p = f + g$ et $q = g - f$ valor ipsius V evanescat.

Analysis pro investiganda relatione inter p et q.

§. 98. Quia jam invenimus aequationem finitam inter x et y, ex ea quoque ponendo $y = \frac{p+q}{2}$ et $x = \frac{p-q}{2}$ relatio inter litteras p et q derivari posset; verum hoc calculos nimis taediosos postularet, quamobrem aliam viam ineamus istam relationem ex formulis differentialibus deducendi. Cum enim sit $\frac{\partial p}{\partial q} = \frac{\partial y + \partial x}{\partial y - \partial x}$, ob proportionem

$$\partial x : \partial y = \sqrt{X} : \sqrt{Y} \text{ erit } \frac{\partial p}{\partial q} = \frac{\sqrt{Y} + \sqrt{X}}{\sqrt{Y} - \sqrt{X}};$$

supra autem invenimus esse

$$\frac{\sqrt{Y} + \sqrt{X}}{q} = \sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)},$$

ubi Δ eandem denotat constantem, quam modo ante definivimus.

§. 99. Nunc igitur fractio pro $\frac{\partial p}{\partial q}$ inventa supra et infra multiplicetur per $\sqrt{Y} + \sqrt{X}$, et cum sit

$$(\sqrt{Y} + \sqrt{X})^2 = qq(\Delta + \gamma + \delta p + \varepsilon pp),$$

habebimus hanc aequationem

$$\frac{\partial p}{\partial q} = \frac{qq(\Delta + \gamma + \delta p + \varepsilon pp)}{Y - X},$$

cujus denominatorem jam supra §. 83. evolvimus, ubi invenimus esse

$$Y - X = \beta q + \gamma p q + \tfrac{1}{4}\delta q(3pp + qq) + \tfrac{1}{2}\varepsilon pq(pp + qq),$$

quo valore substituto erit

$$\frac{\partial p}{\partial q} = \frac{q(\Delta + \gamma + \delta p + \varepsilon pp)}{\beta + \gamma p + \frac{1}{4}\delta(3pp + qq) + \frac{1}{2}\varepsilon p(pp + qq)},$$

quae reducitur ad hanc formam

$$2q\partial q = \frac{[2\beta + 2\gamma p + \frac{1}{2}\delta(3pp + qq) + \varepsilon p(pp + qq)]\partial p}{\Delta + \gamma + \delta p + \varepsilon pp}.$$

100. Transferamus terminos qui continent qq a dextra in sinistram partem ut obtineamus hanc aequationem

$$2q\partial q - \frac{qq\partial p(\frac{1}{2}\delta + \varepsilon p)}{\Delta + \gamma + \delta p + \varepsilon pp} = \frac{(2\beta + 2\gamma p + \frac{3}{2}\delta pp + \varepsilon p^3)\partial p}{\Delta + \gamma + \delta p + \varepsilon pp}.$$

Membrum hujus aequationis sinistrum integrabile reddi potest, si per certam functionem ipsius p, quae sit $= \Pi$, multiplicetur, quando fuerit

$$\frac{\partial \Pi}{\Pi} = -\frac{\partial p(\frac{1}{2}\delta + \varepsilon p)}{\Delta + \gamma + \delta p + \varepsilon pp},$$

quae aequatio integrata dat

$$l\Pi = -\tfrac{1}{2}l(\Delta + \gamma + \delta p + \varepsilon pp).$$

Sicque erit multiplicator iste

$$\Pi = \frac{1}{\sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)}};$$

tum autem integrale quaesitum erit

$$\frac{qq}{\sqrt{(\Delta + \gamma + \delta p + \varepsilon pp)}} = \int \frac{(2\beta + 2\gamma p + \frac{3}{2}\delta pp + \varepsilon p^3)\partial p}{(\Delta + \gamma + \delta p + \varepsilon pp)^{\frac{3}{2}}}.$$

§. 101. Hoc postremum integrale manifesto continet formam $\frac{pp}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$, quippe cujus differentiale est

$$\frac{(2\Delta p+2\gamma p+\frac{3}{2}\delta pp+\varepsilon p^3)\partial p}{(\Delta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}};$$

quare integrale ita potest repraesentari

$$\frac{qq}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{pp}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$$
$$+\int\frac{(2\beta-2\Delta p)\partial p}{(\Delta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}},$$

quod postremum integrale statuatur $=\frac{m+np}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}$, hujus enim differentiale est

$$\frac{[(\Delta+\gamma)n-\frac{1}{2}\delta m+(\frac{1}{2}\delta n-\varepsilon m)p]\partial p}{(\Delta+\gamma+\delta p+\varepsilon pp)^{\frac{3}{2}}},$$

ideoque fieri debet

$$(\Delta+\gamma)n-\tfrac{1}{2}\delta m=2\beta \text{ et}$$
$$\tfrac{1}{2}\delta n-\varepsilon m=-2\Delta,$$

unde deducuntur valores

$$m=\frac{4\beta\delta+8\Delta\Delta+8\Delta\gamma}{4\Delta\varepsilon+4\gamma\varepsilon-\delta\delta} \text{ et } n=\frac{8\beta\varepsilon+4\Delta\delta}{4\Delta\varepsilon+4\gamma\varepsilon-\delta\delta},$$

quarum fractionum loco in calculo retineamus litteras m et n, consequenter adjecta constante aequatio integralis ita se habebit

$$qq=pp+np+m+C\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}.$$

§. 102. Ista autem constans ita definiri debet, ut posito $p=f+g$ fiat $q=g-f$, ex quo quantitas illa constans ita determinabitur

$$C=-\frac{4fg-n(f+g)-m}{\sqrt{[\Delta+\gamma+\delta(f+g)+\varepsilon(f+g)^2]}}.$$

Hoc ergo valore invento, facile assignari poterunt valores non solum ipsius qq sed etiam ejus potestatum parium q^4, q^6, q^8, etc., quibus indigemus. Atque hinc intelligitur pro inveniendo valore ipsius V alias formulas integrales non occurrere nisi quae involvant quantitatem radicalem $\surd(\Delta + \gamma + \delta p + \varepsilon pp)$, quarum ergo integratio, nisi algebraice institui queat, semper per logarithmos et arcus circulares expediri poterit. Evidens autem est, casu quo $\varepsilon = 0$ omnia integralia algebraica exprimi posse.

§. 103. Quod si ergo pro priori curva O Z fuerit

$$\Pi : z = \int \frac{\partial z}{\surd(\alpha + \beta z + \gamma z^2 + \delta z^3)},$$

pro altera vero curva

$$\Phi : z = \int \frac{\partial z(\mathfrak{A} + \mathfrak{B}z + \mathfrak{C}zz + \mathfrak{D}z^3 + etc.)}{\surd(\alpha + \beta z + \gamma zz + \delta z^3)},$$

tum sumtis in priori curva arcubus aequalibus F G et X Y, iis in altera curva respondebunt arcus $\mathfrak{F}\mathfrak{G}$ et $\mathfrak{X}\mathfrak{Y}$, quorum differentia semper geometrice assignari poterit. Interdum etiam fieri potest, ut differentia V in nihilum abeat, id quod quidem semper evenit, sumto $x = f$.

§. 104. Praeterea vero etiam datur alius casus maxime memorabilis, quod differentia illa V algebraice exprimi poterit, qui scilicet semper locum habebit, quando tam in denominatore quam in numeratore tantum potestates pares ipsius z occurrunt, hoc est si fuerit pro curva priore

$$\Pi : z = \int \frac{\partial z}{\surd(\alpha + \gamma zz + \varepsilon z^4)},$$

pro altera vero curva

$$\Phi : z = \int \frac{\partial z(\mathfrak{A} + \mathfrak{C}zz + \mathfrak{E}z^4 + \mathfrak{G}z^6 + etc.)}{\surd(\alpha + \gamma zz + \varepsilon z^4)}.$$

His enim casibus, si in priore curva arcus aequales F G et X Y abscindantur, tum arcuum in altera curva respondentium

$\mathfrak{F}\,\mathfrak{G}$ et $\mathfrak{X}\,\mathfrak{Y}$ differentia semper algebraice seu geometrice exhiberi poterit, ad quotcunque terminos etiam numerator $\mathfrak{A} + \mathfrak{C}\,z\,z + \mathfrak{E}\,z^4 +$ etc. continuetur, atque hic est casus, quem olim tam in calculo integrali quam alibi fusius pertractavi.

§. 105. Ad hoc ostendendum, quia habemus tam $\delta = 0$. quam $\beta = 0$, primo erit

$$q\,q = p\,p + m + C\,\sqrt{(\Delta + \gamma + \varepsilon\,p\,p)},$$

ita ut hic tantum potestates pares ipsius p occurrant, tum autem pro litteris germanicis $\mathfrak{C}$, $\mathfrak{E}$, $\mathfrak{G}$, etc. formulae integrandae sequenti modo se habebunt:

Pro littera $\mathfrak{C}$ . . . . . $\int \frac{p\,\partial q}{\sqrt{(\Delta + \gamma + \varepsilon\,p\,p)}}$,

quae per se est absolute integrabilis.

Pro littera $\mathfrak{E}$ . . . . . $\int \frac{p\,(p\,p + q\,q)\,\partial p}{\sqrt{(\Delta + \gamma + \varepsilon\,p\,p)}}$,

quae loco $q\,q$ substituto valore induet hanc formam

$$\int \frac{p\,(2\,p\,p + m)\,\partial p}{\sqrt{(\Delta + \gamma + \varepsilon\,p\,p)}} + C\int p\,\partial p,$$

ubi integratio est manifesta, quod etiam usu venit pro sequentibus formulis litteris $\mathfrak{G}$, $\mathfrak{F}$, affectis. Evidens enim est, si ponatur $\sqrt{(\Delta + \gamma + \varepsilon\,p\,p)} = s$ fieri

$$p\,p = \frac{s\,s - \Delta - \gamma}{\varepsilon}, \text{ et } p\,\partial p = \frac{s\,\partial s}{\varepsilon}, \text{ ideoque}$$

$$\frac{p\,\partial p}{\sqrt{(\Delta + \gamma + \varepsilon\,p\,p)}} = \frac{\partial s}{\varepsilon},$$

qua substitutione omnes formulae integrandae fiunt rationales et integrae.

§. 106. Cum autem iste posterior casus jam satis prolixe sit tractatus, ac pluribus exemplis a rectificatione Ellipsis et Hyperbolae desumtis illustratus, casus prior quo tantum erat $\varepsilon = 0$ eo majore attentione est dignus, quod quantum equidem scio, a nemine adhuc est observatus, cujus ergo evolutio novae huic me-

thodo unice accepta est referenda. Quemadmodum autem haec deducta sunt ex relatione inter p et q, ita etiam relatio elegantissima erui potest inter has quantitates $p = x + y$ et $u = x y$, quam hic subjungamus.

Analysis
pro investiganda relatione inter p et u.

§. 107. Hic pariter primo in relationem inter ∂p et ∂u inquiramus, et cum sit

$$\frac{\partial p}{\partial u} = \frac{\partial x + \partial y}{y \partial x + x \partial y}, \text{ ob}$$

$$\partial x : \partial y = \sqrt{X} : \sqrt{Y} \text{ erit}$$

$$\frac{\partial p}{\partial u} = \frac{\sqrt{X} + \sqrt{Y}}{y\sqrt{X} + x\sqrt{Y}},$$

et sumtis quadratis

$$\frac{\partial p^2}{\partial u^2} = \frac{X + Y + 2\sqrt{XY}}{yyX + xxY + 2xy\sqrt{XY}}.$$

Supra autem vidimus esse

$$(\sqrt{X} + \sqrt{Y})^2 = qq(\Delta + \gamma + \delta p + \varepsilon pp), \text{ existente } q = y - x.$$

Pro denominatore autem utamur relatione §. 87. inventa

$$\Delta = \frac{2\alpha + \beta(x+y) + 2\gamma xy + \delta xy(x+y) + 2\varepsilon xxyy + 2\sqrt{XY}}{(y-x)^2},$$

unde fit

$$2\sqrt{XY} = \Delta qq - 2\alpha - \beta p - 2\gamma u - \delta pu - 2\varepsilon uu,$$

quo valore substituto aequatio nostra erit

$$\frac{\partial p^2}{\partial u^2} = \frac{qq(\Delta + \gamma + \delta p + \varepsilon pp)}{yyX + xxY + \Delta qqu - 2\alpha u - \beta pu - 2\gamma uu - \delta puu - 2\varepsilon u^3}.$$

§. 108. Hic autem substitutis loco X et Y valoribus, habebimus primo

$$yyX + xxY = \alpha(xx + yy) + \beta xy(x+y) + 2\gamma xxyy$$
$$+ \delta xxyy(x+y) + \varepsilon xxyy(xx+yy),$$

quae ob $x+y=p$, $xy=u$ et $xx+yy=pp-2u$, erit

$$yyX+xxY=\alpha(pp-2u)+\beta pu+2\gamma uu+\delta puu$$
$$+\varepsilon uu(pp-2u),$$

unde totus denominator reperietur fore

$$\alpha(pp+4u)+\varepsilon uu(pp-4u)+\Delta qqu,$$

quare cum sit $pp-4u=qq$, nostra fractio erit

$$\frac{\partial p^2}{\partial u^2}=\frac{\Delta+\gamma+\delta p+\varepsilon pp}{\Delta u+\alpha+\varepsilon uu},$$

unde sequitur haec aequatio separata

$$\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon uu)}}:$$

unde deducitur hoc

Theorema memorabile.

§. 109. Si inter binas variabiles x et y habeatur haec aequatio differentialis

$$\frac{\partial x}{\sqrt{(\alpha+\beta x+\gamma xx+\delta x^3+\varepsilon x^4)}}=\frac{\partial y}{\sqrt{(\alpha+\beta y+\gamma yy+\delta y^3+\varepsilon y^4)}},$$

tum posito $x+y=p$ et $xy=u$, inter has variabiles p et u semper locum habebit haec aequatio differentialis

$$\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}}=\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon uu)}},$$

ubi Δ quidem est constans arbitraria in aequationem posteriorem ingressa, contra vero etiam prior aequatio continet constantem arbitrariam β in altera non occurrentem.

§. 110. Aequationis autem posterioris integratio in promptu est. Si enim utrinque multiplicemus per $\sqrt{\varepsilon}$, integrale per logarithmos ita exprimitur

$$l[p\sqrt{\varepsilon}+\frac{\delta}{2\sqrt{\varepsilon}}+\sqrt{(\Delta+\gamma+\delta p+\varepsilon pp)}]=$$
$$l[u\sqrt{\varepsilon}+\frac{\Delta}{2\sqrt{\varepsilon}}+\sqrt{(\alpha+\Delta u+\varepsilon uu)}]+l\Gamma,$$

ideoque integrale ita algebraice exprimetur

$$\varepsilon p + \tfrac{1}{2}\delta + \sqrt{\varepsilon}(\Delta + \gamma + \delta p + \varepsilon pp) = \Gamma[\varepsilon u + \tfrac{1}{2}\Delta + \sqrt{\varepsilon}(\alpha + \Delta u + \varepsilon uu)].$$

Ubi constans ista Γ facile definitur ex conditione, quod posito $x = f$ fieri debet $y = g$, hoc est ut posito $p = f + g$ fiat $u = fg$, quippe ex qua conditione constans prior Δ jam est definita.

§. 111. Quo hinc jam facilius sive p per u sive u per p definiri possit, notatur esse

$$\frac{1}{\varepsilon p + \frac{1}{2}\delta + \sqrt{[\varepsilon(\Delta + \gamma + \delta p + \varepsilon pp)]}} = \frac{\varepsilon p + \frac{1}{2}\delta - \sqrt{[\varepsilon(\Delta + \gamma + \delta p + \varepsilon pp)]}}{\frac{1}{4}\delta\delta - \varepsilon(\Delta + \gamma)} \text{ et}$$

$$\frac{1}{\varepsilon u + \frac{1}{2}\Delta + \sqrt{[\varepsilon(\alpha + \Delta u + \varepsilon uu)]}} = \frac{\varepsilon u + \frac{1}{2}\Delta - \sqrt{[\varepsilon(\alpha + \Delta u + \varepsilon uu)]}}{\frac{1}{4}\Delta\Delta - \alpha\varepsilon}.$$

Hinc igitur per inversionem sequens aequatio resultabit

$$\frac{\varepsilon p + \frac{1}{2}\delta - \sqrt{\varepsilon}(\Delta + \gamma + \delta p + \varepsilon pp)}{\frac{1}{4}\delta\delta - \varepsilon(\Delta + \gamma)} = \frac{1}{\Gamma} \cdot \frac{\varepsilon u + \frac{1}{2}\Delta - \sqrt{\varepsilon}(\alpha + \Delta u + \varepsilon uu)}{\frac{1}{4}\Delta\Delta - \alpha\varepsilon}, \text{ sive}$$

$$\varepsilon p + \tfrac{1}{2}\delta - \sqrt{\varepsilon}(\Delta + \gamma + \delta p + \varepsilon pp) = \frac{\frac{1}{4}\delta\delta - \varepsilon(\Delta + \gamma)}{\Gamma(\frac{1}{4}\Delta\Delta - \alpha\varepsilon)} \times [\varepsilon u + \tfrac{1}{2}\Delta - \sqrt{\varepsilon}(\alpha + \Delta u + \varepsilon uu)],$$

ex quibus duabus aequationibus sine alio negotio sive p per u sive u per p exprimi poterit.

§. 112. Hoc igitur modo loco variabilis p pro invenienda quantitate V facile introduci posset variabilis u, si quidem loco formulae $\frac{\partial p}{\sqrt{(\Delta+\gamma+\delta p+\varepsilon p p)}}$ substituatur formula ipsi aequalis $\frac{\partial u}{\sqrt{(\alpha+\Delta u+\varepsilon u u)}}$. Verum hoc modo casus illi, quibus quantitas V fieri potest algebraica, non tam facile patescent; interim tamen etiam hoc modo certi erimus, tam casibus quibus $\varepsilon=0$, quam quo $\beta=0$, $\delta=0$ etc. in serie $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc. tantum potestates pares occurrunt, omnes integrationes algebraice succedere debere. Coronidis loco adhuc aliam relationem inter quantitates p et u investigemus, cujus contemplatio insigne incrementum in integratione aequationum polliceri videtur.

Alia Analysis
pro investigatione relationis inter p et u.

§. 113. Cum sit ut ante vidimus $\frac{\partial p}{\partial u}=\frac{\sqrt{X}+\sqrt{Y}}{y\sqrt{X}+x\sqrt{Y}}$, multiplicemus supra et infra per $\sqrt{X}+\sqrt{Y}$, ut numerator evadat

$$(\sqrt{X}+\sqrt{Y})^2=qq(\Delta+\gamma+\delta p+\varepsilon pp);$$

tum autem denominatur prodibit

$$yX+xY+(x+y)\sqrt{XY},$$

ubi denominatoris pars rationalis dat

$$\alpha p+2\beta xy+\gamma xy(x+y)+\delta xy(xx+yy)+\varepsilon xy(x^3+y^3),$$

quae expressio, ob $x+y=p$, $y-x=q$, et $xy=u$, abit in

$$\alpha p+2\beta u+\gamma pu+\delta u(pp-2u)+\varepsilon pu(pp-3u).$$

Deinde ante vidimus esse

$$2\sqrt{XY}=\Delta qq-2\alpha-\beta p-2\gamma u-\delta pu-2\varepsilon uu,$$

quod ductum in $\frac{1}{2}p$ et superiori additum praebet

$\frac{1}{2}\Delta pqq - \frac{1}{2}\beta(pp - 4u) + \frac{1}{2}\delta u(pp - 4u) + \varepsilon pu(pp - 4u)$,

quae denominator ob $pp - 4u = qq$ induet hanc formam

$$\tfrac{1}{2}\Delta pqq - \tfrac{1}{2}\beta qq + \tfrac{1}{2}\delta uqq + \varepsilon puqq:$$

hinc aequatio erit

$$\frac{\partial p}{\partial u} = \frac{\Delta + \gamma + \delta p + \varepsilon pp}{\frac{1}{2}\Delta p - \frac{1}{2}\beta + \frac{1}{2}\delta u + \varepsilon pu},$$

unde deducitur

$$\partial p(\tfrac{1}{2}\Delta p - \tfrac{1}{2}\beta + \tfrac{1}{2}\delta u + \varepsilon pu) = \partial u(\Delta + \gamma + \delta p + \varepsilon pp),$$

quae ergo certe est integrabilis; id quod adeo inde patet, quod altera variabilis u nusquam ultra primam dimensionem exsurgit.

§. 114. Verum adhuc alio modo relatio inter p et u investigari potest; scilicet aequatio primo inventa

$$\frac{\partial p}{\partial u} = \frac{\sqrt{X} + \sqrt{Y}}{y\sqrt{X} + x\sqrt{Y}},$$

si supra et infra multiplicetur per $\sqrt{Y} - \sqrt{X}$ dabit

$$\frac{\partial p}{\partial u} = \frac{Y - X}{-yX + xY + \sqrt{XY}(y - x)}.$$

Nunc igitur pro numeratore habebimus

$$\beta q + \gamma pq + \delta q(pp - u) + \varepsilon pq(pp - 2u).$$

Pro denominatore vero pars rationalis erit

$$-\alpha q + \gamma qu + \delta pqu + \varepsilon qu(pp - u),$$

pars vero irrationalis

$$\tfrac{1}{2}\Delta q^3 - \alpha q - \tfrac{1}{2}\beta pq - \gamma qu - \tfrac{1}{2}\delta pqu - \varepsilon quu,$$

unde totus denominator conficitur

$$\tfrac{1}{2}\Delta q^3 - \alpha q - \tfrac{1}{2}\beta p q + \tfrac{1}{2}\delta p q u + \varepsilon q u (pp - 2u),$$

unde sequitur haec aequatio differentialis

$$\frac{\partial p}{\partial u} = \frac{\beta + \gamma p + \delta (pp - u) + \varepsilon p (pp - 2u)}{\tfrac{1}{2}\Delta (pp - 4u) - 2\alpha - \tfrac{1}{2}\beta p + \tfrac{1}{2}\delta p u + \varepsilon u (pp - 2u)},$$

quae in ordinem redacta ita se habebit

$$\partial p [\Delta (pp - 4u) - 4\alpha - \beta p + \delta p u + 2\varepsilon u (pp - 2u)] = 2\partial [\beta + \gamma p + \delta (pp - u) + \varepsilon p (pp - 2u)],$$

quae jam ita est comparata, ut nulla via ejus integrationem instituenda perspici queat, etiamsi ejus integrale revera exhibere queamus.

§. 115. Alio insuper modo relationem inter p et u definire licet, si aequationis

$$\frac{\partial p}{\partial u} = \frac{\sqrt{X} + \sqrt{Y}}{y\sqrt{X} + x\sqrt{Y}}$$

posterius membrum supra et infra multiplicemus per $y\sqrt{X} - x\sqrt{Y}$ ut prodeat

$$\frac{\partial p}{\partial u} = \frac{yX - xY + (y - x)\sqrt{XY}}{yyX - xxY}.$$

Nunc enim denominator evadet

$$\alpha p q + \beta q u + \delta q u u - \varepsilon p q u u.$$

Pro numeratore autem pars rationalis praebet

$$\alpha q - \gamma q u - \delta p q u - \varepsilon q u (pp - u),$$

et pars irrationalis

$$\tfrac{1}{2}\Delta q^3 - \alpha q - \tfrac{1}{2}\beta p q - \gamma q u - \tfrac{1}{2}\delta p q u - \varepsilon q u u,$$

totus igitur numerator erit

$$\tfrac{1}{2}\Delta q^3 - \tfrac{1}{2}\beta p q - 2\gamma q u - \tfrac{3}{2}\delta p q u - \varepsilon q u p p,$$

ideoque

$$\frac{\partial p}{\partial u} = \frac{\frac{1}{2}\Delta(pp-4u)-\frac{1}{2}\beta p-2\gamma u-\frac{3}{2}\delta pu-\varepsilon ppu}{\alpha p+\beta u-\delta uu-\varepsilon puu},$$

sive

$$2\,\partial p\,(\alpha p+\beta u-\delta uu-\varepsilon puu) =$$
$$\partial u\,[\Delta(pp-4u)-\beta p-4\gamma u-3\delta pu-2\varepsilon ppu].$$

Hic autem penitus non patet, quomodo multiplicator hanc aequationem integrabilem reddens investigari debeat, unde nullum est dubium, quin ista contemplatio haud parum ad limites analyseos prolatandos conferre possit.

SUPPLEMENTUM IX.

AD SECT. I. TOM. II.

DE RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM SECUNDI GRADUS, DUAS TANTUM VARIABILES INVOLVENTIUM.

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1). Methodus singularis resolvendi aequationes differentiales secundi gradus. *M. S. Academiae exhib. die* 19 *Jan.* 1779.

§. 1. Si p et q fuerint functiones quaecunque ipsius x, atque proposita fuerit haec aequatio inter binas variabiles x et z

$$2p\partial z + z\partial p = \frac{\partial x}{q}\int\frac{z\partial x}{q},$$

evidens est ejus integrale facile inveniri posse, si ea multiplicetur per z, ut habeatur

$$2pz\partial z + zz\partial p = \frac{z\partial x}{q}\int\frac{z\partial x}{q}.$$

Prioris enim membri integrale est pzz, posterius vero membrum posito $\int\frac{z\partial x}{q} = v$, abit in $v\partial v$, cujus integrale est $\frac{1}{2}vv + C$, ita ut hinc nanciscamur istam aequationem integralem $pzz = \frac{1}{2}vv + C$, unde fit $vv = 2pzz - 2C$, hincque

$$v = \int\frac{z\partial x}{q} = \sqrt{(2ppzz - C)},$$

quae differentiata dat

$$\frac{z\partial x}{q} = \frac{2pz\partial z + zz\partial p}{\sqrt{(2pzz - C)}},$$

facto ergo divisione per z, erit

$$\frac{\partial x}{q} = \frac{2p\partial z + z\partial p}{\sqrt{(2pzz - C)}},$$

quemadmodum autem hinc valor ipsius z per x, ejusque functiones p et q exprimi queat, non liquet. Ut autem istum scopum obtineamus, posito ut fecimus $\int \frac{z\partial x}{q} = v$ ut sit $vv = 2pzz - C$, retineamus quantitatem v in calculo, et cum sit

$$\frac{z\partial x}{q} = \partial v, \text{ erit } Z = \frac{q\partial v}{\partial x},$$

quo valore substituto habebimus

$$vv = \frac{2pqq\partial v^2}{\partial x^2} - C,$$

unde colligitur

$$\partial v = \frac{\partial x\sqrt{[vv + C]}}{q\sqrt{2p}},$$

quae sponte separationem admittit, cum sit

$$\frac{\partial v}{\sqrt{(vv + C)}} = \frac{\partial x}{q\sqrt{2p}}, \text{ ideoque}$$

$$\int \frac{\partial v}{\sqrt{(vv + C)}} = \int \frac{\partial x}{q\sqrt{2p}},$$

cujus valor, quoniam p et q sunt functiones ipsius x, tanquam cognitus spectari potest.

§. 2. Statuamus ergo hoc integrale

$$\int \frac{\partial x}{q\sqrt{2p}} = l\,X,$$

ut habeamus

$$\int \frac{-\partial v}{\sqrt{(vv + C)}} = l\,X,$$

quare cum constet esse

$$\int \frac{\partial v}{\sqrt{(vv + C)}} = l\,[v + \sqrt{(vv + C)}], \text{ erit}$$

$$v + \sqrt{(vv + C)} = X,$$

unde colligitur $v = \frac{X^2 - C}{2X}$, ideoque per quantitatem X definitur.

§. 3. Cum igitur supra invenerimus $2pzz = vv + C$, erit

$$2pzz = \frac{(X^2 - C)^2}{4XX} + C = \frac{(XX + C)^2}{4XX},$$

consequenter erit

$$z\sqrt{2p} = \frac{X^2 + C}{2X},$$

sicque quantitas z ita per X exprimitur, ut sit

$$z = \frac{X^2 + C}{2X\sqrt{2p}},$$

ubi meminisse oportet esse

$$lX = \int \frac{\partial x}{q\sqrt{2p}}, \text{ sive } X = e^{\int \frac{\partial x}{q\sqrt{2p}}}.$$

§. 4. Manifestum autem est, aequationem nostram propositam, si a signo integrali liberetur, abire in aequationem differentialem secundi gradus, cujus ergo integrale completum modo elicuimus. Facta enim multiplicatione per q fiet

$$2pq\partial z + qz\partial p = \partial x \int \frac{z\partial x}{q},$$

et differentiatio sumto elemento ∂x constante praebebit sequentem aequationem differentialem secundi gradus

$$\left.\begin{array}{r} 2pq\partial\partial z + 2p\partial q\partial z + z\partial q\partial p \\ + 3q\partial p\partial z + qz\partial\partial p \end{array}\right\} = \frac{z\partial x^2}{q},$$

cujus ergo aequationis non parum abstrusae novimus esse integrale completum

$$z = \frac{X^2 + C}{2X\sqrt{2p}}, \text{ existente } X = e^{\int \frac{\partial x}{q\sqrt{2p}}},$$

ita ut ista quantitas X etiam constantem arbitrariam involvat.

§. 5. Cum autem haec aequatio non parum sit complicata, sequenti modo concinnius repraesentari potest; cum enim sit

$$\frac{q}{z}\partial . pzz = \partial x \int \frac{z\partial x}{q},$$

erit differentiationem tantum indicando

$$\partial . \frac{q\partial . pzz}{z} = \frac{z\partial x^2}{q},$$

quae manifesto integrabilis evadit, si multiplicetur per $\frac{2q\partial . pzz}{z}$, quodsi enim brevitatis gratia statuatur $\frac{q\partial . pzz}{z} = s\partial x$, membrum sinistrum fit

$$2s\partial x . \partial s\partial x = 2\int \partial s\partial x^2,$$

ejusque ergo integrale $ss\partial x^2$: at vero ex parte dextra habebimus $2\partial x^2 \partial . pzz$, cujus igitur integrale est

$$2pzz\partial x^2 + C\partial x^2,$$

ita ut integratio nobis praebeat $ss = 2pzz + C$.

§. 6. Quo nunc hanc aequationem penitius evolvamus, statuamus ut ante $pzz = v$, ita ut sit $\frac{q\partial v}{z} = s\partial x$, eritque nostrum integrale inventum

$$ss = \frac{qq\partial v^2}{zz\partial x^2} = 2v + C,$$

quae ob $zz = \frac{v}{p}$ abit in hanc

$$\frac{pqq\partial v^2}{v\partial x^2} = 2v + C,$$

unde eruitur propemodum ut ante

$$\frac{\partial v}{\sqrt{v(2v+C)}} = \frac{\partial x}{q\sqrt{p}},$$

quae a forma ante inventa non discrepat.

§. 7. Simili modo etiam aliae aequationes differentiales magis complicatae resolvi poterunt, veluti si proponatur

ista aequatio

$$3p\partial z + z\partial p = \frac{\partial x}{q}\int\frac{zz\partial x}{q},$$

ubi iterum p et q denotant functiones quascunque ipsius x. Cum enim sit

$$3p\partial z + z\partial p = \frac{\partial \cdot pz^3}{zz},$$

erit per zz multiplicando

$$\partial \cdot pz^3 = \frac{zz\partial x}{q}\int\frac{zz\partial x}{q},$$

quae posito $\int\frac{zz\partial x}{q} = v$ abit in $\partial \cdot pz^3 = v\partial v$, ideoque integrando $2pz^3 = vv + C$.

§. 8. Quoniam autem posuimus $\int\frac{zz\partial x}{q} = v$, erit $zz = \frac{q\partial v}{\partial x}$, hincque $z^3 = \frac{q\partial v}{\partial x}\sqrt{\frac{q\partial v}{\partial x}}$,

unde fit

$$\frac{2pq\partial v}{\partial x}\sqrt{\frac{q\partial v}{\partial x}} = vv + C.$$

Sumtis ergo quadratis erit

$$\frac{4ppq^3\partial v^3}{\partial x^3} = (vv + C)^2, \text{ ideoque}$$

$$\frac{\partial v^3}{(vv + C)^2} = \frac{\partial x^3}{4ppq^3},$$

cujus radix cubica praebet

$$\frac{\partial v}{\sqrt[3]{(vv + C)^2}} = \frac{\partial x}{q\sqrt[3]{4pp}}.$$

Hinc igitur quantitas v per x definitur, ita ut jam v spectare queamus tanquam veram functionem ipsius x, qua inventa erit

$$z^3 = \frac{vv + C}{2p}, \text{ hincque } z = \sqrt[3]{\frac{vv + C}{2p}}.$$

§. 9. Eadem ista aequatio adhuc alio modo resolvi poterit, quandoquidem per q multiplicata ita repraesentatur

$$\frac{q\partial\cdot pz^3}{zz} = \partial x \int \frac{zz\partial x}{q}, \text{ sive}$$

$$\partial\cdot\frac{q\partial\cdot pz^3}{zz} = \frac{zz\partial x^2}{q},$$

quae manifesto integrabilis redditur, multiplicando per $\frac{2q\partial\cdot pz^3}{zz}$, prodit enim

$$\left(\frac{q\partial\cdot pz^3}{zz}\right) = 2pz^3\partial x^2 + C\partial x^2.$$

§. 10. Jam ponatur $pz^3 = v$, ita ut sit

$$z^3 = \frac{v}{p}, \text{ et } z^4 = \frac{v}{p}\sqrt[3]{\frac{v}{p}},$$

quo valore substituto habebimus

$$\frac{pqq\partial v^2\sqrt[3]{p}}{v\sqrt[3]{v}} = 2v\partial x^2 + C\partial x^2,$$

unde concluditur

$$\frac{\partial v^2}{v(2v+C)\sqrt[3]{v}} = \frac{\partial x^2}{pqq\sqrt[3]{p}}, \text{ sive}$$

$$\frac{\partial v}{\sqrt{v(2v+C)\sqrt[3]{v}}} = \frac{\partial v}{v^{\frac{2}{3}}\sqrt{(2v+C)}} = \frac{\partial x}{q\sqrt[3]{pp}},$$

haec aequatio simplicior evadit, ponendo $v = u^3$, scilicet

$$\frac{3\partial u}{\sqrt{(2u^3+C)}} = \frac{\partial x}{q\sqrt[3]{pp}}.$$

Hinc intelligitur, innumerabilia exempla per has formulas expediri posse.

§. 11. Quin etiam hujusmodi aequationes multo generaliores tractari poterunt; namque aequatio generalior ita potest repraesentari

$$\frac{\partial . p z^m}{z^n} = \frac{\partial x}{q}\int\frac{z^n \partial x}{q},$$

quae evoluta dat

$$m p z^{m-n-1} \partial z + z^{m-n} \partial p = \frac{\partial x}{q}\int\frac{z^n \partial x}{q}.$$

Facta autem multiplicatione per z^n, prodit aequatio sponte integrabilis

$$\partial . p z^m = \frac{z^n \partial x}{q}\int\frac{z^n \partial x}{q},$$

si quidem prodit

$$2 p z^m = \left(\int\frac{z^n \partial x}{q}\right)^2 + C.$$

§. 12. Ad hanc aequationem ulterius evolvendam statuamus

$$\int\frac{z^n \partial x}{q} = v, \text{ eritque } z^n = \frac{q \partial v}{\partial x},$$

unde primo $2 p z^m = v v + C$, et hinc porro

$$(2 p)^{\frac{n}{m}} . z^n = (2 p)^{\frac{n}{m}} . \frac{q \partial v}{\partial x} = (v v + C)^{\frac{n}{m}},$$

quae cum sponte sit separabilis, dabit

$$\frac{\partial v}{(v v + C)^{\frac{n}{m}}} = \frac{\partial x}{q (2 p)^{\frac{n}{m}}},$$

unde ergo quantitas v per x determinabitur, qua inventa ipsa quantitas quaesita z ita exprimetur, ut sit $z^m = \frac{v v + C}{2 p}$.

§. 13. Illustremus haec unico exemplo a primo casu petito, sumendo scilicet $p = 1 + xx$ et $q = \sqrt{2}$, ita ut aequatio proposita sit

$$2\,\partial z(1 + xx) + 2\,z x\,\partial x = \frac{\partial x}{\frac{1}{2}}\int z\,\partial x,$$

quae in hanc aequationem secundi gradus evolvitur

$$4\,\partial\partial z(1 + xx) + 12\,x\,\partial x\,\partial z + 3\,z\,\partial x^2 = 0,$$

cujus ergo integrale quaeritur.

§. 14. Faciamus ergo applicationem solutionis supra §. 3. inventae, ubi cum hic sit $p = 1 + xx$ et $q = \sqrt{2}$, erit

$$l\mathrm{X} = \tfrac{1}{2}\int\frac{\partial x}{\sqrt{(1 + xx)}} = \tfrac{1}{2}l[x + \sqrt{(1 + xx)}] - \tfrac{1}{2}l\,a,$$

unde fit

$$\mathrm{X} = \frac{\sqrt{[x + \sqrt{(1 + xx)}]}}{\sqrt{a}},$$

hoc igitur valore substituto habebimus

$$z = \frac{a\mathrm{C} + x + \sqrt{(1 + xx)}}{2\sqrt{2a(1 + xx)[x + \sqrt{(1 + xx)}]}},$$

quae hoc modo simplicius exprimitur

$$z = \frac{[a\mathrm{C} + x + \sqrt{(1 + xx)}]\sqrt{[-x + \sqrt{(1 + xx)}]}}{2\sqrt{2a}(1 + xx)}.$$

Ubi ergo duae quantitates constantes arbitrariae sunt involutae, atque adeo hoc integrale completum algebraice determinetur. Posito ergo $\mathrm{C} = 0$, integrale particulare erit ex prima forma petitum

$$z = \frac{\sqrt{[x + \sqrt{(1 + xx)}]}}{2\sqrt{2a}\,(1 + xx)}.$$

§. 15. Aliud integrale particulare hinc exhiberi potest, constantes ita sumendo ut sit $a\,\mathrm{C}$ infinitum, at vero $\mathrm{C}\sqrt{a}$ finitum $= b$, tum enim erit

$$z = \frac{a\mathrm{C}}{2\sqrt{2a(1 + xx)[x + \sqrt{(1 + xx)}]}} = \frac{b}{2\sqrt{2(1 + xx)[x + \sqrt{(1 + xx)}]}},$$

quae forma redigitur ad hanc

$$z = \frac{\alpha \sqrt{[-x + \sqrt{(1 + xx)}]}}{\sqrt{(1 + xx)}}.$$

2.) Methodus nova investigandi omnes casus, quibus hanc aequationem differentio-differentialem

$$\partial \partial y (1 - a x x) - b x \partial x \partial y - c y \partial x^2 = 0$$

resolvere licet. *M. S. Academiae exhib. die* 13 *Januarii*, 1780.

§. 16. Hic quidem in usum vocari posset methodus a me et ab aliis jam passim exposita, qua valor ipsius y per seriem infinitam exprimitur. Tunc enim omnibus casibus, quibus haec series alicubi abrumpitur, habebitur integrale particulare aequationis propositae; unde quidem haud difficulter integrale completum erui poterit. Verum etsi hoc modo infiniti casus integrabiles reperiuntur, tamen non omnes innotescunt, sed dantur praeterea infiniti alii casus, qui resolutionem admittunt. Quamobrem hic methodum prorsus singularem proponam, cujus ope omnes plane casus integrabiles elici poterunt. Haec autem methodus ita est comparata, ut cognito casu quocunque resolutionem admittente, ex eo innumerabiles alii deduci queant.

§. 17. Statim autem se offerunt duo casus simplicissimi, quibus resolutio succedit, quorum alter est, si $c = 0$, alter vero si $b = a$, quos ergo binos casus principales ante omnia evolvi oportet.

Casus prior principalis quo $c = 0$.

§. 18. Hoc igitur casu aequatio nostra erit

$$\partial \partial y (1 - a x x) = b x \partial x \partial y,$$

quae posito $\partial y = p \partial x$, abit in hanc

$$\partial p (1 - a x x) = b p x \partial x, \text{ sive}$$

$$\frac{\partial p}{p} = \frac{b x \partial x}{1 - a x x},$$

cujus integrale est

$$l p = -\frac{b}{2a} l (1 - a x x) + l\, C,$$

sicque erit

$$p = C (1 - a x x)^{-\frac{b}{2a}} = \frac{\partial y}{\partial x},$$

unde obtinetur

$$y = C \int \partial x (1 - a x x)^{-\frac{b}{2a}}:$$

ubi notasse juvabit istum valorem fieri algebraicum quoties fuerit $-\frac{b}{2a}$ numerus integer positivus, sive $b = -2ia$ denotante i numerum integrum quemcunque. Tum vero valor integralis etiam algebraicus evadit, quando fuerit $-\frac{b}{2a}$, sive $-\frac{3}{2}$, sive $-\frac{5}{2}$, sive $-\frac{7}{2}$, etc. ideoque in genere $\frac{b}{a} = 2i + 1$, ubi esse nequit $i = 0$.

Casus principalis alter quo $b = a$.

§. 19. Hoc ergo casu aequatio nostra per $2 \partial y$ multiplicata erit

$$2 \partial y \partial \partial y (1 - a x x) - 2 a x \partial x \partial y^2 - 2 c y \partial y \partial x^2 = 0,$$

quae sponte est integrabilis, ejus enim integrale erit

$$\partial y^2 (1 - a x x) - c y y \partial x^2 = C \partial x^2.$$

Ex hac igitur aequatione erit

$$\partial y \sqrt{(1 - a x x)} = \partial x \sqrt{(C + c y y)},$$

separatione ergo facta erit

$$\frac{\partial x}{\sqrt{(1 - a x x)}} = \frac{\partial y}{\sqrt{(C + c y y)}}.$$

In hac ergo forma iterum continentur casus algebraici, ad quos eruendos faciamus $a = -\alpha\alpha$, $c = \gamma\gamma$ et $C = \beta\beta$, ut habeamus

$$\frac{\partial x}{\sqrt{(1 + \alpha\alpha x x)}} = \frac{\partial y}{\sqrt{(\beta\beta + \gamma\gamma y y)}},$$

cujus integrale est

$$\frac{1}{\alpha} l [\alpha x + \sqrt{(1 + \alpha\alpha x x)}] = \frac{1}{\gamma} l [\gamma y + \sqrt{(\beta\beta + \gamma\gamma y y)}] - \frac{\gamma}{1} l \Delta,$$

unde ad numeros ascendendo erit

$$\gamma y + \sqrt{(\beta\beta + \gamma\gamma y y)} = \Delta [\alpha x + \sqrt{(1 + \alpha\alpha x x)}]^{\frac{\gamma}{\alpha}}.$$

Posito ergo V pro hac expressione posteriore erit

$$V - \gamma y = \sqrt{(\beta\beta + \gamma\gamma y y)},$$

et sumtis quadratis $y = \frac{VV - \beta\beta}{2\gamma V}$. Cum igitur sit

$$V = \Delta [\alpha x + \sqrt{(1 + \alpha\alpha x x)}]^{\frac{\gamma}{\alpha}}, \text{ erit}$$

$$2\gamma y = \Delta [\alpha x + \sqrt{(1 + \alpha\alpha x x)}]^{\frac{\gamma}{\alpha}} -$$

$$\frac{\beta\beta}{\Delta} [\alpha x + \sqrt{(1 + \alpha\alpha x x)}]^{-\frac{\gamma}{\alpha}},$$

ubi est $\beta\beta = C$, exponens vero $\frac{\gamma}{\alpha} = \sqrt{\frac{c}{a}}$, sicque, quoties $\sqrt{\frac{c}{a}}$ fuerit numerus rationalis, integrale semper erit algebraicum.

§. 20. His duobus casibus principalibus expeditis duplicem tradam viam aequationem propositam in infinitas alias ejusdem generis transformandi, ita ut semper aequatio hujus formae

$$\partial\partial Y(1-axx)-Bx\partial x\partial Y-CY\partial x^2=0$$

prodeat, quae cum resolutionem admittat casibus vel $C=0$ vel $B=a$, iisdem casibus etiam ipsa aequatio proposita erit resolubilis. Duplices igitur hasce transformationes jam sum expositurus.

Transformationes prioris ordinis.

§. 21. Statuo $y=\frac{\partial v}{\partial x}$, unde ob

$$\partial y=\frac{\partial\partial v}{\partial x} \text{ et } \partial\partial y=\frac{\partial^3 v}{\partial x},$$

aequatio nostra induet hanc formam

$$\partial^3 v(1-axx)-bx\partial x\partial\partial v-c\partial x^2\partial v=0,$$

cujus singuli termini integrationem admittunt: erit enim

$$\int\partial x^2\partial v=v\partial x^2,$$
$$\int x\partial x\partial\partial v=x\partial x\partial v-v\partial x^2,$$
$$\int\partial^3 v(1-axx)=\partial\partial v(1-axx)$$
$$+2ax\partial x\partial v-2av\partial x^2.$$

His partibus colligendis, aequatio nostra erit

$$\partial\partial v(1-axx)-(b-2a)x\partial x\partial v-(c-b+2a)v\partial x^2,$$

quae cum propositae prorsus sit similis, integrabilis erit his duobus casibus $c-b+2a=0$ et $b=3a$, sive quoties fuerit $c=b-2a$ vel $b=3a$, atque integratione pro utroque casu instituta, ita ut v exprimatur per x, tum pro ipsa aequatione proposita erit $y=\frac{\partial v}{\partial x}$; unde patet, si integralia pro v inventa fuerint algebraica, fore quoque valorem ipsius y algebraicum.

§. 22. Quod si ulterius simili modo statuamus $v = \frac{\partial v'}{\partial x}$, quoniam per operationem praecedentem litterae b et c transibunt in $b - 2a$ et $c - b + 2a$, nunc ista aequatio proveniet

$$\partial \partial v' (1 - a x x) - (b - 4 a) x \partial x \partial v' \\ - (c - 2 b + 6 a) v' \partial x^2 = 0,$$

quae ergo integrabilis erit, si fuerit vel $b = 4 a$ vel $c = 2 b - 6 a$. Atque inventis valoribus pro v' fiet $y = \frac{\partial \partial v'}{\partial x^2}$, scilicet differentialia secunda ipsius v' dabunt y: sicque, si pro v' valor algebraicus prodierit, etiam y adipiscetur valorem algebraicum.

§. 23. Quod si eandem substitutionem denuo repetamus ponendo $v' = \frac{\partial v''}{\partial x}$, pro litteris initialibus b et c jam habebimus $b - 6 a$ et $c - 3 b + 12 a$, et aequatio resultans erit

$$\partial \partial v'' (1 - a x x) - (b - 6 a) x \partial x \partial v'' \\ - (c - 3 b + 12 a) v'' \partial x^2 = 0,$$

quae ergo resolutionem admittet, quoties fuerit vel $b = 6 a$ vel $c = 3 b - 12 a$, quibus ergo casibus etiam ipsa aequatio proposita resolutionem admittat necesse est, cum sit $y = \frac{\partial^3 v''}{\partial x^3}$.

§. 24. Quod si ergo easdem has operationes continuo repetamus, perpetuo ad aequationes ejusdem formae perveniemus; ubi notasse sufficiet ambos valores, quos pro litteris b et c in qualibet operatione obtinuerimus, quos una cum valoribus ipsius y in sequenti tabula ob oculos ponamus

	b	c	y
Operatio I.	$b - 2a$	$c - b + 2a$	$\frac{\partial v}{\partial x}$
II.	$b - 4a$	$c - 2b + 6a$	$\frac{\partial\partial v'}{\partial x^2}$
III.	$b - 6a$	$c - 3b\;12a$	$\frac{\partial^3 v''}{\partial x^3}$
IV.	$b - 8a$	$c - 4b + 20a$	$\frac{\partial^4 v'''}{\partial x^4}$
—	- - -	- - - - - -	- - -
—	- - -	- - - - - -	- - -
—	- - -	- - - - - -	- - -
i	$b - 2ia$	$c - ib + i(i+1)a$	$\frac{\partial^i v^{[i-1]}}{\partial x^i}$

§. 25. Hinc igitur in genere patet, aequationem propositam semper resolutionem admittere, quoties fuerit vel $b = 2ia + a$, vel $c = ib - i(i+1)a$, ubi pro i omnes numeros integros positivos accipere licet, ita ut hinc duos ordines innumerabilium casuum integrabilium nanciscamur, quorum posteriores tantum per methodum serierum initio indicatam reperiuntur, priores vero huic methodo prorsus sint inaccessi.

Transformationes posterioris ordinis.

§. 26. Quemadmodum hic per differentialia sumus progressi, nunc per integralia regrediamur, ac primo quidem ponamus $y = \int z \partial x$, et aequatio proposita evadet

$$\partial z(1 - axx) - bxz\partial x - c\partial x \int z \partial x = 0,$$

quae differentiata ad formam propositam reducitur

$$\partial\partial z(1 - axx) - (b + 2a)x\partial x\partial z - (c + b)z\partial x^2 = 0,$$

quae ergo secundum casus principales integrationem admittet, casibus $c + b = 0$ et $b + 2a = a$, sive $c = -b$ et $b = -a$.

Integralibus igitur inventis erit $y = \int z \partial x$; unde patet etiamsi haec integralia fuerint algebraica, tamen valores ipsius y fieri transcendentes.

§. 27. Simili modo statuamus porro $z = \int z' \partial x$, et quia per praecedentem operationem loco b et c adepti sumus $b + 2a$ et $c + b$, nunc perveniemus ad hanc aequationem

$$\partial\partial z'(1 - axx) - (b + 4a)x\partial x\partial z' - (c + 2b + 2a)z'\partial x^2 = 0,$$

quae ergo integrationem admittet, si fuerit vel $c + 2b + 2a = 0$, vel $b + 4a = a$, sive $c = -2b - 2a$ et $b = -3a$. Integralibus autem hinc inventis pro y habebimus $y = \int \partial x \int z' \partial x$, quae ita ad signum integrale simplex reducitur, ut sit

$$y = x \int z' \partial x - \int z' x \partial x.$$

§. 28. Simili modo statuamus porro $z' = \int z'' \partial x$, atque nunc deducemur ad hanc aequationem

$$\partial\partial z''(1 - axx) - (b + 6a)x\partial x\partial z'' - (c + 3b + 6a)z''\partial x^2 = 0,$$

quae igitur integrabilis erit, si fuerit vel $c + 3b + 6a = 0$, vel $b + 6a = a$, hoc est si $c = -3b - 6a$ et $b = -5a$; atque ex his integralibus fiet $y = \int \partial x \int \partial x \int z'' \partial x$, qui valor ex praecedente reduci potest, si is per ∂x multiplicatus denuo integretur et loco z' scribatur z'', obtinetur enim

$$y = \tfrac{1}{2}xx \int z'' \partial x - x \int x z'' \partial x + \tfrac{1}{2} \int xxz'' \partial x.$$

§. 29. Quod si jam has operationes ulterius continuemus, totum negotium huc redibit, ut formulae, quae loco b et c sunt prodituraе, rite formentur, simulque valores ipsius y assignentur, quemadmodum sequens tabula indicabit

	b	c	y
Operat. I.	$b+2a$	$c+b$	$\int z\,\partial x$
II.	$b+4a$	$c+2b+2a$	$\int \partial x \int z' \partial x$
III.	$b+6a$	$c+3b+6a$	$\int \partial x \int \partial x \int z'' \partial x$
IV.	$b+8a$	$c+4b+12a$	$\int \partial x \int \partial x \int \partial x \int z''' \partial x$
—	- - -	- - - - -	- - - - - - - -
—	- - -	- - - - -	- - - - - - - -
—	- - -	- - - - -	- - - - - - - -
i	$b+2ia$	$c+ib+i(i-1)a$	$\int \partial x \int \partial x \ldots \int z^{(i-1)} \partial x$

§. 30. Ex antecedentibus satis manifestum est, quomodo integralia ista complicata ad simplicia reduci queant, unde tantum sequentem tabulam subjungemus

$$\int \partial x \int z' \partial x = x \int z' \partial x - \int z' x \partial x$$

$$\int \partial x \int \partial x \int z'' \partial x = \tfrac{1}{2}(xx \int z'' \partial x - 2x \int z'' x \partial x + \int z'' xx \partial x)$$

$$\int \partial x \int \partial x \int \partial x \int z''' \partial x = \tfrac{1}{6}\left\{\begin{array}{l} x^3 \int z''' \partial x - 3xx \int z''' x \partial x \\ + 3x \int z''' xx \partial x - \int z''' x^3 \partial x \end{array}\right\}$$

$$\int \partial x \int \partial x \int \partial x \int \partial x \int z^{\mathrm{IV}} \partial x = \tfrac{1}{24}\left\{\begin{array}{l} x^4 \int z^{\mathrm{IV}} \partial x - 4x^3 \int z^{\mathrm{IV}} x \partial x \\ + 6xx \int z^{\mathrm{IV}} xx \partial x - 4x \int z^{\mathrm{IV}} x^3 \partial x \\ + \int z^{\mathrm{IV}} x^4 \partial x \end{array}\right\}.$$

etc. etc.

§. 31. Quod si jam has operationes secundum numerum indefinitum i continuemus, et loco b, c, z, scribamus B, C, Z, aequatio resultans erit

$$\partial\partial Z(1 - axx) - Bx\partial x\partial Z - CZ\partial x^2 = 0,$$

ubi erit, uti jam indicavimus

$$B = b + 2ia \quad \text{et} \quad C = c + ib + 2ia;$$

quamobrem haec aequatio integrationem admittet, quoties fuerit vel $C = 0$ hoc est $c = -ib - i(i-i)a$, vel $B = a$ hoc est $b = -(2i-1)a$: quae formulae ab illis quas supra pro priori transformationum ordine invenimus, tantum in hoc discrepant, quod hic littera i valorem negativum accepit; unde adjungatur sequens

Conclusio generalis.

§. 32. Si littera i hic denotet omnes numeros integros sive positivos sive negativos, aequatio proposita differentio-differentialis

$$\partial\partial y(1-axx) - bx\partial x\partial y - cy\partial x^2 = 0$$

semper integrationem sive resolutionem admittet, quoties fuerit

1°.) $0 = ib - i(i+1)a$, vel

2°.) $b = (2i+1)a$:

ubi asseverare licet, omnes plane casus resolubiles in hac duplici forma contineri, ita ut nullus plane casus integrationem admittens exhiberi queat, qui non in alterutra harum duarum formularum comprehendatur, dum contra methodus per series procedens, cujus initio mentionem fecimus, tantum casus integrabiles priores ostendit, ita ut inde infinitus numerus casuum pariter resolubilium inde excludatur.

Corollarium 1.

§. 33. Transformetur aequatio proposita in aequationem differentialem primi gradus ponendo $y = e^{\int u\partial x}$, ac perveniemus ad hanc aequationem

$$\partial u + uu\partial x - \frac{bux\partial x + c\partial x}{1-axx} = 0,$$

quae ergo etiam integrationem admittet casibus quibus vel $b =$

$(2i+1)a$ vel $c = ib - i(i+1)a$, denotante i numerum quemcunque integrum sive positivum sive negativum.

Corollarium 2.

§. 34. Quod si porro ponatur $u = (1-axx)^n v$, posito brevitatis gratia $n = -\frac{b}{2a}$, pervenietur ad hanc aequationem ad genus *Riccatianum* referendam

$$(1-axx)^n \partial v + (1-axx)^{2n} vv \partial x = \frac{c\partial x}{1-axx},$$

quae per $(1-axx)^n$ divisa abit in hanc

$$\partial v + (1-axx)^n vv \partial x = \frac{c\partial x}{(1-axx)^{n+1}},$$

quae ergo iisdem casibus integrationem admittet.

Corollarium 3.

§. 35. Quod si sumamus $a = 0$, orietur ista aequatio

$$\partial u + uu\partial x = bux\partial x + c\partial x,$$

quae ergo integrabilis erit, si fuerit vel $b = 0$ vel $c = ib$, quorum quidem prior casus per se est manifestus, quia tum erit $\partial x = \frac{\partial u}{c-uu}$. Haec forma autem commodius exprimi poterit, ponendo

$$u = \tfrac{1}{2}bx + v, \text{ unde } \partial v + vv\partial x = (c - \tfrac{1}{2}b)\partial x + \tfrac{1}{4}bbxx\partial x,$$

sive ponendo $b = 2f$, ut fiat

$$\partial v + vv\partial x = (c-f)\partial x + ffxx\partial x,$$

eritque haec aequatio integrabilis, quoties fuerit $c = 2if$, ita ut sequens aequatio semper integrationem admittat

$$\partial v + vv\partial x = (2i-1)f\partial x + ffxx\partial x,$$

quicunque numerus integer sive positivus sive negativus pro i accipiatur; hoc est, si in penultimo termino f multiplicetur per numerum imparem quemcunque sive positivum sive negativum, qui

casus eo erunt abstrusiores, quo major accipiatur numerus i; atque adeo vix alia via patere videtur integralia eruendi, nisi ut ad aequationem differentialem secundi gradus propositam regrediamur atque easdem operationes instituamus quas supra docuimus. Interim tamen observavi, omnes istos casus etiam immediate ex ipsa aequatione per fractiones continuas derivari posse. Si enim proposita fuerit haec aequatio

$$\partial v + vv\partial x = g\partial x + ffxx\partial x,$$

valor ipsius v duplici modo per fractionem continuam exprimi potest. Est enim priori modo

$$v = fx + \cfrac{g-f}{2fx + \cfrac{g-3f}{2fx + \cfrac{g-5f}{2fx + \text{etc.}}}}$$

Altero vero modo est

$$v = -fx - \cfrac{(g-f)}{2fx + \cfrac{g+3f}{2fx + \cfrac{g+5f}{2fx + \cfrac{g+7f}{2fx + \text{etc.}}}}}$$

quarum prior abrumpitur, quoties fuerit $g = (2i+1)f$, posterior vero, quoties fuerit $g = -(2i+1)f$, qui sunt ipsi casus integrabiles ante inventi.

3.) De formulis integralibus implicatis, earumque evolutione et transformatione. *M. S. Academiae exhib. die* 20 *Aprilis* 1778.

§. 36. Talium formularum implicatarum forma generalis ita exhiberi potest

$$\int p\,\partial x \int q\,\partial x \int r\,\partial x \int s\,\partial x \text{ etc.}$$

ubi quodvis signum integrale omnia sequentia in se complectitur. Ita ad valorem hujus expressionis inveniendum a fine est incipiendum, positoque integrali $\int s\,\partial x = S$ erit

$$\int r\,\partial x \int s\,\partial x = \int S\,r\,\partial x,$$

cujus valor si ponatur $= R$, erit

$$\int q\,\partial x \int r\,\partial x \int s\,\partial x = \int R\,q\,\partial x,$$

quod integrale si ponatur $= Q$, valor ipsius formulae propositae erit $= \int Q\,p\,\partial x$, ubi per se intelligitur, in qualibet integratione more solito constantem arbitrariam in calculum introduci posse.

§. 37. Hic scilicet probe tenendum est, istam expressionem $\int p\,\partial x \int q\,\partial x$ non significare productum ex formula $\int p\,\partial x$ in formulam $\int q\,\partial x$, sed integrale quod oritur, si tota formula differentialis $p\,\partial x \int q\,\partial x$ integretur: at vero si velimus productum talium duarum formularum integralium designare, id interpositione puncti fieri solet hoc modo $\int p\,\partial x \,.\, \int q\,\partial x$, ubi scilicet punctum declarat praecedentia signa integralia non ultra hunc terminum extendi debere, ita haec forma

$$\int p\,\partial x \int q\,\partial x \,.\, \int r\,\partial x \int s\,\partial x$$

exprimit productum, quod oritur si formula $\int p\,\partial x \int q\,\partial x$ multiplicetur per $\int r\,\partial x \int s\,\partial x$.

§. 38. Hic igitur signandi nos prorsus contrarius usu est receptus, atque in formulis differentialibus observari solet, ubi talis expressio $\partial x \partial y \partial z$ denotat productum trium differentialium ∂x, ∂y et ∂z, ita ut singula signa differentiationis tantum litteras immediate sequentes afficiant: at si velimus verbi gratia differentiale hujus expressionis $x \partial y \partial z$ exprimere, hoc interpositione puncti fieri solet $\partial . x \partial y \partial z$, ubi punctum significat, praefixum ∂ complecti totam expressionem sequentem.

§. 39. Tales autem formulae integrales implicatae potissimum nascuntur ex continua integratione aequationum integralium linearium, quarum forma in genere est

$$p z + \frac{q \partial z}{\partial x} + \frac{r \partial \partial z}{\partial x^2} + \frac{s \partial^3 z}{\partial x^3} + \text{etc.} = X,$$

ubi litterae p, q, r, s, etc. sunt functiones datae variabilis x, cujus etiam functio quaecunque sit littera X, altera vero variabilis z ubique unam tantum tenet dimensionem, prouti haec forma generalis hic exhibetur, ad ordinem tertium differentialium refertur, ideoque ternas integrationes postulat, totidemque constantes arbitrarias involvere est censenda, hic scilicet ad methodum integrandi maxime naturalem respicio, quae per ternas integrationes successivas integrale desideratum producat.

§. 40. Tali scilicet aequatione proposita ante omnia nosse oportet multiplicatorem, quo ea reddatur integrabilis, quem ergo supponamus esse $= \partial P$, atque integratione peracta prodeat ista aequatio

$$p' z + \frac{q' \partial z}{\partial x} + \frac{r' \partial \partial z}{\partial x^2} = \int X \partial P,$$

quae aequatio jam est ordinis secundi; quodsi jam ponamus hujus multiplicatorem idoneum esse $= \partial P'$, facta integratione oriatur haec aequatio primi ordinis, quae sit

$$p'' z + \frac{q'' \partial z}{\partial x} = \int \partial P' \int X \partial P,$$

pro qua si $\partial P''$ fuerit multiplicator idoneus, completum integrale induet hanc formam

$$p''' z = \int \partial P'' \int \partial P' \int X \partial P.$$

Sicque quantitas z exprimetur per formulam integralem implicatam.

§. 41. Tali autem forma pro integrali inventa praecipuum negotium huc redit, ut ea ita evolvatur, ut formula continens functionem indefinitam X, quae hic terna signa integralia habet praefixa, plus unico ante se non habeat, quamobrem quemadmodum talis reductio commodissime institui queat, hic ostendere constitui, siquidem nisi certa artificia adhibeantur, hujusmodi operatio calculos maxime molestos postularet.

§. 42. In genere autem hujusmodi formulas implicatas ita repraesentemus

$$\int \partial p \int \partial q \int \partial r \int \partial s \int \partial t \text{ etc.}$$

pro cujus evolutione a casu duorum signorum integralium inchoemus, et quia erit $\int \partial p \int \partial q = \int q \partial p$, reductio vulgaris dat $pq - \int p \partial q$. Jam loco p et q iterum scribamus $\int \partial p$ et $\int \partial q$, atque evolutio ita se habebit

$$\int \partial p \int \partial q = \int \partial p . \int \partial q - \int \partial q \int \partial p,$$

ubi in genere hanc aequalitatem notasse juvabit

$$\int \partial p \int \partial q - \int \partial p . \int \partial q + \int \partial q \int \partial p = 0.$$

§. 43. Consideremus nunc formulam tria signa integralia involventem $= \int \partial p \int \partial q \int \partial r$, et quia ut modo vidimus est $\int \partial q \int \partial r = q r - \int q \partial r$, nostra formula in has partes discerpitur $\int q r \partial p - \int \partial p \int q \partial r$, quae posterior pars reducitur ad hanc formam $p \int q \partial r - \int p q \partial r$, sicque formula nostra erit

$\int q\,r\,\partial p - p\int q\,\partial r + \int p\,q\,\partial r$. Quoniam nunc requiritur, ut elementum ∂r in singulis partibus unicum tantum signum integrale habeat praefixum; ponamus $q\,\partial p = \partial v$ ut sit

$$v = \int q\,\partial p = \int \partial p \int \partial q\text{, eritque}$$

$$\int q\,r\,\partial p = \int r\,\partial v = r\,v - \int v\,\partial r\text{,}$$

hincque colligitur

$$\int p\,q\,\partial r - \int v\,\partial r = \int \partial r\,(p\,q - v) = \int \partial r \int p\,\partial q.$$

Jam loco litterarum finitarum differentialia rursus introducentur, atque valor quaesitus formulae $\int \partial p \int \partial q \int \partial r$ sequenti modo exprimetur

$$\int \partial p \int \partial q\,.\int \partial r - \int \partial p\,.\int \partial r \int \partial q + \int \partial r \int \partial q \int \partial p,$$

ubi in singulis membris elemento ∂r unicum signum integrale est praefixum.

§. 44. Inter terna igitur elementa ∂p, ∂q et ∂r sequentem relationem notari operae erit pretium

$$\int \partial p \int \partial q \int \partial r - \int \partial p \int \partial q\,.\int \partial r + \int \partial p\,.\int \partial r \int \partial q - \int \partial r \int \partial q \int \partial p = 0,$$

quodsi autem similem reductionem pro casibus plurium signorum integralium exsequi vellemus, in calculos molestissimos ac taediosissimos delaberemur; interim tamen totum hoc negotium per sequentia theoremata facillime et planissime expedietur, et quoniam singula membra ope puncti in duos factores resolvi convenit, ubi talis factor deest, ejus locum unitate supplebimus.

Theorema 1.

§. 45. Pro unico elemento ∂p haec relatio habetur $\int \partial p\,.\,1 - 1\,.\int \partial p = 0$, maxime obvia.

Theorema 2.

§. 46. Inter bina elementa ∂p et ∂q semper locum habebit haec relatio

$$\int \partial p \int \partial q \,.\, 1 - \int \partial p \,.\, \int \partial q + 1 \,.\, \int \partial q \int \partial p = 0.$$

Demonstratio.

Ad hoc demonstrandum sufficiet ostendisse, differentiale hujus aequationis esse $= 0$, quoniam vero singula membra binis constant factoribus, seorsim considerentur differentialia ex factoribus prioribus et posterioribus oriunda, hic igitur ex factoribus prioribus oritur differentiale $\partial p\,(\int \partial q \,.\, 1 - 1 \,.\, \int \partial q) = 0$ per theorema 1. At ex factoribus posterioribus oritur differentiale $-\partial q\,(\int \partial p \,.\, 1 - 1 \,.\, \int \partial p) = 0$.

Theorema 3.

§. 47. Inter terna elementa ∂p, ∂q et ∂r semper haec relatio locum habet

$$\int \partial p \int \partial q \int \partial r \,.\, 1 - \int \partial p \int \partial q \,.\, \int \partial r + \int \partial p \,.\, \int \partial r \int \partial q - 1 \,.\, \int \partial r \int \partial q \int \partial p = 0.$$

Demonstratio.

Hic iterum seorsim perpendantur differentialia tam ex prioribus quam ex posterioribus factoribus oriunda; ex prioribus autem oritur

$$\partial p\,(\int \partial q \int \partial r \int \,.\, 1 - \int \partial q \,.\, \int \partial r + 1 \,.\, \int \partial r \int \partial q),$$

cujus valor manifesto ad nihilum redigitur per theorema 2. si scilicet litterae p et q uno gradu promoveantur; tum vero differentiale ex factoribus posterioribus ortum est

$$-\partial r\,(\int \partial p \int \partial q \,.\, 1 - \int \partial p \,.\, \int \partial q + 1 \,.\, \int \partial q \int \partial p),$$

cujus valor pariter per theorema praecedens evanescit, quoniam

igitur ambo differentialia sunt $= 0$, etiam ipsa forma nihilo vel etiam constanti aequalis esse debet, evidens autem est constantem sponte involvi in signis integralibus.

Theorema 4.

§. 48. Inter quaterna elementa ∂p, ∂q, ∂r et ∂s semper ista relatio locum habet

$$\left.\begin{array}{l}\int\partial p\int\partial q\int\partial r\int\partial s\,.\,1-\int\partial p\int\partial q\int\partial r\,.\int\partial s\\ \quad+\int\partial p\int\partial q\,.\int\partial s\int\partial r-\int\partial p\,.\int\partial s\int\partial r\int\partial q\\ \quad+1\,.\int\partial s\int\partial r\int\partial q\int\partial p\end{array}\right\}=0.$$

Demonstratio.

Differentiatio factorum priorum suppeditat sequentem expressionem

$$\partial p\left(\int\partial q\int\partial r\int\partial s\,.\,1-\int\partial q\int\partial r\,.\int\partial s+\int\partial q\,.\int\partial r\int\partial s-1\int\partial q\int\partial r\int\partial s\right),$$

quae ob theorema praecedens ad nihilum reducitur. Simili modo differentiatio factorum posteriorum praebet hanc expressionem

$$-\partial s\left(\int\partial p\int\partial q\int\partial r\,.\,1-\int\partial p\int\partial q\,.\int\partial r+\int\partial p\,.\int\partial r\int\partial q-1\,.\int\partial r\int\partial q\int\partial p\right),$$

quae ob theorema 3. iterum est $= 0$.

Theorema 5.

§. 49. Inter quina elementa ∂p, ∂q, ∂r, ∂s et ∂t semper haec relatio locum habet

$$\left.\begin{array}{l}\int\partial p\int\partial q\int\partial r\int\partial s\int\partial t\,.\,1-\int\partial p\int\partial q\int\partial r\int\partial s\,.\int\partial t\\ \quad+\int\partial p\int\partial q\int\partial r\,.\int\partial t\int\partial s-\int\partial p\int\partial q\,.\int\partial t\int\partial s\int\partial r\\ \quad+\int\partial p\,.\int\partial t\int\partial s\int\partial r\int\partial q-1\,.\int\partial t\int\partial s\int\partial r\int\partial q\int\partial p\end{array}\right\}=0.$$

Demonstratio.

Hujus theorematis demonstratio prorsus eodem modo se habet ac theorematum praecedentium; sicque clarissime jam est evictum tales relationes perpetuo veritati esse consentaneas, quotcunque etiam elementis fuerint composita.

§. 50. Quo vis horum theorematum clarius perspiciatur, operae pretium erit, ea per exempla determinata illustrasse; ponamus igitur esse

$$\partial p = x^{\alpha-1}\,\partial x,\ \partial q = x^{\beta-1}\,\partial x,\ \partial r = x^{\gamma-1}\,\partial x,$$
$$\partial s = x^{\delta-1}\,\partial x,\ \partial t = x^{\epsilon-1}\,\partial x,$$

atque ex theoremate primo statim aequatio identica nascitur $\frac{x^\alpha}{\alpha} - \frac{x^\alpha}{\alpha} = 0$. Verum theorema secundum nobis praebet hanc aequationem

$$\frac{x^{\alpha+\beta}}{\beta(\alpha+\beta)} - \frac{x^{\alpha+\beta}}{\alpha\beta} + \frac{x^{\alpha+\beta}}{\alpha(\alpha+\beta)} = 0,$$

unde per $x^{\alpha+\beta}$ dividendo prodit haec aequalitas

$$\frac{1}{\beta(\alpha+\beta)} - \frac{1}{\alpha\beta} + \frac{1}{\alpha(\alpha+\beta)} = 0,$$

cujus veritas satis facile in oculos incurrit.

§. 51. Hae porro positiones in theoremate tertio introductae producent hanc aequationem

$$\frac{x^{\alpha+\beta+\gamma}}{\gamma(\alpha+\beta+\gamma)(\beta+\gamma)} - \frac{x^{\alpha+\beta+\gamma}}{\beta\gamma(\alpha+\beta)}$$
$$+ \frac{x^{\alpha+\beta+\gamma}}{\alpha\beta(\beta+\gamma)} - \frac{x^{\alpha+\beta+\gamma}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)},$$

unde per $x^{\alpha+\beta+\gamma}$ dividendo prodit haec egregia aequalitas

$$\frac{1}{\gamma(\beta+\gamma)(\alpha+\beta+\gamma)} - \frac{1}{\beta\gamma(\alpha+\beta)} + \frac{1}{\alpha\beta(\beta+\gamma)} - \frac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)} = 0,$$

§. 52. Hae positiones iterum in theoremate quarto substitutae dant hanc aequationem

$$\left.\begin{array}{l}\dfrac{x^{\alpha+\beta+\gamma+\delta}}{\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}-\dfrac{x^{\alpha+\beta+\gamma+\delta}}{\gamma\delta(\gamma+\beta)(\gamma+\beta+\alpha)}\\ +\dfrac{x^{\alpha+\beta+\gamma+\delta}}{\beta\gamma(\beta+\alpha)(\gamma+\delta)}-\dfrac{x^{\alpha+\beta+\gamma+\delta}}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)}\\ +\dfrac{x^{\alpha+\beta+\gamma+\delta}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)}\end{array}\right\}=0,$$

quae per $x^{\alpha+\beta+\gamma+\delta}$ divisa producit hanc aequationem

$$\left.\begin{array}{l}\dfrac{1}{\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}-\dfrac{1}{\delta\gamma(\gamma+\beta)(\gamma+\beta+\alpha)}\\ +\dfrac{1}{\gamma\beta(\beta+\alpha)(\gamma+\delta)}-\dfrac{1}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)}\\ +\dfrac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)}\end{array}\right\}=0.$$

§. 53. Denique eaedem positiones in theoremate quinto substitutae producunt hanc aequationem

$$\left.\begin{array}{l}\dfrac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\varepsilon(\varepsilon+\delta)(\varepsilon+\delta+\gamma)(\varepsilon+\delta+\gamma+\beta)(\varepsilon+\delta+\gamma+\beta+\alpha)}\\ -\dfrac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\varepsilon\delta(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}\\ +\dfrac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\delta\gamma(\gamma+\beta)(\gamma+\beta+\alpha)(\delta+\varepsilon)}\\ -\dfrac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\beta\gamma(\beta+\alpha)(\gamma+\delta)(\gamma+\delta+\varepsilon)}\\ +\dfrac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)(\beta+\gamma+\delta+\varepsilon)}\\ -\dfrac{x^{\alpha+\beta+\gamma+\delta+\varepsilon}}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)(\alpha+\beta+\gamma+\delta+\varepsilon)}\end{array}\right\}=0,$$

quae per $x^{\alpha+\beta+\gamma+\delta+\varepsilon}$ divisa dat hanc aequationem maxime notatu dignam

$$\left.\begin{aligned}&\frac{1}{\varepsilon(\varepsilon+\delta)(\varepsilon+\delta+\gamma)(\varepsilon+\delta+\gamma+\beta)(\varepsilon+\delta+\gamma+\beta+\alpha)}\\ &-\frac{1}{\delta\varepsilon(\delta+\gamma)(\delta+\gamma+\beta)(\delta+\gamma+\beta+\alpha)}\\ &+\frac{1}{\gamma\delta(\gamma+\beta)(\gamma+\beta+\alpha)(\delta+\varepsilon)}\\ &-\frac{1}{\beta\gamma(\beta+\alpha)(\gamma+\delta)(\gamma+\delta+\varepsilon)}\\ &+\frac{1}{\alpha\beta(\beta+\gamma)(\beta+\gamma+\delta)(\beta+\gamma+\delta+\varepsilon)}\\ &-\frac{1}{\alpha(\alpha+\beta)(\alpha+\beta+\gamma)(\alpha+\beta+\gamma+\delta)(\alpha+\beta+\gamma+\delta+\varepsilon)}\end{aligned}\right\}=0.$$

§. 54. Haec theorematis eo magis sunt memorabilia, quod eorum veritas non nisi per plures ambages in numeris explorari potest, ideoque multo majorem attentionem merentur, quam aliud simile theorema, ad quod nuper sum perductus, quippe cujus demonstratio haud difficulter exhiberi potest, quod ita se habet.

Theorema numericum.

Sumtis pro lubitu quotcunque numeris veluti quatuor $\alpha, \beta, \gamma, \delta$, si hinc totidem alii sequenti modo formentur

$$a=\alpha,\ b=\alpha+\beta,$$
$$c=\alpha+\beta+\gamma \text{ et } d=\alpha+\beta+\gamma+\delta,$$

similique modo etiam isti

$$D=\delta,\ C=\delta+\gamma$$
$$B=\delta+\gamma+\beta \text{ et } A=\delta+\gamma+\beta+\alpha,$$

tam semper erit

$$\frac{1}{abcd}-\frac{1}{abcD}+\frac{1}{abCD}-\frac{1}{aBCD}+\frac{1}{ABCD}=0.$$

Demonstratio.

§. 55. Binae fractiones priores inventae, ob $D - d = - c$, dant fractionem $- \frac{1}{abdD}$, quae cum tertia conjuncta producit $\frac{1}{adCD}$, cui quarta fractio juncta dat $- \frac{1}{dBCD}$, quae [ob $d = A$] a termino ultimo penitus destruitur.

§. 56. Ope superiorum theorematum omnes formulae integrales implicatae, ad quas integratio aequationum linearum perducere solet, facile resolvi poterunt. Pervenitur autem plerumque ad tales formas:

$$Z = \int \partial q \int X \partial p, \; Z = \int \partial r \int \partial q \int X \partial p,$$

$$Z = \int \partial s \int \partial r \int \partial q \int X \partial p, \; Z = \int \partial t \int \partial s \int \partial r \int \partial q \int X \partial p \text{ etc.}$$

ubi litterae p, q, r, s, t, etc. sunt functiones datae ipsius X, at vero X functio quaecunque ipsius x; atque hic tota resolutio ita institui debet, ut in singulis membris functio haec indefinita X unicum tantum signum integrale habeat praefixum: hoc igitur, ope superiorum theorematum, facile praestari poterit, si modo ibi loco elementi ∂p scribamus $X \partial p$, quo observato singulae reductiones sequenti modo se habebunt.

I. Resolutio formulae integralis $\int \partial q \int X \partial p$.

§. 57. Si loco ∂p scribamus $X \partial p$ theorema secundum §. 46. nobis suppeditat hanc aequationem:

$$\int X \partial p \int \partial q - \int X \partial p . \int \partial q + \int \partial q \int X \partial p = 0,$$

cujus postremum membrum est ipsa nostra forma reducenda Z, consequenter resolutio statim dat

$$Z = \int \partial q . \int X \partial p - \int X \partial p \int \partial q,$$

ideoque ob $\int \partial q = q$ habebimus

$$Z = q \int X \partial p - \int X q \partial p.$$

Corollarium.

§. 58. Si fuerit $q = p$, erit

$$Z = p \int X \partial p - \int X p \partial p.$$

II. Resolutio formulae implicatae

§. 59. Pro hoc casu sumamus theorema 3. §. 47. unde, si loco ∂p scribatur $X \partial p$, deducimus hanc aequationem

$$\int X \partial p \int \partial q \int \partial r - \int X \partial p \int \partial q . \int \partial r + \int X \partial p . \int \partial r \int \partial q - \int \partial r \int \partial q \int X \partial p = 0,$$

cujus postremum membrum est ipsa forma reducenda Z, hincque adeoque colligitur

$$Z = \int \partial r \int \partial q . \int X \partial p - \int \partial r . \int X \partial p \int \partial q + \int X \partial p \int \partial q \int \partial r,$$

quae ergo reducta dat

$$Z = \int q \partial r . \int X \partial p - r \int X q \partial p + \int X \partial p \int r \partial q.$$

Corollarium.

§. 60. Si ergo hic fuerit $q = r = p$, prodibit ista resolutio:

$$Z = \int \partial p \int \partial p \int X \partial p = \tfrac{1}{2} pp \int X \partial p - p \int X p \partial p + \tfrac{1}{2} \int X p p \partial p.$$

III. Resolutio hujus formulae implicatae

$$Z = \int \partial s \int \partial r \int \partial q \int X \partial p.$$

§. 61. Pro hoc casu sumamus theorema 4. §. 48. unde si loco ∂p scribatur $X \partial p$ deducimus hanc aequationem

$$\left.\begin{array}{r}\int X\partial p\int\partial q\int\partial r\int\partial s-\int X\partial p\int\partial q\int\partial r.\int\partial s+\int X\partial p\int\partial q.\int\partial s\int\partial r\\ -\int X\partial p.\int\partial s\int\partial r\int\partial q+\int\partial s\int\partial r\int\partial q\int X\partial p\end{array}\right\}=0,$$

cujus postremum membrum est ipsa nostra formula reducenda Z; hincque adeo colligimus

$$Z=\left\{\begin{array}{l}\int\partial s\int\partial r\int\partial q.\int X\partial p-\int\partial s\int\partial r.\int X\partial p\int\partial q\\ \quad+\int\partial s.\int X\partial p\int\partial q\int\partial r-\int X\partial p\int\partial q\int\partial r\int\partial s,\end{array}\right.$$

quae ergo reducta praebet

$$Z=\left\{\begin{array}{l}\int\partial s\int q\partial r.\int X\partial p-\int r\partial s.\int Xq\partial p+s\int X\partial q\int r\partial q\\ \qquad\qquad -\int X\partial p\int\partial q\int s\partial r.\end{array}\right.$$

Corollarium.

§. 62. Si ponatur $s=r=q=p$, tum prodibit ista resolutio

$$Z=\left\{\begin{array}{l}\frac{1}{6}p^3\int X\partial p-\frac{1}{2}pp\int Xp\partial p+\frac{1}{2}p\int Xpp\partial p\\ \qquad -\frac{1}{6}\int Xp^3\partial p.\end{array}\right.$$

IV. Resolutio hujus formulae implicatae.

$$Z=\int\partial t\int\partial s\int\partial r\int\partial q\int X\partial p.$$

§. 63. Pro hoc casu sumamus theorema 5. §. 49. unde si loco ∂p scribatur $X\partial p$, prodibit ista aequatio

$$\left.\begin{array}{l}\int X\partial p\int\partial q\int\partial r\int\partial s\int\partial t-\int X\partial p\int\partial q\int\partial r\int\partial s.\int\partial t\\ \quad+\int X\partial p\int\partial q\int\partial r.\int\partial t\int\partial s-\int X\partial p\int\partial q.\int\partial t\int\partial s\int\partial r\\ \quad+\int X\partial p.\int\partial t\int\partial s\int\partial r\partial q-\int\partial t\int\partial s\int\partial r\int\partial q\int X\partial p\end{array}\right\}=0,$$

cujus postremum membrum est ipsa nostra forma reducenda Z, unde ergo prodit

$$Z=\begin{cases}\int\partial t\int\partial s\int\partial r\int\partial q\,.\int X\partial p-\int\partial t\int\partial s\int\partial r\,.\int X\partial p\int\partial q\\ +\int\partial t\int\partial s\,.\int X\partial p\int\partial q\int\partial r-\int\partial t\,.\int X\partial p\int\partial q\int\partial r\int\partial s\\ +\int X\partial p\int\partial q\int\partial r\int\partial s\int\partial t,\end{cases}$$

quae ergo reducta praebet

$$Z=\begin{cases}\int\partial t\int\partial s\int q\partial r\,.\int X\partial p-\int\partial t\int r\partial s\,.\int X q\partial p\\ +\int s\partial t\,.\int X\partial p\int r\partial q-t\int X\partial p\int\partial q\int s\partial r\\ +\int X\partial p\int\partial q\int\partial r\int t\partial s.\end{cases}$$

Corollarium.

§. 64. Si hic sumatur $t=s=r=q=p$, tum prodibit ista resolutio

$$Z=\begin{cases}\frac{1}{24}p^4\int X\partial p-\frac{1}{6}p^3\int Xp\partial p+\frac{1}{4}pp\int Xpp\partial p\\ \frac{1}{6}p\int Xp^3\partial p+\frac{1}{24}\int Xp^4\partial p.\end{cases}$$

§. 65. Quo indoles harum resolutionum clarius perspiciatur, quoniam litterae p, q, r, s, t, functiones datas ipsius x denotant, ideoque omnes expressiones ex iis formatae pariter ut cognitae spectari possunt, statuamus brevitatis gratia

$$\partial p\int\partial q=\partial p';\ \partial p\int\partial q\int\partial r=\partial p'';\ \partial p\int\partial q\int\partial r\int\partial s=\partial p''';$$
$$\partial p\int\partial q\int\partial r\int\partial s\int\partial t=\partial p'''';\ \text{etc.}$$

hocque modo postrema resolutio ita referetur

$$Z=\int\partial t\int\partial s\int\partial r\int\partial q\,.\int X\partial p-\int\partial t\int\partial s\int\partial r\,.\int X\partial p'$$
$$+\int\partial t\int\partial s\,.\int X\partial p''-\int\partial t\,.\int X\partial p'''+\int X\partial p''''.$$

Quod si hic porro statuamus

$$\int\partial t\int\partial s=\int s\partial t=t';\ \int\partial t\int\partial s\int\partial r=t'';\ \int\partial t\int\partial s\int\partial r\int\partial q=t''';$$

tota resolutio hoc modo conncinne repraesentabitur

$$Z=t'''\int X\partial p-t''\int X\partial p'+t'\int X\partial p''-t\int X\partial p'''$$
$$+\int X\partial p'''',$$

quam repraesentationem etiam ad praecedentes resolutiones accommodasse juvabit.

§. 66. Cum igitur integratio formulae implicatae

$$Z = \int \partial t \int \partial s \int \partial r \int \partial q \int X \partial p$$

reducatur ad integrationem sequentium formularum integralium simplicium: $\int x \partial p$; $\int x \partial p'$; $\int x \partial p''$; $\int x \partial p'''$; $\int x \partial p''''$; quaestio hinc oritur non parum curiosa: quemadmodum ex his formulis simplicibus vicissim quantitates q, r, s et t concludi queant? quod sequenti modo facile praestabitur. Cum sit $\partial p' = \partial p \int \partial q$, erit $\int \partial q = q = \frac{\partial p'}{\partial p}$. Ponatur nunc porro $\frac{\partial p''}{\partial p} = q'$; $\frac{\partial p'''}{\partial p} = q''$; $\frac{\partial p''''}{\partial p} = q'''$; etc. quibus valoribus introductis habebimus

$$q' = \int \partial q \int \partial r;\quad q'' = \int \partial q \int \partial r \int \partial s;$$
$$q''' = \int \partial q \int \partial r \int \partial s \int \partial t;\ \text{etc.}$$

Quoniam igitur hi valores q, q', q'', q''', q'''' sunt dati, ex prima statim colligimus $\int \partial r = \frac{\partial q'}{\partial q} = r$. Ponamus autem porro $\frac{\partial q''}{\partial q} = r'$; $\frac{\partial q'''}{\partial q} = r''$; etc. eruntque etiam hi valores, r, r', r'', etc. dati, quibus substitutis habebitur $r' = \int \partial r \int \partial s$; $r'' = \int \partial r \int \partial s \int \partial t$; ex quarum prima sequitur $\int \partial s = s = \frac{\partial r'}{\partial r}$. Quare si porro fiat $s' = \frac{\partial r''}{\partial r}$, erit quoque $s' = \int \partial s \int \partial t$, hincque $\int \partial t = t = \frac{\partial s'}{\partial s}$. Ex his clare intelligitur, quomodo hae formulae inveniri queant pro casibus adhuc magis complicatis.

§. 67. Superest, ut etiam de transformatione talium formularum integralium implicatarum pauca adjiciamus, quod totum negotium sequenti problemate includi potest.

Problema.

§. 68. *Proposita formula implicata terna signa summatoria involvente* $\int \partial p \int \partial q \int \partial r$, *investigare aliam similem formulam*

$$\int \partial P \int \partial Q \int \partial R,$$

illi aequalem.

Solutio.

Per theorema 2. supra allatum formula proposita ita est resoluta

$$\int \partial q \int \partial r = \int \partial q . \int \partial r - \int \partial r \int \partial q = q \int \partial r - \int q \partial r.$$

Simili modo pro formula quaesita erit

$$\int \partial Q \int \partial R = Q \int \partial R - \int Q \partial R,$$

requiritur igitur ut sit

$$q \partial p \int \partial r - \partial p \int q \partial r = Q \partial P \int \partial R - \partial P \int Q \partial R,$$

quae aequalitas adimpleretur, sumendo $P = p$, $Q = q$ et $R = r$; verum permutandis membris statuamus

$$Q \partial P \int \partial R = - \partial p \int q \partial r \text{ et } \partial P \int Q \partial R = - q \partial p \int \partial r,$$

atque ex priore aequatione deducimus $Q \partial P = - \partial p$, ideoque $\partial P = \frac{\partial p}{Q}$, tum vero $\partial R = q \partial r$; ex altera vero aequatione habemus $\partial P = - q \partial p$ et $Q \partial R = \partial r$. Cum igitur esset $\partial P = - \frac{\partial p}{Q}$, erit $Q = \frac{1}{q}$, hincque porro $\partial R = q \partial r$, unde ob $Q = \frac{1}{q}$, erit $\partial Q = \frac{-\partial q}{q q}$. Consequenter formula integralis quaesita proposita $\int \partial p \int \partial q \int \partial r$ aequalis erit

$$\int q \partial p \int \frac{\partial q}{q q} \int q \partial r,$$

unde patet perpetuo loco formulae $\int \partial p \int \partial q \int \partial r$, scribi posse istam: $\int q \partial p \int \frac{\partial q}{q q} \int q \partial r$.

Corollarium 1.

§. 69. Quando igitur plura signa integralia sibi invicem fuerint involuta, veluti si habeamus $\int \partial p \int \partial q \int \partial r \int \partial s$, ista transformatio in quibusvis ternis signis se mutuo sequentibus institui poterit, unde in hac formula proposita duplex transformatio adhiberi poterit; prior scilicet in ternis signis prioribus praebebit

$$\int q \partial p \int \frac{\partial q}{qq} \int q \partial r \int \partial s,$$

at vero in ternis posterioribus haec transformatio adhibita dabit

$$\int \partial p \int r \partial q \int \frac{\partial r}{rr} \int r \partial s,$$

Corollarium 2.

§. 70. Hinc porro ope ejusdem transformationis aliae insuper fieri possunt, veluti ex postrema forma

$$\int \partial p \int r \partial q \int \frac{\partial r}{rr} \int r \partial s,$$

ut in ternis prioribus signis res expediri queat, loco $r \partial q$ scribamus ∂v, ut habeamus

$$\int \partial p \int \partial v \int \frac{\partial r}{rr} \int r \partial s,$$

quae transformatur in hanc

$$\int v \partial p \int \frac{\partial v}{vv} \int \frac{v \partial r}{rr} \int r \partial s,$$

quae omnes formulae ipsi propositae sunt prorsus aequales.

§. 71. Ut rem exemplo illustremus, sumamus esse $p = x^\alpha$; $q = x^\beta$; $r = x^\gamma$, ita ut formula proposita sit

$$\alpha\beta\gamma \int x^{\alpha-1} \partial x \int x^{\beta-1} \partial x \int x^{\gamma-1} \partial x = \frac{\alpha\,\beta\, x^{\gamma+\beta+\alpha}}{(\gamma+\beta)(\gamma+\beta+\alpha)}.$$

Jam pro transformatione erit primo

$\int q\,\partial r = \frac{\gamma\, x^{\beta+\gamma}}{\beta+\gamma}$, ideoque ob $\frac{\partial q}{q q} = \frac{\beta\,\partial x}{x^{\beta+1}}$, erit

$$\int \frac{\partial q}{q q} \int q\,\partial r = \frac{\beta\, x^{\gamma}}{\beta+\gamma},$$

quod ductum in $q\,\partial p$ et integratum producit

$$\frac{\alpha\,\beta\, x^{\alpha+\beta+\gamma}}{(\beta+\gamma)(\alpha+\beta+\gamma)}.$$

Patet igitur hanc transformationem latissime patere, atque ad omnes formulas implicatas accommodari posse eo pluribus diversis modis, quo plura signa integralia invicem involvantur.

§. 72. Haud abs re fore judico resolutiones supra traditas ad summationem serierum potestatum reciprocarum applicare, quod fiet si loco X sumamus fractionem $\frac{x}{1-x}$, tum vero pro singulis elementis ∂p, ∂q, ∂r, ∂s, scribamus $\frac{\partial x}{x}$, unde corollaria subnexa in usum vocari poterunt, ubi scilicet erit $p = l\,x$.

§. 73. Cum sit per seriem infinitam

$$X = x + x x + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \text{etc.}$$

erit

$$\int X\,\partial p = \int \frac{X\,\partial x}{x} = x + \tfrac{1}{2} x x + \tfrac{1}{3} x^3 + \tfrac{1}{4} x^4 + \tfrac{1}{5} x^5 + \tfrac{1}{6} x^6 + \text{etc.}$$

quam seriem constat exprimere logarithmum fractionis $\frac{1}{1-x}$, quandoquidem est

$$\int \frac{X\,\partial x}{x} = -\,l\,(1-x) = l\,\frac{1}{1-x}.$$

§. 74. Multiplicetur haec series porro per $\frac{\partial x}{x}$ et integretur, prodibitque

$$\int\frac{\partial x}{x}\int\frac{X\partial x}{x}=x+\tfrac{1}{4}xx+\tfrac{1}{9}x^3+\tfrac{1}{16}x^4+\tfrac{1}{25}x^5+\text{etc.}$$

at vero hujus formulae integralis resolutio supra §. 57. data praebet

$$\int\frac{\partial x}{x}\int\frac{X\partial x}{x}=lx\int\frac{\partial x}{1-x}-\int\frac{\partial x\, lx}{1-x},$$

quae quidem integralia ita accipi supponuntur, ut posito $x=0$ evanescant; hic autem imprimis notetur, casu quo sumitur $x=1$, ob $l\,1=0$, hujus seriei

$$1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\tfrac{1}{25}+\text{etc.}$$

summam fore $-\int\frac{\partial x\, lx}{1-x}$, cujus valorem olim primus inveni esse $=\frac{\pi\pi}{6}$.

§. 75. Ducamus superiorem seriem denuo in $\frac{\partial x}{x}$ et integrando obtinebimus:

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x}=x+\tfrac{1}{2^3}xx+\tfrac{1}{3^3}x^3+\tfrac{1}{4^3}+x^4+\tfrac{1}{5^3}x^5+\text{etc.}$$

Formula autem haec implicata per §. 59. ita resolvitur

$$\tfrac{1}{2}(lx)^2\int\frac{\partial x}{1-x}-lx\int\frac{\partial x\, lx}{1-x}+\tfrac{1}{2}\int\frac{\partial x\,(lx)^2}{1-x}.$$

Casu igitur quo $x=1$, summa seriei reciprocae cuborum

$$1+\frac{1}{2^3}+\frac{1}{3^3}+\frac{1}{4^3}+\frac{1}{5^3}+\text{etc.}$$

erit $=\frac{1}{2}\int\frac{\partial x\,(lx)^2}{1-x}$.

§ 76. Simili modo superiorem seriem per $\frac{\partial x}{x}$ multiplicemus et integremus, tum prodibit

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x}=x+\tfrac{1}{2^4}xx+\tfrac{1}{3^4}x^3+\tfrac{1}{4^4}x^4+\text{etc.}$$

At vero haec formula implicata per §. 61. reducitur ad hanc formam

$$\tfrac{1}{6}(lx)^3\int\frac{\partial x}{1-x}-\tfrac{1}{2}(lx)^2\int\frac{\partial x\, lx}{1-x}+\tfrac{1}{2}lx\int\frac{\partial x\,(lx)^2}{1-x}$$
$$-\tfrac{1}{6}\int\frac{\partial x\,(lx)^3}{1-x}.$$

Pro casu ergo quo $x = 1$ hujus seriei reciprocae biquadratorum summa erit $-\frac{1}{6}\int\frac{\partial x (l x)^3}{1-x}$, cujus valorem olim ostendi esse $\frac{\pi^4}{90}$.

§. 77. Multiplicatione denuo per $\frac{\partial x}{x}$ instituta et integratione peracta habebimus:

$$\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{x}\int\frac{\partial x}{1-x} = x + \frac{1}{2^4}xx + \frac{1}{3^4}x^3 + \frac{1}{4^4}x^4 + \frac{1}{5^4}x^5 + \text{etc.}$$

quae formula implicata per §. 63. reducitur ad hanc formam

$$\frac{1}{24}(l x)^4\int\frac{\partial x}{x} - \frac{1}{6}(l x)^3\int\frac{\partial x\, l x}{1-x} + \frac{1}{4}(l x)^2\int\frac{\partial x (l x)^2}{1-x} - \frac{1}{6}l x\int\frac{\partial x (l x)^3}{1-x} + \frac{1}{24}\int\frac{\partial x (l x)^4}{1-x}.$$

Hinc ergo casu $x = 1$ hujus seriei reciprocae potestatum quintarum summa erit $\frac{1}{24}\int\frac{\partial x (l x)^4}{1-x}$.

§. 78. Colligamus omnes istas series pro casu $x = 1$, earumque summae sequenti modo per formulam integralem simplicem exprimentur:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \text{etc.} = \int\frac{\partial x}{1-x} = \infty,$$

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \text{etc.} = -\int\frac{\partial x\, l x}{1-x} = \frac{\pi\pi}{6},$$

$$1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \frac{1}{5^3} + \text{etc.} = \frac{1}{2}\int\frac{\partial x (l x)^2}{1-x},$$

$$1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \text{etc.} = -\frac{1}{6}\int\frac{\partial x (l x)^3}{1-x} = \frac{\pi^4}{90},$$

$$1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} + \text{etc.} = \frac{1}{24}\int\frac{\partial x (l x)^4}{1-x},$$

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \text{etc.} = -\frac{1}{120}\int\frac{\partial x (l x)^5}{1-x} = \frac{\pi^6}{945},$$

$$1 + \frac{1}{2^7} + \frac{1}{3^7} + \frac{1}{4^7} + \frac{1}{5^7} + \text{etc.} = \frac{1}{720}\int\frac{\partial x (l x)^6}{1-x},$$

$$1 + \frac{1}{2^8} + \frac{1}{3^8} + \frac{1}{4^8} + \frac{1}{5^8} + \text{etc.} = -\frac{1}{5040}\int\frac{\partial x (l x)^7}{1-x} = \frac{\pi^8}{9450},$$

etc. etc.

§. 79. In genere igitur hujus seriei

$$1 + \frac{1}{2^n} + \frac{1}{3^n} + \frac{1}{4^n} + \frac{1}{5^n} + \frac{1}{6^n} + \text{etc.}$$

in infinitum continuatae summa ita exprimetur

$$\pm \frac{1}{1.\,2.\,3.\,\ldots\,(n-1)} \int \frac{\partial x\,(l\,x)^{n-1}}{1-x},$$

ubi signum superius $+$ valet, quando exponens n est impar, inferius vero, quando est par. Istas summationes, jam pridem quidem repertas, ideo hic afferre visum est, quod non ita pridem Celeberr. Lorgna easdem has summationes per formulas continuo magis implicatas expressas exhibuit, cum sine dubio istae formulae integrales simplices longe praeferendae videantur.

SUPPLEMENTUM X.

AD SECT. II. TOM. II.

DE

RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM TERTII ALIORUMQUE GRADUUM, QUAE DUAS TANTUM VARIABILES INVOLVUNT.

1.) De aequationibus differentialibus cujuscunque gradus, quae denuo differentiatae integrari possunt. *M. S. Academiae exhib. die* 8 *Octobris* 1781.

§. 1. Sint x et y binae variabiles, inter quas earumque differentialia cujuscunque gradus aequationes propositae subsistant. Ad formam differentialium tollendam ponatur more solito

$$\partial y = p\,\partial x,\ \partial p = q\,\partial x,\ \partial q = r\,\partial x,\ \partial r = s\,\partial x, \text{ etc.}$$

ita ut, sumto elemento ∂x constante, sit

$$p = \frac{\partial y}{\partial x},\ q = \frac{\partial\partial y}{\partial x^2},\ r = \frac{\partial^3 y}{\partial x^3},\ s = \frac{\partial^4 y}{\partial x^4}, \text{ etc.}$$

Sint porro P et $\mathfrak{P}$ functiones quaecunque ipsius p; Q et $\mathfrak{Q}$ functiones quaecunque ipsius q; R et $\mathfrak{R}$ ipsius r; S et $\mathfrak{S}$ ipsius s etc. quae functiones non solum esse possunt rationales, sed etiam irrationales, atque adeo transcendentes.

§. 2. His positis duo aequationum genera per differentiationem integrare docebo, quarum primum istas continet aequationes

$$y - p x = \mathrm{P},\ p - q r = \mathrm{Q},\ q - r x = \mathrm{R},\ r - s x = \mathrm{S},\ \text{etc.}$$

quarum prima involvere potest functiones quascunque ipsius ∂y, tam rationales quam irrationales, quin etiam functiones transcendentes; secunda tales functiones ipsius $\partial \partial y$ involvere potest; tertia ipsius $\partial^3 y$; quarta ipsius $\partial^4 y$; et ita porro, cujusmodi aequationum integratio certe nemini adhuc in mentem venire potuit.

§. 3. Alterum genus aequationum, quarum integrationem per differentiationem expedire docebo, sequentes complectitur aequationes

$$y + \mathfrak{P} x = \mathrm{P},\ p + \mathfrak{Q} x = \mathrm{Q},\ q + \mathfrak{R} x = \mathrm{R},\ r + \mathfrak{S} x = \mathrm{S},\ \text{etc.}$$

quae duas functiones quascunque involvunt. Evidens autem est has aequationes praecedentes in se comprehendere, quando scilicet est

$$\mathfrak{P} = -p,\ \mathfrak{Q} = -q,\ \mathfrak{R} = -r,\ \mathfrak{S} = -s,\ \text{etc.}$$

Ceterum patet, has aequationes adeo complicatas esse posse, ut nemo certe earum integrationem suscipere voluerit.

De aequationibus prioris generis.

Problema 1.

§. 4. *Proposita aequatione differentiali primi gradus* $y - p x = \mathrm{P}$, *ejus integrale completum invenire.*

Solutio.

Cum sit $\partial y = p \partial x$, si aequatio proposita differentietur, prodibit haec $-x \partial p = \partial \mathrm{P}$, unde, posito $\partial \mathrm{P} = \mathrm{P}' \partial p$,

colligitur $x = - P'$. Quod si jam p tanquam novam variabilem spectemus, per eam tam x quam y exprimere poterimus. Cum enim sit $y = p x + P$, erit $y = P - p P'$, unde, eliminando p, quoties quidem calculus id permittet, conflari poterit aequatio inter x et y, quae autem tantum ut integrale particulare spectari debet, quia nullam involvit constantem arbitrariam. At vero, quoniam aequationem per differentiationem erutam $- x \partial p = P' \partial p$ per ∂q dividere licuit, iste factor nihilo aequatus integrale completum suppeditare est censendus. Posito enim $\partial p = 0$, erit $p =$ const. $= \alpha$, ideoque $y = \int p \partial x = \alpha x + \beta$. Haec quidem aequatio duas constantes arbitrarias involvere videtur; at vero altera per ipsam aequationem propositam determinatur, cum facta substitutione fiat

$$\alpha x + \beta - \alpha x = P, \text{ ideoque } \beta = P = f : \alpha.$$

Problema 2.

§. 5. *Proposita aequatione differentiali secundi gradus* $p - q x = Q$, *ejus integrale completum assignare.*

Solutio.

Si haec aequatio differentietur et loco ∂p scribatur $q \partial x$, prodibit ista $- x \partial q = \partial Q$, sive, posito $\partial Q = Q' \partial q$, erit $- x \partial q = Q' \partial q$. Hinc factor communis ∂q nihilo aequatus praebet $q =$ const. $= 2 \alpha$, unde fit

$$p = \int q \partial x = 2 \alpha x + \beta, \text{ hincque}$$
$$y = \int p \partial x = \alpha x x + \beta x + \gamma,$$

quarum trium constantium α, β, γ, una per aequationem propositam determinatur. Facta autem divisione per ∂q habebi-

mus $x = -Q'$, unde colligitur

$$p = Q + q x = Q - q Q',$$

hincque ob $\partial x = -\partial Q' = Q'' \partial q$, erit

$$y = \int p \partial x = \int Q'' \partial q (Q' q - Q) + b.$$

Exemplum.

§. 6. Sit $Q = a q^m$, erit

$$Q' = m a q^{m-1} \text{ atque}$$

$$Q'' = m (m-1) a q^{m-2}.$$

Hoc ergo casu erit

$$x = -Q' = -m a q^{m-1} \text{ et}$$

$$y = m (m-1)^2 a a \int q^{2m-2} \partial q + b, \text{ sive}$$

$$y = \frac{m(m-1)}{2} a a q^{2m-1} + b.$$

Est vero $q^{m-1} = -\frac{x}{m a}$, ita ut valor ipsius y facile per x exprimi poterit, quo facto habebitur integrale completum hujus aequationis differentio-differentialis

$$\frac{\partial y}{\partial x} - \frac{x \partial \partial y}{\partial x^2} = \frac{a \partial \partial y^m}{\partial x^{2m}}.$$

Problema

§. 7. *Proposita aequatione differentiali tertii gradus* $q - r x = R$, *ejus integrale completum investigare.*

Solutio.

Haec aequatio differentiata, ob $\partial q = r \partial x$, dat $-x \partial r = \partial R = R' \partial r$, cujus aequationis factor ∂r nihilo ae-

quatus hanc suppeditabit aequationem

$$y = \alpha x^3 + \beta x^2 + \gamma x + \delta,$$

ubi quatuor constantium α, β, γ, δ, una ex ipsa aequatione proposita determinata habebitur. Cum enim hinc sit

$$p = 3 \alpha x^2 + 2 \beta x + \gamma, \quad q = 6 \alpha x + 2 \beta, \quad r = 6 \alpha,$$

erit substituendo $2 \beta = \mathrm{R}$, ita ut tres tantum constantes arbitrariae in calculo relinquantur, uti natura hujusmodi aequationum postulat. Facta autem divisione per ∂r satisfaciet aequatio $x = -\mathrm{R}'$, unde colligitur $q = \mathrm{R} - r \mathrm{R}'$. Hinc, ab

$$\partial x = -\partial \mathrm{R}' = -\mathrm{R}'' \partial r,$$

reperietur

$$p = \int q \partial x = \int \mathrm{R}'' \partial r (r \mathrm{R}' - \mathrm{R}),$$

ac denique $y = \int p \partial x$, ubi ob duplicem integrationem duae constantes arbitrariae inferuntur.

Exemplum.

§. 8. Sit $\mathrm{R} = a r^m$, erit

$$\mathrm{R}' = m a r^{m-1} \text{ et } \mathrm{R}'' = m(m-1) a r^{m-2},$$

unde colligitur

$$p = \frac{m(m-1)}{2} a a r^{2m-1} + b,$$

atque ob

$$\partial x = -\partial \mathrm{R}' = -\mathrm{R}'' \partial r = -m(m-1) a r^{m-2} \partial r,$$

nanciscimur

$$y = \int p \partial x = -\frac{m^2(m-1)}{2 \cdot 3} a^3 r^{3m-2} - \frac{m(m-1)}{m-2} a b r^{m-1} + c,$$

unde ob $r^{m-1} = -\frac{x}{ma}$ facile obtinetur aequatio finita inter x et y, haecque erit integrale completum hujus aequationis differentialis tertii gradus

$$\frac{\partial \partial y}{\partial x^2} - \frac{x \partial^3 y}{\partial x^3} = \frac{a(\partial^3 y)^m}{\partial x^{3m}}.$$

Problema 4.

§. 9. *Proposita aequatione differentiali quarti gradus* $r - sx = S$, *ejus integrale completum indagare.*

Solutio.

Ob $\partial r = s \partial x$ fiet, aequationem propositam differentiando, $-x \partial s = \partial S = S' \partial s$, cujus aequationis factor ∂s praebet aequationem finitam

$$y = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon,$$

ubi una constantium per ipsam aequationem propositam determinatur. Porro satisfacit aequatio $x = -S'$; unde colligitur $r = S - s S'$, hincque, ob

$$\partial x = -\partial S' = -S'' \partial s$$

reperitur

$$q = \int r \partial x, \; p = \int q \partial x \text{ et } y = \int p \partial x, \text{ sive}$$

$$y = \int \partial x \int \partial x \int r \partial x,$$

ubi ob triplicem integrationem tres adjiciendae sunt constantes arbitrariae. Simili modo ad aequationes altiorum graduum progredi licet.

De aequationibus secundi generis.

Problema 5.

§. 10. *Proposita aequatione differentiali primi gradus hujusmodi* $y + \mathfrak{P} x = P$, *ejus integrale completum investigare.*

Solutio.

Si ista aequatio $y + \mathfrak{P} x = P$ differentietur, et loco ∂y scribatur $p \partial x$, prodit haec

$$p \partial x + \mathfrak{P} \partial x + x \partial \mathfrak{P} = \partial P,$$

sive posito $\partial P = P' \partial p$, erit

$$(p + \mathfrak{P}) \partial x + x \partial \mathfrak{P} = P' \partial p,$$

quae per $p + \mathfrak{P}$ divisa dat

$$\partial x + x . \partial . l (p + \mathfrak{P}) - \frac{x \partial p}{p + \mathfrak{P}} = \frac{P' \partial p}{p + \mathfrak{P}}.$$

[Est enim $\frac{x \partial \mathfrak{P}}{p + \mathfrak{P}} = x . \partial . l (p + \mathfrak{P}) - \frac{x \partial p}{p + \mathfrak{P}}$].

Quod si jam ponamus $\int \frac{\partial p}{p + \mathfrak{P}} = z$, aequatio illa integrabilis reddetur multiplicando per $e^{-z} (p + \mathfrak{P})$. Prodit enim

$$(p + \mathfrak{P}) e^{-z} \partial x + (p + \mathfrak{P}) x e^{-z} \partial . l (p + \mathfrak{P})$$
$$- x \partial z e^{-z} (p + \mathfrak{P}) = e^{-z} P' \partial p,$$

cujus integrale manifesto est

$$x e^{-z} (p + \mathfrak{P}) = \int e^{-z} P' \partial p,$$

unde colligitur

$$x = \frac{e^z}{p + \mathfrak{P}} \int e^{-z} P' \partial p = \frac{e^z}{p + \mathfrak{P}} \int e^{-z} \partial P,$$

unde statim fit

$$y = P - \frac{\mathfrak{P} e^z}{p + \mathfrak{P}} \int e^{-z} \partial P,$$

ubi e^z est etiam functio ipsius p, ita ut ambae variabiles x et y per unam eandemque variabilem p exprimantur, quae expressiones jam constantem arbitrariam per se involvunt, ita ut ejus adjectione non amplius opus sit.

Exemplum.

§. 11. Sit $P = a p^m$ et $\mathfrak{P} = b p^n$, ita ut aequatio integranda sit $y + b p^n = a p^m$. Hic igitur erit

$$z = \int \frac{\partial p}{p (1 + b p^{n-1})} = \int \frac{\partial p}{p} - b \int \frac{p^{n-2} \partial p}{1 + b p^{n-1}},$$

unde colligitur actu integrando

$$z = lp - \frac{1}{n-1} l(1 + bp^{n-1}),$$

ex quo fit

$$e^z = \frac{p}{(1 + bp^{n-1})^{\frac{1}{n-1}}}, \text{ et } e^{-z} = \frac{(1 + bp^{n-1})^{\frac{1}{n-1}}}{p},$$

quamobrem habebimus

$$\int e^{-z} \partial P = am \int p^{m-2} (1 + bp^{n-1})^{\frac{1}{n-1}} \partial p,$$

in qua expressione nullae quantitates transcendentes insunt, ita ut x et y facile definiantur, hocque modo obtinetur integrale completum istius aequationis differentialis primi gradus

$$y + bx \frac{\partial y^n}{\partial x^n} = a \frac{\partial y^m}{\partial x^m}.$$

Problema 6.

§. 12. *Proposita hac aequatione differentiali secundi gradus,* $p + \mathfrak{Q} x = Q$, *ejus integrale completum invenire.*

Solutio.

Attendenti mox patebit, hanc aequationem ex praecedente oriri, si loco y, P, $\mathfrak{P}$, scribantur litterae p, Q, $\mathfrak{Q}$, quandoquidem litterae y, p, q, r, etc. uniformi lege progrediuntur; quamobrem facta hac immutatione ex praecedente solutione statim habebimus

$$x = \frac{e^z}{q + \mathfrak{Q}} \int e^{-z} \partial Q, \text{ existente } z = \int \frac{\partial q}{q + \mathfrak{Q}};$$

sicque x hic erit functio solius quantitatis q, ex qua fit

$$\partial x = \frac{Q' - x\mathfrak{Q}'}{q + \mathfrak{Q}} \partial q.$$

Deinde nunc etiam p per solam variabilem q definietur: erit enim per §. 10.

$$p = Q - \frac{\mathfrak{Q}\, e^z}{q + \mathfrak{Q}} \int e^{-z} \partial Q.$$

Cum igitur sit $y = \int p \partial x$, etiam quantitas y per solam functionem ipsius q exprimetur, hocque modo problema perfecte solutum est censendum.

Problema 7.

§. 13. *Proposita aequatione differentiali tertii gradus hac $q + \mathfrak{R} x = R$, ejus integrale completum assignare.*

Solutio.

Haec solutio simili modo ex problemate primo hujus secundi generis (§. 10.) derivari potest, dum loco y, P, $\mathfrak{P}$, scribatur q, R, $\mathfrak{R}$, id quod si primo in aequatione pro x fuerit factum, suppeditabit hanc expressionem

$$x = \frac{e^z}{r + \mathfrak{R}} \int e^{-z} \partial R, \text{ existente } z = \int \frac{\partial r}{r + \mathfrak{R}},$$

sicque x erit functio solius variabilis r; tum vero erit

$$\partial x = \frac{R' - x \mathfrak{R}'}{r + \mathfrak{R}} \partial r.$$

Formula porro ibi pro y inventa et huc translata dabit pro q hanc expressionem

$$q = R - \frac{\mathfrak{R}\, e^z}{r + R} \int e^{-z} \partial R,$$

quae etiam tantum variabilem r ejusque functiones involvit. Quia igitur $p = \int q \partial x$ et $y = \int p \partial x$, erit $y = \int \partial x \int q \partial x$, sicque etiam y per solam variabilem r exprimetur.

Problema 8.

§. 14. *Proposita aequatione differentiali quarti gradus* $r + \mathfrak{S} x = S$, *ejus integrale investigare.*

Solutio.

Hic erit

$$x = \frac{e^z}{s + \mathfrak{S}} \int e^{-z} \partial S, \text{ existente } z = \int \frac{\partial s}{s + \mathfrak{S}}.$$

Porro erit

$$\partial x = \frac{S' - x \mathfrak{S}'}{s + \mathfrak{S}} \partial s, \; r = S - \frac{\mathfrak{S} e^z}{s + \mathfrak{S}} \int e^{-z} \partial S,$$

$$q = \int r \partial x, \; p = \int \partial x \int r \partial x, \text{ et}$$

$$y = \int p \partial x = \int \partial x \int \partial x \int r \partial x,$$

ubi omnia per solam variabilem s determinantur.

§. 15. Quin etiam istas aequationes differentiales, quarum integralia hic exhibuimus, certo modo inter se conjungere licet, ut integratio eadem methodo, qua hic usi sumus, institui queat. Hoc modo nanciscemur innumera nova genera hujusmodi aequationum differentialium, quae etiam differentiando ad integrationem perduci poterunt, quod argumentum in sequentibus problematibus pertractemus.

Problema 9.

§. 16. *Posito* $p + f q = t$, *sint* T *et* $\mathfrak{T}$ *functiones quaecunque ipsius* t, *sive algebraicae sive transcendentes, ac proposita fuerit haec aequatio differentialis secundi gradus* $y + f p + \mathfrak{T} x = T$, *ejus integrale completum investigare.*

Solutio.

Ponatur $y + fp = z$, erit

$$\partial z = \partial x (p + fg), \text{ ergo } \partial z = t \partial x.$$

Quare cum nunc aequatio proposita sit $z + \mathfrak{T} x = \mathrm{T}$, differentiando prodit

$$\partial z + \mathfrak{T} \partial x + x \partial \mathfrak{T} = \partial \mathrm{T}, \text{ sive}$$

$$(t + \mathfrak{T}) \partial x + x \partial \mathfrak{T} = \partial \mathrm{T},$$

unde colligitur haec aequatio

$$\partial x + \frac{x \partial \mathfrak{T}}{t + \mathfrak{T}} = \frac{\partial \mathrm{T}}{t + \mathfrak{T}},$$

ad quam integrandam ponatur $\int \frac{\partial t}{t + \mathfrak{T}} = u$, eritque

$$\int \frac{\partial \mathfrak{T}}{t + \mathfrak{T}} = l(t + \mathfrak{T}) - u,$$

tum vero aequatio nostra integrabilis reddetur, si eam multiplicemus per $e^{-u}(t + \mathfrak{T})$: integrale enim erit

$$x e^{-u} (t + \mathfrak{T}) = \int e^{-u} \partial \mathrm{T},$$

ex quo deducitur

$$x = \frac{e^u}{t + \mathfrak{T}} \int e^{-u} \partial \mathrm{T},$$

sicque x aequetur certae functioni ipsius t, quam hoc modo per integrationem invenire licet, ejusque differentiale erit

$$\partial x = \frac{\partial \mathrm{T} - x \partial \mathfrak{T}}{t + \mathfrak{T}}.$$

Hinc igitur prodit $z = \mathrm{T} - \mathfrak{T} x$. Cum nunc sit

$$y + fp = z, \text{ erit } y \partial x + f \partial y = z \partial x,$$

unde colligitur

$$\partial y + \frac{y \partial x}{f} = \frac{z \partial x}{f},$$

quae aequatio multiplicata per $e^{\frac{x}{f}}$ dat integrale

$$y e^{\frac{x}{f}} = \frac{1}{f} \int e^{\frac{x}{f}} z \partial x,$$

ubi cum tam z quam x sint functiones ipsius t, erit etiam y functio ipsius t tantum, cum sit

$$y = \frac{e^{-\frac{x}{f}}}{f} \int e^{\frac{x}{f}} z \partial x,$$

Problema 10.

§. 17. *Posito $p + fg + gz = t$, si fuerint T et $\mathfrak{T}$ functiones quaecunque ipsius t, sive algebraicae sive transcendentes, ac proposita fuerit haec aequatio differentialis tertii gradus: $y + fp + gq + \mathfrak{T}x = \mathrm{T}$, ejus integrale completum invenire.*

Solutio.

Ponatur $y + fp + gq = z$, eritque differentiando

$$\partial z = \partial x (p + fq + gr) = t \partial x,$$

sicque nostra aequatio integranda erit $z + \mathfrak{T}x = \mathrm{T}$, pro qua erit ut ante

$$x = \frac{e^{u}}{t + \mathfrak{T}} \int e^{-u} \partial \mathrm{T}, \text{ et } z = \mathrm{T} - \mathfrak{T}x,$$

posito scilicet $\int \frac{\partial t}{t + \mathfrak{T}} = u$. Ambae igitur illae expressiones functiones erunt solius variabilis t, unde etiam ∂x per eandem variabilem exprimetur. Tantum igitur superest ut etiam altera variabilis principalis y indagetur. Cum autem sit $y + fp + gq = z$, loco litterarum p et q scribantur valores initio assumti $\frac{\partial y}{\partial x}$ et $\frac{\partial \partial y}{\partial x^2}$, eritque, si tota aequatio per ∂x^2 multiplicetur, haec aequatio integranda

$$y \partial x^2 + f \partial x \partial y + g \partial \partial y = z \partial x^2,$$

in qua cum tam x quam z sint functiones solius t, etiam z

tanquam functionem ipsius t tractare licebit. Jam olim autem a me aliisque ostensum est, quomodo talis aequatio tractari debeat, quam ergo evolutionem hic repetere superfluum foret. Sufficiat enim notasse, valorem ipsius y per terminos hujus formae $\int e^{\lambda x} z \partial x$ assignari, eum igitur per solam variabilem t exprimere licebit, sicque etiam y per functionem ipsius t definietur.

Problema 11.

§. 18. *Posito $p + f q + g r + h s = t$, si fuerint T et $\mathfrak{T}$ functiones quaecunque ipsius t, sive algebraicae sive transcendentes, ac proposita fuerit talis aequatio differentialis quarti gradus*

$$y + f p + g q + h r + \mathfrak{T} x = \mathrm{T},$$

in ejus integrale completum inquirere.

Solutio.

Sit $y + f p + g q + h r = z$, eritque differentiando

$$\partial z = \partial x (p + f q + g r + h s) = t \partial x,$$

atque aequatio integranda fiet $z + \mathfrak{T} x = \mathrm{T}$, pro qua iterum, sumto $\int \frac{\partial t}{t + \mathfrak{T}} = u$, erit

$$x = \frac{e^u}{t + \mathfrak{T}} \int e^u \partial \mathrm{T}, \text{ atque } z = \mathrm{T} - \mathfrak{T} x,$$

ita ut tam x quam z per solam variabilem t exprimantur. His inventis, si in aequatione initio assumta loco p, q, r, s, eorum valores substituantur, prodibit haec aequatio tertii gradus

$$y \partial x^3 + f \partial x^2 \partial y + g \partial x \partial \partial y + h \partial^3 y = z \partial x^3,$$

cujus integrale completum per ea quae circa hujusmodi aequationes sunt prolata, tanquam cognitum spectare licet, ita ut etiam hoc casu ambae variabiles x et y per novam variabilem t

exprimantur. Facile autem patet hoc modo ad aequationes differentiales adhuc altiorum graduum progredi licere. Hac igitur ratione calculo integrali haud contemnendum incrementum allatum est censendum. Cum igitur hic praecipuum negotium versetur in integratione completa hujusmodi aequationis

$$y + \frac{f\,\partial y}{\partial x} + \frac{g\,\partial\partial y}{\partial x^2} + \frac{h\,\partial^3 y}{\partial x^3} + \text{etc.} = z,$$

ubi z est functio quaecunque ipsius x, ejus resolutionem jam passim exhibitam huc accommodemus et breviter ostendamus. Formetur haec aequatio

$$1 + f u + g u^2 + h u^3 + i u^4 + \text{etc.} = 0,$$

cujus radices u designentur litteris α, β, γ, δ, etc. quibus inventis erit uti jam olim ostendi

$$y = \frac{e^{\alpha x}\int e^{-\alpha x} z\,\partial x}{f + 2g\alpha + 3h\alpha^2 + 4i\alpha^3 + \text{etc.}} + \frac{e^{\beta x}\int e^{-\beta x} z\,\partial x}{f + 2g\beta + 3h\beta^2 + 4i\beta^3 + \text{etc.}} + \text{etc.}$$

Hae scilicet formulae ex singulis radicibus α, β, γ, δ, etc. formatae et junctim sumtae dabunt valorem ipsius y atque adeo integrale completum, quia singulae formulae integrales constantem arbitrariam involvunt.

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2) Specimen aequationum differentialium indefiniti gradus earumque integrationis. *M. S. Academiae exhib. die* 13 *Decembris*, 1781.

§. 19. Quando aequationes differentiales secundum gradus differentialium distinguuntur, ipsa rei natura gradus intermedios excludere videtur: cum enim totidem integrationibus opus sit, harum numerus certe non integer esse non potest. Incidi tamen

nuper in aequationem differentialem indefiniti gradus, cujus exponens etiam numerus fractus esse potest, atque adeo mihi licuit ejus integrale assignare; quod cum omni attentione dignum videatur, totam analysin, qua sum usus, hic dilucide exponam.

§. 20. Cum miras proprietates unciarum potestatum binomii, quas hoc charactere indicare soleo $\left(\frac{p}{q}\right)$, cujus valor est hoc productum

$$\frac{p}{1} \cdot \frac{p-1}{2} \cdot \frac{p-2}{3} \cdot \cdots \cdot \frac{p-q+1}{q},$$

considerassem, in mentem mihi venit valorem hujusmodi formulae $\left(\frac{p}{q}\right)$ ad formulam integralem revocare, unde etiam casus, quibus p et q non sunt numeri integri, assignari queant. Directe quidem talem reductionem non succedere observavi, unde ejus valorem reciprocum $\frac{1}{\left(\frac{p}{q}\right)}$ sum contemplatus, cujus valor est

$$\frac{1}{p} \cdot \frac{2}{p-1} \cdot \frac{3}{p-2} \cdot \cdots \cdot \frac{q}{p-q+1}.$$

Hunc in finem statuo

$$\frac{1.\,2.\,3.\ldots.q \times x^p}{p(p-1)(p-2)\ldots.(p-q+1)} = s,$$

ita ut posito $x = 1$ desideratus valor ipsius $1 : \left(\frac{p}{q}\right)$ obtineatur.

§. 21. Sit nunc brevitatis gratia $1.\,2.\,3\ldots q = \mathrm{N}$, ut habeatur $s = \frac{\mathrm{N}\,x^p}{p\ldots(p-q+1)}$, in cujus denominatore tenendum est factores continuo unitate decrescere. Quod si jam ista formula differentietur, prodibit

$$\frac{\partial s}{\partial x} = \frac{N x^{p-1}}{(p-1) \ldots . (p-q+1)},$$

sicque primus factor denominatoris est sublatus, ac differentiatione denuo instituta prodibit

$$\frac{\partial \partial s}{\partial x^2} = \frac{N x^{p-1}}{(p-2) \ldots . (p-q+1)}.$$

Hoc igitur modo continuo differentiando, omnes factores denominatoris tollentur, ac pervenietur tandem ad hanc aequationem

$$\frac{\partial^q s}{\partial x^q} = N x^{p-q}.$$

§. 22. Pervenimus igitur, loco N valorem suum substituendo, ad hanc aequationem differentialem

$$\frac{\partial^q s}{1 \ldots . . q \, \partial x^q} = x^{p-q},$$

quam ergo tot vicibus integrari oporteret, quot q continet unitates, atque singulae integrationes ita sunt instituendae ut, posito $x = 0$ integralia evanescant, et postquam omnes integrationes fuerint absolutae, loco x scribi debebit unitas, hocque modo valor ipsius s resultans dabit valorem formulae $1 : \left(\frac{p}{q}\right)$. Quo autem istas integrationes generalius expediamus, loco x^{p-q} scribamus X, ut habeamus hanc aequationem resolvendam

$$\frac{\partial^q s}{1.2 \ldots . . q \, \partial x^q} = X.$$

§. 23. Hanc aequationem primo multiplicemus per ∂x, ejusque integrale dabit

$$\frac{\partial^{q-1} s}{1.2.3 \ldots . q \, \partial x^{q-1}} = \int X \partial x.$$

Istam aequationem ducamus in $1.\,\partial x$, eritque integrando

$$\frac{\partial^{q-2} s}{2.3\ldots.q.\partial x^{q-2}} = \int \partial x \int X \partial x = x \int X \partial x - \int X x \partial x.$$

Per notas enim reductiones ejusmodi integralia repetita ad simplicia reduci possunt. Haec aequatio jam per $2\,\partial x$ multiplicata eodemque modo integrata praebebit

$$\frac{\partial^{q-3} s}{3.4\ldots.q.\partial x^{q-3}} = x^2 \int X \partial x - 2x \int X x \partial x + \int X x^2 \partial x.$$

Nunc per $3\,\partial x$ multiplicando et integrando proveniet

$$\frac{\partial^{q-4} s}{4.5\ldots.q.\partial x^{q-4}} = x^3 \int X \partial x - 3x^2 \int X x \partial x + 3x \int X x^2 \partial x - \int X x^3 \partial x.$$

Eodem modo reperietur

$$\frac{\partial^{q-5} s}{5.6\ldots.q.\partial x^{q-5}} = x^4 \int X \partial x - 4x^3 \int X x \partial x + 6x^2 \int X x^2 \partial x$$
$$- 4x \int X x^3 \partial x + \int X x^4 \partial x,$$

sicque in genere nostros characteres in usum vocando erit

$$\frac{\partial^{q-n} s}{n(n+1)\ldots..q.\partial x^{q+n}} = x^{n-1} \int X \partial x - \left(\frac{n-1}{1}\right) x^{n-2} \int X x \partial x$$
$$+ \left(\frac{n-1}{2}\right) x^{n-3} \int X x^2 \partial x - \left(\frac{n-1}{3}\right) x^{n-4} \int X x^3 \partial x + \text{etc.}$$

§. 24. Statuamus nunc $n = q$, et cum sit $\partial^0 s = s$, orietur haec aequatio finita

$$\frac{s}{q} = x^{q-1} \int X \partial x - \left(\frac{q-1}{1}\right) x^{q-2} \int X x \partial x$$
$$+ \left(\frac{q-1}{2}\right) x^{q-3} \int X x^2 \partial x - \text{etc.}$$

cujus singula membra ita integrari debent, ut posito $x = 0$ evanescant, quod quidem semper eveniet, si modo sit $q - 1 > 0$, quamobrem ipsae formulae integrales $\int X \partial x$, $\int X x \partial x$, etc. tantum sive adjectione constantis integrari debent. Etsi enim hoc

modo x forte in denominatorem ingrediatur, per potestatem ipsius x, qua multiplicari debent, iterum tolletur.

§. 25. His circa singula integralia observatis extra signa summatoria jam ponere licebit $x = 1$, quippe qui est casus quaestionis propositae; sicque reperietur

$$1 : q\left(\frac{p}{q}\right) = \int X \partial x \left[1 - \left(\frac{q-1}{1}\right) x + \left(\frac{q-1}{2}\right) x^2 - \left(\frac{q-1}{3}\right) x^3 + \text{etc.}\right],$$

cujus seriei valor manifesto est $(1-x)^{q-1}$, ita ut habeamus hanc expressionem determinatam

$$\frac{1}{q\left(\frac{p}{q}\right)} = \int X \partial x (1-x)^{q-1},$$

cujus ergo valor etiam casibus quibus q non est numerus integer per quadraturas exhiberi potest, sicque aequationis differentialis indefiniti gradus $\partial^q s = N X \partial x^q$ integrale feliciter elicuimus, et quia $X = x^{p-q}$, omnes unciae hoc modo ad formas integrales redigentur

$$\left(\frac{p}{q}\right) = \frac{1}{q \int x^{p-q} \partial x (1-x)^{q-1}},$$

et quia exponentes ipsius x et ipsius $1-x$ permutari possunt, erit etiam

$$\left(\frac{p}{q}\right) = \frac{1}{q \int x^{q-1} \partial x (1-x)^{p-q}},$$

hancque formulam ex principio diversissimo non ita pridem sum adeptus.

Theorema 1.

§. 26. Valor hujus characteris $\left(\frac{p}{q}\right)$ reduci potest ad formulam integralem, cum sit

$$\left(\frac{p}{q}\right) = \frac{1}{q \int x^{q-1} \partial x (1-x)^{p-q}},$$

siquidem hoc integrale ab $x = 0$ ad $x = 1$ extendatur.

Corollarium 1.

§. 27. Sumto ergo $p = 0$ erit

$$\left(\frac{0}{q}\right) = \frac{1}{q \int x^{q-1} \partial x (1-x)^{-q}}.$$

Ostendi autem olim esse

$$\int x^{q-1} \partial x (1-x)^{-q} = \frac{\pi}{\sin. \pi q},$$

unde ergo fiet

$$\left(\frac{0}{q}\right) = \frac{\sin. \pi q}{\pi q}.$$

Corollarium 2.

§. 28. Deinde per notam integralium reductionem reperitur

$$\int x^{q-1} \partial x (1-x)^{p-q} = \frac{p}{\sin. \pi q} : \left(\frac{p-q}{p}\right),$$

cujus ergo valor, quoties p est numerus integer, absolute assignari potest, quamobrem in genere erit

$$\left(\frac{p}{q}\right) = \frac{\sin. \pi q}{\pi q} : \left(\frac{p-q}{p}\right).$$

Corollarium 3.

§. 29. Cum igitur vicissim sit

$$\int x^{q-1} \partial x (1-x)^{p-q} = \frac{1}{q \left(\frac{p}{q}\right)},$$

si hic loco $q-1$ scribamus f, et g loco $p-q$, habebimus

$$\int x^{f} \partial x (1-x)^{g} = \frac{1}{(1+f)\left(\frac{f+g+1}{f+1}\right)}.$$

Scholion.

§. 30. Quoniam igitur hanc formulam integralem nacti sumus ex aequatione integrali indefiniti gradus, eandem investigationem latius extendamus in sequente problemate.

Problema 12.

§. 31. *Proposita serie sive finita sive infinita*

$$S = \frac{A}{\left(\frac{p}{q}\right)} + \frac{B}{\left(\frac{p+1}{q}\right)} + \frac{C}{\left(\frac{p+2}{q}\right)} + \frac{D}{\left(\frac{p+3}{q}\right)} + etc.$$

ejus valorem per formulam integralem exprimere.

Solutio.

Tribuamus singulis terminis potestates ipsius x, ac statuamus

$$S = \frac{A\,x^p}{\left(\frac{p}{q}\right)} + \frac{B\,x^{p+1}}{\left(\frac{p+1}{q}\right)} + \frac{C\,x^{p+2}}{\left(\frac{p+2}{q}\right)} + \text{etc.},$$

quae series ergo, posito $x = 1$, praebebit ipsam seriem propositam. Ubi observandum, in omnibus terminis litteram q eundem retinere valorem, alteram vero p continuo unitate augeri, unde productum indefinitum $1.\ 2.\ 3\ \ldots\ldots\ q = N$ in omnibus terminis eundem retinebit valorem. Quare cum supra ex aequatione $s = \frac{x^p}{\left(\frac{p}{q}\right)}$ deduxerimus hanc aequationem differentialem indefiniti gradus

$$\frac{\partial^q s}{\partial x^q} = N\,x^{p-q},$$

ex singulis terminis nostrae seriei idem resultabit differentiale, si modo exponentem p unitate augeamus, unde ergo reperiemus

$$\frac{\partial^q s}{\partial x^q} = N\,A\,x^{p-q} + N\,B\,x^{p-q+1} + \text{etc.}$$

§. 32. Ponamus nunc

$$A + B x + C x^2 + D x^3 + \text{etc.} = V,$$

eritque

$$\frac{\partial^q s}{N \partial x^q} = x^{p-q} V,$$

quamobrem si statuamus $x^{p-q} V = X$, habebimus ipsam aequationem jam ante tractatam

$$\frac{\partial^q s}{1.2 \ldots\ldots q \partial x^q} = X,$$

cujus integratio q vicibus repetita nos perduxit ad hanc expressionem $s = q \int X \partial x (1 - x)^{q-1}$, unde ergo pro X et V valores substituendo nanciscemur summam quaesitam S, scilicet

$$S = q \int x^{p-q} \partial x (A + B x + C x^2 + D x^3 + \text{etc.}) (1 - x)^{q-1},$$

si modo hoc integrale ab $x = 0$ ad $x = 1$ extendatur, vel ut ante innuimus, si modo in integratione nulla constans adjiciatur, deinde vero sumatur $x = 1$.

Exemplum.

§. 33. Sit $V = (1 - x)^n$, ita ut sit

$$A = 1, \; B = -\left(\tfrac{n}{1}\right), \; C = +\left(\tfrac{n}{2}\right), \; D = -\left(\tfrac{n}{3}\right), \; \text{etc.},$$

et series proposita erit

$$S = \frac{1}{\left(\frac{p}{q}\right)} - \frac{\left(\frac{n}{1}\right)}{\left(\frac{p+1}{q}\right)} + \frac{\left(\frac{n}{2}\right)}{\left(\frac{p+2}{q}\right)} - \frac{\left(\frac{n}{3}\right)}{\left(\frac{p+3}{q}\right)} + \text{etc.}$$

tum igitur summa hujus seriei erit

$$S = q \int x^{p-q} \partial x (1 - x)^{q+n-1},$$

sive permutatis exponentibus ipsius x et $1 - x$, erit quoque

$$S = q \int x^{q+n-1} \partial x (1 - x)^{p-q}.$$

Nunc autem evidens est hanc ipsam formulam integralem ite-

rum ad characterem hic usitatum reduci posse ope §. 29. erit enim $f = q + n - 1$ et $g = p - q$, atque hinc prodibit

$$S = \frac{q}{(q+n)\left(\frac{p+n}{q+n}\right)}.$$

Hinc ergo sive formulis integralibus habebimus hanc summationem seriei infinitae maxime notabilem

$$\frac{1}{\left(\frac{p}{q}\right)} - \frac{\left(\frac{n}{1}\right)}{\left(\frac{p+1}{q}\right)} + \frac{\left(\frac{n}{2}\right)}{\left(\frac{p+2}{q}\right)} - \frac{\left(\frac{n}{3}\right)}{\left(\frac{p+3}{q}\right)} + \frac{\left(\frac{n}{4}\right)}{\left(\frac{p+4}{q}\right)} - \text{etc.}$$

$$= \frac{q}{(q+n)\left(\frac{p+n}{q+n}\right)}.$$

Corollarium 1.

§. 34. Si ergo fuerit $n = 0$, oritur aequatio manifeste identica scilicet $\frac{1}{\left(\frac{p}{q}\right)} = \frac{1}{\left(\frac{p}{q}\right)}$. At si $n = 1$ prodit

$$\frac{q}{(q+1)\left(\frac{p+1}{q+1}\right)} = \frac{1}{\left(\frac{p}{q}\right)} - \frac{1}{\left(\frac{p+1}{q+1}\right)}.$$

Si $n = 2$ fiet

$$\frac{q}{(q+2)\left(\frac{p+2}{q+2}\right)} = \frac{1}{\left(\frac{p}{q}\right)} - \frac{2}{\left(\frac{p+1}{q}\right)} + \frac{1}{\left(\frac{p+2}{q}\right)}.$$

Corollarium 2.

§. 35. Quo consensus cum veritate clarius appareat evolvamus casum determinatum, quo $p = 3$, $q = 2$, $n = 4$, eritque

$$\frac{q}{q+n} = \frac{1}{3}, \text{ et } \left(\frac{p+n}{q+n}\right) = \left(\frac{7}{6}\right) = \left(\frac{7}{1}\right) = 7.$$

Deinde fit

$\left(\frac{p}{q}\right) = \left(\frac{3}{2}\right) = 3$; $\left(\frac{p+1}{q}\right) = \left(\frac{4}{2}\right) = 6$; $\left(\frac{p+2}{q}\right) = \left(\frac{5}{2}\right) = 10$; $\left(\frac{p+3}{q}\right) = 15$;

quae est progressio numerorum trigonalium; tum vero erit

$$\left(\frac{n}{1}\right)=4;\ \left(\frac{n}{2}\right)=6;\ \left(\frac{n}{3}\right)=4;\ \left(\frac{n}{4}\right)=1.$$

His igitur valoribus substitutis erit

$$\frac{1}{3\cdot 7}=\frac{1}{3}-\frac{4}{6}+\frac{6}{10}-\frac{4}{15}+\frac{1}{21},$$

quod egregie convenit.

Exemplum. 2.

§. 36. Statuamus $V=(1+x)^{q-1}$, ut fiat

$$S=q\int x^{p-q}\,\partial x\,(1-xx)^{q-1};$$

tum vero erit

$$A=1;\ B=\left(\frac{q-1}{1}\right);\ C=\left(\frac{q-1}{2}\right);\ D=\left(\frac{q-1}{3}\right);\ \text{etc.}$$

sicque series proposita erit

$$S=\frac{1}{\left(\frac{p}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{p+2}{q}\right)}+\frac{\left(\frac{q-1}{3}\right)}{\left(\frac{p+3}{q}\right)}+\text{etc.}$$

Evidens autem est, hanc formulam integralem etiam ad nostros characteres reduci posse. Ponamus enim $xx=y$, erit

$$S=\frac{q}{2}\int y^{\frac{p-q-1}{2}}\,\partial y\,(1-y)^{q-1},$$

sive permutatis exponentibus

$$S=\frac{q}{2}\int y^{q-1}\,\partial y\,(1-y)^{\frac{p-q-1}{2}},$$

quae comparata cum §. 29. dat $f=q-1$, $g=\frac{p-q-1}{2}$, quibus valoribus substitutis colligitur

$$S=\frac{q}{2q\left(\frac{\frac{p+q-1}{2}}{q}\right)}=\frac{1}{2\left(\frac{\frac{p+q-1}{2}}{q}\right)}=\frac{1}{\left(\frac{p}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{q-2}{2}\right)}{\left(\frac{p+2}{q}\right)}+\text{etc.}$$

vel si ponatur $\frac{p+q-1}{2}=r$, erit

$$S=\frac{1}{2\left(\frac{r}{q}\right)}=\frac{1}{\left(\frac{p}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{p+2}{q}\right)}+\text{etc.}$$

Corollarium 1.

§. 37. Hic casu $q = 1$ summa inventa ipsi termino primo aequatur. Sumamus autem $q = 2$, erit

$$\frac{1}{2\left(\frac{\frac{p+1}{2}}{q}\right)} = \frac{1}{\left(\frac{p}{2}\right)} + \frac{1}{\left(\frac{p-1}{2}\right)},$$

hoc est

$$\frac{4}{pp-1} = \frac{2}{p(p-1)} + \frac{2}{p(p+1)},$$

unde patet istam summationem esse veritati consentaneam, de quo quidem nullum superesse potest dubium, quoties q est numerus integer positivus; quamobrem quosdam casus consideremus ubi non est talis.

Corollarium 2.

§. 38. Quo autem evolutio facilior evadat, contemplemur casum quo $r = q$, ut fiat $\left(\frac{r}{q}\right) = 1$, tum autem erit $p = 1 + q$ hincque

$$\left(\frac{p}{q}\right) = 1 + q;\ \left(\frac{p+1}{q}\right) = \frac{q+1}{1}\cdot\frac{q+2}{2};\ \left(\frac{p+2}{q}\right) = \frac{q+1}{1}\cdot\frac{q+2}{2}\cdot\frac{q+3}{3},$$

quibus substitutis orietur haec series

$$\tfrac{1}{2} = \frac{1}{q+1} + \frac{2(q-1)}{(q+1)(q+2)} + \frac{3(q-1)(q-2)}{(q+1)(q+2)(q+3)} + \frac{4(q-1)(q-2)(q-3)}{(q+1)(q+2)(q+3)(q+4)} + \text{etc.}$$

quae series notatu maxime est digna, quia ejus summa semper est $\frac{1}{2}$, quicunque valores litterae q tribuantur. Si enim sit $q = 0$, habebitur

$$\tfrac{1}{2} = 1 - 1 + 1 - 1 + 1 - \text{etc.}$$

quae est series notissima. Sit nunc $q = -1$, et ob $q + 1 = 0$ multiplicemus omnes terminos per $q + 1$, prodibitque haec series

$$0 = 1 - 4 + 9 - 16 + 25 - \text{etc.}$$

uti differentias sumendo facile patet. Ponamus $q = \frac{1}{2}$, et haec

SUPPLEMENTUM XI.

AD FINEM TOM. III.

DE

CALCULO VARIATIONUM.

Methodus nova et facilis calculum variationum tractandi. *Nov. Comment. Tom. XVI. Pag.* 35 — 70.

§. 1. Si detur aequatio quaecunque inter binas variabiles x et y, seu quod eodem redit, si y fuerit functio quaecunque ipsius x, tum omnes expressiones quomodocunque ex his duabus quantitatibus x et y formatae et compositae, tanquam functiones unius variabilis x spectari poterunt, ita ut pro quovis valore determinato ipsius x, determinatos quoque valores sortiantur.

§. 2. Hujusmodi autem expressionum ex quantitatibus x et y formatarum, tria genera constitui convenit; ad quorum primum referimus omnes illas expressiones, in quibus tantum ipsae quantitates x et y occurrunt et per operationes quascunque sive algebraicas sive etiam transcendentes inter se sunt complicatae, cujusmodi sunt $\alpha x^3 + \beta x y + \gamma y^3$, item $e^{\alpha x}$ Arc. sin. y, in qua posteriore operationes transcendentes cernuntur. Secundum autem genus eas complectitur expressiones, in quibus praeter ipsas quantitates x et y etiam ratio differentialium occurrit, quam rationem adeo ad differentialia cujusque gradus extendimus, cujusmodi expressionum indolem quo clarius perspiciamus, ponatur

more solito

$$\partial y = p\,\partial x;\ \partial p = q\,\partial x;\ \partial q = r\,\partial x;\ \text{etc.}$$

ac tales expressiones erunt functiones quantitatum x, y, p, q, r, etc. Tertium denique genus ejusmodi expressiones continet in quibus praeterea formulae integrales involvuntur, quorsum pertinent expressiones illae in calculo variationum imprimis consideratae, quae hac forma sunt repraesentatae $\int V\,\partial x$, ubi V est functio quaecunque non solum ipsarum x et y; sed etiam quantitatum p, q, r, etc., quin etiam ea alias insuper formulas integrales involvere potest.

§. 3. His circa terna hujusmodi expressionum genera constitutis, facilius indolem calculi variationum explicare poterimus. Totum enim negotium huc redit, ut si proposita fuerit relatio quaecunque inter x et y, eaque aliquantillum varietur, seu ejus loco alia quaepiam relatio inter x et y ab illa infinite parum quomodocunque discrepans adhibeatur, investigari oporteat, quantam mutationem omnes illae expressiones, tam primi, quam secundi et tertii generis sint subiturae, ad quod inveniendum in calculo variationum prouti equidem eum olim tractavi, praeter differentiale ∂y, quo quantitas y augetur dum x in $x + \partial x$ abit, ipsi quantitati y aliud incrementum δy tribuitur, penitus ab arbitrio nostro pendens neque per x determinatum, cui incremento variationis nomen indideram, atque methodum exposueram, variationes inde in singula expressionum genera redundantes inveniendi.

§. 4. Videbatur igitur calculus variationum omnino singulare calculi genus constituere, verum postquam ejus indolem accuratius essem perscrutatus, universum hunc calculum perspexi levi facta immutatione ad secundam partem calculi integralis, cujus ele-

menta in tertio volumine operis mei de hoc argumento exposui, reduci posse. Pertractavi autem in ista secunda parte eas integrationes, quae circa functiones duarum variabilium versantur, in quo calculi genere etiam nunc vix ultra prima elementa progredi licuit.

§. 5. Illius scilicet incrementi loco, quod variationem appellavi, ipsam quantitatem y non amplius tanquam functionem solius variabilis x considero, sed eam tanquam functionem binarum variabilium x et t in calculum introduco, sic enim dum $\partial x \left(\frac{\partial y}{\partial x}\right)$ significat verum differentiale ipsius y, haec formula $\partial t . \left(\frac{\partial y}{\partial t}\right)$ idem significare poterit, quod antea signo $\delta\, y$ indicavimus. Quo haec reddantur clariora concipiamus y ut applicatam cujuspiam curvae abscissae x respondentem, atque in calculo variationum alia relatio requiritur, quae omnes alias curvas huic saltem proximas complectatur, omnes autem hujusmodi curvas, si X denotet illam functionem cui y aequatur, tali aequatione contineri posse $y = X + t\,V$ manifestum est; denotante V functionem quamcunque ipsius x. Sumta enim t infinite parva haec aequatio omnes omnino lineas curvas propositae proximas in se comprehendet, atque adeo hanc formam multo generaliorem reddere licet, ita ut pro y functio quaecunque binarum variabilium x et t usurpari possit, dummodo ea ita fuerit comparata, ut posito $t = 0$, prodeat ipsa functio proposita $y = X$.

§. 6. Pro variatione igitur invenienda, quantitas x ut constans spectari, ipsius vero y differentiale tantum ex variabilitate ipsius t desumi debet; unde si expressio proposita fuerit primi generis, functio scilicet ipsarum x et y tantum, quam littera Z designemus, ponamus differentiatione consueta pro-

dire $M\,\partial x + N\,\partial y$, atque nunc pro variatione invenienda fiat $\partial x = 0$, at loco ∂y scribatur $\partial t \left(\frac{\partial y}{\partial t}\right)$, quippe quod est incrementum ex sola variabilitate t oriundum. Quo facto variatio quaesita hujus expressionis Z erit $= N\,\partial t \left(\frac{\partial y}{\partial t}\right)$. Quare si ipsa variatio simili modo per $\partial t \left(\frac{\partial Z}{\partial t}\right)$ indicetur, habebimus $\left(\frac{\partial Z}{\partial t}\right) = N \left(\frac{\partial y}{\partial t}\right)$.

§. 7. Nunc ad expressiones secundi generis progrediamur, in quibus quum praeter x et y occurrant quantitates p, q, r, etc. harum variationes quatenus y etiam a variabili t pendet, per legem generalem his formulis exprimentur

$$\partial t \left(\frac{\partial p}{\partial t}\right); \quad \partial t \left(\frac{\partial q}{\partial t}\right); \quad \partial t \left(\frac{\partial r}{\partial t}\right); \quad \text{etc.}$$

Quum autem pro sola variabili x, sit

$$p = \left(\frac{\partial y}{\partial x}\right); \quad q = \left(\frac{\partial p}{\partial x}\right) = \left(\frac{\partial\,\partial y}{\partial x^2}\right); \quad \text{et}$$

$$r = \left(\frac{\partial q}{\partial x}\right) = \left(\frac{\partial\,\partial p}{\partial x^2}\right) = \left(\frac{\partial^3 y}{\partial x^3}\right); \quad \text{etc.}$$

erit per regulas generales differentiandi functiones duarum variabilium

$$\left(\frac{\partial p}{\partial t}\right) = \left(\frac{\partial\,\partial y}{\partial x\,\partial t}\right); \quad \left(\frac{\partial q}{\partial t}\right) = \left(\frac{\partial^3 y}{\partial x^2\,\partial t}\right); \quad \left(\frac{\partial r}{\partial t}\right) = \left(\frac{\partial^4 y}{\partial x^3\,\partial t}\right); \quad \text{etc.}$$

ubi meminisse juvabit formulam verbi gratia $\left(\frac{\partial^3 y}{\partial x^2 . \partial t}\right)$ prodire, si functio y ter differentietur, et duabus vicibus sola x, una vice autem sola t variabilis sumatur, tum vero qualibet differentiatione differentialia simplicia ∂x vel ∂t abjiciantur.

§. 8. His expeditis sit jam Z functio quaecunque ipsarum x, y, p, q, r, etc., hic quidem nullo adhuc respectu habito ad variabilem t, quippe quae tantum in subsidium variationis introducitur, atque differentiatione more solito facta prodeat

$$\partial Z = M\,\partial x + N\,\partial y + P\,\partial p + Q\,\partial q + R\,\partial r + \text{etc.}$$

nunc igitur pro variatione seu $\partial t\left(\frac{\partial Z}{\partial t}\right)$ invenienda scribi debebit ut sequitur

$$\partial x = 0;\ \partial y = \partial t\left(\frac{\partial y}{\partial t}\right);\ \partial p = \partial t\left(\frac{\partial p}{\partial t}\right) = \partial t\left(\frac{\partial\partial y}{\partial x\partial t}\right);$$

$$\partial q = \partial t\left(\frac{\partial^3 y}{\partial x^2\partial t}\right);\ \partial r = \partial t\left(\frac{\partial^4 y}{\partial x^3\partial t}\right);\ \text{etc.}$$

atque variatio quaesita erit

$$\partial t\left(\frac{\partial Z}{\partial t}\right) = N\partial t\left(\frac{\partial y}{\partial t}\right) + P\partial t\left(\frac{\partial\partial y}{\partial x\partial t}\right) + Q\partial t\left(\frac{\partial^3 y}{\partial x^2\partial t}\right) + R\partial t\left(\frac{\partial^4 y}{\partial x^3\partial t}\right) + \text{etc.}$$

unde sequitur divisione per ∂t facta fore

$$\left(\frac{\partial Z}{\partial t}\right) = N\left(\frac{\partial y}{\partial t}\right) + P\left(\frac{\partial\partial y}{\partial x\partial t}\right) + Q\left(\frac{\partial^3 y}{\partial x^2\partial t}\right) + R\left(\frac{\partial^4 y}{\partial x^3\partial t}\right) + \text{etc.}$$

§. 9. Sit nunc etiam expressio quaecunque tertii generis proposita $\int Z\partial x$, ubi Z sit functio quaecunque ipsarum x, y, p, q, r, etc. ita ut per differentiationem ordinariam habeatur

$$\partial Z = M\partial x + N\partial y + P\partial p + Q\partial q + R\partial r + \text{etc.}$$

ubi quidem hactenus nulla ratio novae variabilis t est habita, atque integratio formulae propositae $\int Z\partial x$ per solam variabilem x est expedienda, quo observato, quaestio huc redit, ut si jam y ut functio binarum variabilium x et t consideretur et ubique quantitas y elemento $\partial t\left(\frac{\partial y}{\partial t}\right)$ augeatur, augmentum quod ipsa formula integralis $\int Z\partial x$ inde capiet definiatur, hoc enim augmentum ipsa erit variatio formulae integralis propositae.

§. 10. Quare ad hanc variationem inveniendam in functione illa Z ubique loco y scribatur ejus valor auctus $y + \partial t\left(\frac{\partial y}{\partial t}\right)$, sicque ut ante vidimus, ipsa functio Z augmentum capiet $\partial t\left(\frac{\partial Z}{\partial t}\right)$ ex quo ipsa formula integralis augmentum capiet hoc $\int\partial t\left(\frac{\partial Z}{\partial t}\right)\partial x$, quod erit ipsa variatio quaesita. Quoniam vero in hac integratione sola x pro variabili habetur elementum ∂t ante signum poni poterit ita ut jam variatio futura sit $= \partial t\int\partial x\left(\frac{\partial Z}{\partial t}\right)$.

§. 11. Quoniam igitur in §. 8. valor ipsius $\left(\frac{\partial Z}{\partial t}\right)$ jam evolutus habetur, si ille hic substituatur, formulae $\int Z \partial x$ variatio prodibit ita expressa

$$\partial t \int \partial x \left[N\left(\frac{\partial y}{\partial t}\right) + P\left(\frac{\partial \partial y}{\partial x \partial t}\right) + Q\left(\frac{\partial^3 y}{\partial x^2 \partial t}\right) + R\left(\frac{\partial^4 y}{\partial x^3 . \partial t}\right) + \text{etc.}\right]$$

quam etiam sequenti modo per partes repraesentasse juvabit

$$\partial t \int N \partial x \left(\frac{\partial y}{\partial t}\right) + \partial t \int P \partial x \left(\frac{\partial \partial y}{\partial x \partial t}\right) + \partial t \int Q \partial x \left(\frac{\partial^3 y}{\partial x^2 . \partial t}\right) + \partial t \int R \partial x \left(\frac{\partial^4 y}{\partial x^3 \partial t}\right) + \text{etc.}$$

qua expressione contenti esse possemus, si quaestio circa casum aliquem determinatum institueretur, ubi y non solum functioni cuipiam datae ipsius x aequaretur, sed etiam nova variabilis t modo determinato introduceretur; tum enim omnes istas formulas $\left(\frac{\partial y}{\partial t}\right)$; $\left(\frac{\partial \partial y}{\partial x \partial t}\right)$; $\left(\frac{\partial^3 y}{\partial x^2 \partial t}\right)$; etc. actu evolvere liceret, ita ut tum elementum ∂x per solam functionem ipsius x afficeretur, siquidem uti initio innuimus, evolutione facta, iterum poni debet $t = 0$.

§. 12. At vero tales quaestiones determinatae nunquam occurrere solent; sed potius relatio inter y et x semper incognita esse solet, inde demum determinanda, quod variatio in nihilum abire debeat, quippe in quo methodus maximorum et minimorum versatur. Hujusmodi quaestiones ergo ita enunciari convenit: qualis relatio inter quantitates x et y intercedere debeat, ut formulae integralis propositae $\int Z \partial x$ variatio in nihilum abeat, quomodocunque etiam nova variabilis t in calculum introducatur? Quodsi autem quaestio hac ratione instituatur, perspicuum est formulis $\left(\frac{\partial y}{\partial t}\right)$; $\left(\frac{\partial \partial y}{\partial x \partial t}\right)$; $\left(\frac{\partial^3 y}{\partial x^2 \partial t}\right)$; etc. nullos certos valores tribui posse.

§. 13. Verum hic prorsus singulare artificium in subsidium vocari potest, cujus ope formulas integrales posteriores in §. 11. ad formam priores reducere licet, ita ut in omnibus

eadem formula $\left(\frac{\partial y}{\partial t}\right)$ occurrat. Quum enim $\partial x\left(\frac{\partial\partial y}{\partial x\partial t}\right)$ sit differentiale formulae $\left(\frac{\partial y}{\partial t}\right)$ sumta sola x variabili, erit per consuetam integralium reductionem

$$\int P\partial x\left(\frac{\partial\partial y}{\partial x\partial t}\right)=P\left(\frac{\partial y}{\partial t}\right)-\int\partial x\left(\frac{\partial P}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right),$$

simili modo quia $\partial x\left(\frac{\partial^3 y}{\partial x^2.\partial t}\right)$ est differentiale formulae $\left(\frac{\partial\partial y}{\partial x\partial t}\right)$, habebimus statim hanc reductionem

$$\int Q\partial x\left(\frac{\partial^3 y}{\partial x^2.\partial t}\right)=Q\left(\frac{\partial\partial y}{\partial x\partial t}\right)-\int\partial x\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial\partial y}{\partial x\partial t}\right),$$

nunc vero per praecedentem reductionem fit

$$\int\partial x\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial\partial y}{\partial x\partial t}\right)=\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)-\int\partial x\left(\frac{\partial\partial Q}{\partial x^2}\right)\left(\frac{\partial y}{\partial t}\right),$$

sicque omnino habebimus

$$\int Q\partial x\left(\frac{\partial^3 y}{\partial x^2.\partial t}\right)=Q\left(\frac{\partial\partial y}{\partial x.\partial t}\right)-\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)+\int\partial x\left(\frac{\partial\partial Q}{\partial x^2}\right)\left(\frac{\partial y}{\partial t}\right),$$

atque nunc satis perspicuum est, sequentem formulam integralem ita reductam iri

$$\int R\partial x\left(\frac{\partial^4 y}{\partial x^3\partial t}\right)=R\left(\frac{\partial^3 y}{\partial x^2.\partial t}\right)-\left(\frac{\partial R}{\partial x}\right)\left(\frac{\partial\partial y}{\partial x\partial t}\right)+\left(\frac{\partial\partial R}{\partial x^2}\right)\left(\frac{\partial y}{\partial t}\right)$$
$$-\int\partial x\left(\frac{\partial^3 R}{\partial x^3}\right)\left(\frac{\partial y}{\partial t}\right),$$

ac si insuper talis formula adesset, foret

$$\int S\partial x\left(\frac{\partial^5 y}{\partial x^4\partial t}\right)=S\left(\frac{\partial^4 y}{\partial x^3.\partial t}\right)-\left(\frac{\partial S}{\partial x}\right)\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)+\left(\frac{\partial\partial S}{\partial x^2}\right)\left(\frac{\partial\partial y}{\partial x\partial t}\right)$$
$$-\left(\frac{\partial^3 S}{\partial x^3}\right)\left(\frac{\partial y}{\partial t}\right)+\int\partial x\left(\frac{\partial^4 S}{\partial x^4}\right)\left(\frac{\partial y}{\partial t}\right).$$

§. 14. Quodsi nunc has formulas reductas substituamus in expressione variationis quaesitae formulae $\int Z\partial x$, tum haec variatio non solum formulis constabit integralibus, sed etiam continebit partes absolutas, quarum aliae formulam $\left(\frac{\partial y}{\partial t}\right)$, aliae hanc $\left(\frac{\partial\partial y}{\partial x\partial t}\right)$, aliae vero hanc $\left(\frac{\partial^3 y}{\partial x^2\partial t}\right)$ etc. continebunt; dum contra omnes integrales eandem formulam $\left(\frac{\partial y}{\partial t}\right)$ involvunt, quocirca variatio quaesita formulae propositae $\int Z\partial x$, sequenti modo habebitur expressa

$$\partial t \int \partial x \left(\frac{\partial y}{\partial t}\right) \left[N - \left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial \partial Q}{\partial x^2}\right) - \left(\frac{\partial^3 R}{\partial x^3}\right) + \left(\frac{\partial^4 S}{\partial x^4}\right) - \text{etc.}\right]$$
$$+ \partial t \left(\frac{\partial y}{\partial t}\right) \quad \left[P - \left(\frac{\partial Q}{\partial x}\right) + \left(\frac{\partial \partial R}{\partial x^2}\right) - \left(\frac{\partial^3 S}{\partial x^3}\right) + \text{etc.}\right]$$
$$+ \partial t \left(\frac{\partial \partial y}{\partial x \partial t}\right) \left[Q - \left(\frac{\partial R}{\partial x}\right) + \left(\frac{\partial \partial S}{\partial x^2}\right) - \text{etc.}\right]$$
$$+ \partial t \left(\frac{\partial^3 y}{\partial x^2 \partial t}\right) \left[R - \left(\frac{\partial S}{\partial x}\right) + \text{etc.}\right]$$
$$+ \partial t \left(\frac{\partial^3 y}{\partial x^3 \partial t}\right) \left[S - \text{etc.}\right]$$
$$+ \text{etc.}$$

§. 15. Quamquam hic meum institutum non est methodum maximorum et minimorum pertractare, quoniam hoc alibi jam satis copiose est factum; tamen hic praetermittere non possum, quin observem, si variatio formulae $\int Z \partial x$ evanescere debeat, quomodocunque etiam nova variabilis t in calculum ingrediatur, id nullo modo fieri posse, nisi tota pars prima integralis seorsim evanescat, ex quo necesse est, inter x et y hanc aequationem constitui

$$0 = N - \left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial \partial Q}{\partial x^2}\right) - \left(\frac{\partial^3 R}{\partial x^3}\right) + \left(\frac{\partial^4 S}{\partial x^4}\right) - \text{etc.}$$

et quia nunc variabilis t nulla amplius ratio habetur, sicque tantum unica adhuc variabilis x superest, clausulis omissis hanc habebimus aequationem

$$0 = N - \frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} \frac{\partial^4 S}{\partial x^4} - \text{etc.}$$

qua desiderata relatio inter x et y exprimitur. Partes autem absolutae, tantum ad terminos extremos referuntur, circa quas ea observari debent, quae jam alibi fusius sunt praecepta.

§. 16. Hic etiam non immoror iis casibus, quibus quantitas Z ipsa insuper formulas integrales involvit, quoniam etiam hoc argumentum alibi satis est pertractatum, verum hic opus multo magis arduum molior, dum eandem hanc methodum ad functiones adeo duarum variabilium extendere conabor, quod equidem in dis-

sertatione illa, quam olim de calculo variationum conscripseram, tunc temporis praestare non potui, multitudine tot quantitatum diversi generis deterritus.

Applicatio methodi praecedentis ad functiones duarum variabilium.

§. 17. Si habeatur aequatio quaecunque inter ternas variabiles x, y et z, ea naturam cujuspiam superficiei exprimi censemus, ubi quidem binas coordinatas x et y in plano horizontali constitui intelligamus, tertiam vero z verticalem, sicque haec tertia z, ut functio spectari potest binarum x et y; unde more solito duplicia incrementa consideranda occurrunt, quatenus scilicet a variabilitate ipsius x, vel ipsius y nascuntur. Illud nempe incrementum ipsius z quod ex variatione ipsius x oritur hac formula $\partial x \left(\frac{\partial z}{\partial x}\right)$, hoc vero ex variatione ipsius y oriundum ista $\partial y \left(\frac{\partial z}{\partial y}\right)$ indicari solet.

§. 18. Quodsi jam haec superficies aequatione inter x, y et z expressa, cum aliis quibuscunque superficiebus ipsi proximis comparari debeat, id commodissime fiet novam variabilem t introducendo, ita ut jam z spectanda sit ut functio trium variabilium, x, y et t, quae quidem sumto $t = 0$, in functionem superiorem abeat, at dum ipsi t valores infinite parvi tribuuntur, omnes superficies proximas complectatur, quo posito perspicuum est, quoniam variabiles x et y a nova t neutiquam pendent earum differentialia ∂x et ∂y nullo modo cum ∂t permisceri, sola vero coordinata z triplicis generis incrementa capere potest, praeter bina enim jam ante commemorata, quae vel ab x vel ab y proficiscuntur, accipere poterit incrementum a variabilitate ipsius t oriundum, quod tali formula $\partial t \left(\frac{\partial z}{\partial t}\right)$ est repraesentandum.

§. 19. Ponamus nunc V esse expressionem utcunque ex ipsius coordinatis x, y et z compositam, sive per meras operationes algebraicas, sive etiam transcendentes formatas, quae more solito differentiata praebeat

$$\partial V = L \partial x + M \partial y + N \partial z,$$

atque si ejusdem incrementum desideretur a nova variabili t sola oriundum, manifestum est, statui debere $\partial x = 0$ et $\partial y = 0$, at loco ∂z scribi debere $\partial t \left(\frac{\partial z}{\partial t}\right)$, sicque hoc signandi modo usurpato habebimus

$$\partial t \left(\frac{\partial V}{\partial t}\right) = N \partial t \left(\frac{\partial z}{\partial t}\right) \text{ ideoque } \left(\frac{\partial V}{\partial t}\right) = N \left(\frac{\partial z}{\partial t}\right).$$

Tales autem expressiones ut ante primum genus constituunt.

§. 20. Progrediamur ergo ad secundum genus, quo expressio v praeter ipsas coordinatas x, y, z etiam rationes differentialium earum involvat; atque hic quidem ante omnia formam hujusmodi expressionum accuratius perpendi oportet. Quoniam autem hic statim quantitas z duplicia incrementa capere potest, (hic enim nondum ad novam variabilem t respicimus) ponamus brevitatis gratia

$$\left(\frac{\partial z}{\partial x}\right) = p \text{ et } \left(\frac{\partial z}{\partial y}\right) = p',$$

quae duae litterae differentialia primi gradus comprehendunt deinde pro differentialibus secundi gradus ponamus

$$\left(\frac{\partial \partial z}{\partial x^2}\right) = q; \left(\frac{\partial \partial z}{\partial x \partial y}\right) = q'; \left(\frac{\partial \partial z}{\partial y^2}\right) = q'';$$

unde sequentes relationes inter has litteras et praecedentes notasse juvabit

$$\left(\frac{\partial p}{\partial x}\right) = q; \left(\frac{\partial p}{\partial y}\right) = \left(\frac{\partial p'}{\partial x}\right) = q'; \left(\frac{\partial p'}{\partial y}\right) = q'';$$

simili modo differentialia tertii gradus his formulis complectamur

$$\left(\frac{\partial^3 z}{\partial x^3}\right) = r; \left(\frac{\partial^3 z}{\partial x^2 \partial y}\right) = r'; \left(\frac{\partial^3 z}{\partial x \partial y^2}\right) = r''; \left(\frac{\partial^3 z}{\partial y^3}\right) = r''';$$

ubi hae relationes sunt notandae

$$r = \left(\frac{\partial q}{\partial x}\right);\ r' = \left(\frac{\partial q}{\partial y}\right) = \left(\frac{\partial q'}{\partial x}\right);\ r'' = \left(\frac{\partial q'}{\partial y}\right) = \left(\frac{\partial q''}{\partial x}\right);\ r''' = \left(\frac{\partial q''}{\partial y}\right);$$

quarta autem differentialia has formulas praebent

$$s = \left(\frac{\partial^4 z}{\partial x^4}\right);\ s' = \left(\frac{\partial^4 z}{\partial x^3 \partial y}\right);\ s'' = \left(\frac{\partial^4 z}{\partial x^2 \partial y^2}\right);\ s''' = \left(\frac{\partial^4 z}{\partial x \partial y^3}\right);$$
$$s'''' = \left(\frac{\partial^4 z}{\partial y^4}\right);$$

et sic ultra quousque libuerit.

§. 21. His explicatis, expressiones secundi generis, praeter ipsas coordinatas x, y et z, etiam quantitates p, p', q, q', q'', r, r', r'', r''', etc. utcunque involvere possunt, ex quo si V denotat quamcunque hujusmodi expressionem, ejus differentiale more solito sumtum sequenti forma exhibeamus

$$\partial V = L\partial x + M\partial y + N\partial z + \left.\begin{aligned} &P\partial p + Q\,\partial q + R\,\partial r \\ &+ P'\partial p' + Q'\,\partial q' + R'\,\partial r' \\ &\qquad + Q''\,\partial q'' + R''\,\partial r'' \\ &\qquad\qquad + R'''\partial r''' \end{aligned}\right\}\ \text{etc.}$$

quam formam animo imprimi conveniet, ne opus sit eam saepius repetere.

§. 22. Quodsi jam hujusmodi expressionem variatio, seu id incrementum inveniri debeat, quod resultat ex variatione novae variabilis t, quam in valorem coordinatae z introducimus, jam vidimus sumi debere $\partial x = 0$ et $\partial y = 0$, tum vero fieri $\partial z = \partial t \left(\frac{\partial z}{\partial t}\right)$, ob eandem vero rationem sequentia differentialia simili modo erunt exprimenda, quae cum suis transformationibus per se satis claris ita se habebunt

$$\partial p = \partial t \left(\frac{\partial p}{\partial t}\right) = \partial t \left(\frac{\partial \partial z}{\partial x \partial t}\right);\ \partial p' = \partial t \left(\frac{\partial p'}{\partial t}\right) = \left(\frac{\partial \partial z}{\partial y \partial t}\right);$$
$$\partial q = \partial t \left(\frac{\partial q}{\partial t}\right) = \partial t \left(\frac{\partial^3 z}{\partial x^2 \partial t}\right);\ \partial q' = \partial t \left(\frac{\partial q'}{\partial t}\right) = \partial t \left(\frac{\partial^3 z}{\partial x \partial y \partial t}\right);$$
$$\partial q'' = \partial t \left(\frac{\partial q''}{\partial t}\right) = \partial t \left(\frac{\partial^3 z}{\partial y^2 \partial t}\right);$$

$$\partial r = \partial t \left(\frac{\partial r}{\partial t}\right) = \partial t \left(\frac{\partial^4 z}{\partial x^3 \partial t}\right);\ \partial r' = \partial t \left(\frac{\partial r'}{\partial t}\right) = \partial t \left(\frac{\partial^4 z}{\partial x^2 \partial y \partial t}\right);$$

$$\partial r'' = \partial t \left(\frac{\partial r''}{\partial t}\right) = \partial t \left(\frac{\partial^4 z}{\partial x \partial y^2 \partial t}\right);$$

$$\partial r''' = \partial t \left(\frac{\partial r'''}{\partial t}\right) = \partial t \left(\frac{\partial^4 z}{\partial y^3 \partial t}\right);\ \text{etc.}$$

§. 23. Totum ergo negotium huc redit, ut in formula illa differentiali pro ∂V data, loco singulorum differentialium isti valores substituantur, hocque modo prodibit variatio expressionis V ex sola variabilitate ipsius t oriunda, seu valor hujus formulae $\partial t \left(\frac{\partial V}{\partial t}\right)$, quoniam autem singula membra elemento ∂t erunt affecta, eo omisso adipiscimur sequentem formam

$$\begin{aligned}\left(\frac{\partial V}{\partial t}\right) = N\left(\frac{\partial z}{\partial t}\right) &+ P\left(\frac{\partial \partial z}{\partial x \partial t}\right) + Q\left(\frac{\partial^3 z}{\partial x^2 . \partial t}\right) + R\left(\frac{\partial^4 z}{\partial x^3 . \partial t}\right) \\ &+ P'\left(\frac{\partial \partial z}{\partial y \partial t}\right) + Q'\left(\frac{\partial^3 z}{\partial x \partial y \partial t}\right) + R'\left(\frac{\partial^4 z}{\partial x^2 \partial y \partial t}\right) \\ &+ Q''\left(\frac{\partial^3 z}{\partial y^2 . \partial t}\right) + R''\left(\frac{\partial^4 z}{\partial x \partial y^2 \partial t}\right) \\ &+ R'''\left(\frac{\partial^4 z}{\partial y^3 . \partial t}\right)\end{aligned}$$

quae ad variationes quarumcunque expressionum secundi generis inveniendas sufficit.

§. 24. Nunc expressiones tertii generis aggredi poterimus formulas integrales involventes in quibus potissimum vis hujus methodi cernitur. Quando enim quaestio circa maxima vel minima, quae in superficiebus occurrere possunt, versatur, formula illa, quae maximum vel minimum reddi debet, necessario est formula integralis atque adeo formula integralis duplicata, cujus indolem hic paucis explicari convenit. Quemadmodum enim in praecedente parte formulae integrales simplices sunt consideratae, quae ad datam abscissam x sunt relatae, ita hic in superficiebus, quaestiones semper non ad solam abscissam x, sed ad totum quoddam spatium in plano horizontali tanquam basem sunt referendae, cui portio super-

ficiei quae maximi minimive quadam proprietate gaudere debet, immineat. Quare cum talis basis duplicem habeat dimensionem alteram ab x, alteram vero ab y pendentem, hujusmodi formulae integrales erunt duplicatae, hoc modo exprimi solitae $\int\int V\,\partial x\,\partial y$, eae scilicet duplicem integrationem postulant, atque in priore sola coordinata x vel sola y pro variabili habetur, et integratio usque ad terminos basis propositae extenditur, tum vero demum etiam altera variabilis assumitur, atque altera integratio absolvitur. Et quoniam perinde est utra prius pro variabili habeatur, sine discrimine geminam illam integrationem signo duplicato $\int\int$ indicamus; neque vero hic loci est, omnia quae circa hujusmodi integrationes duplicatas sunt observanda, fusius exponere, quippe quod argumentum supra in supplemento VI. pag. 416. seq. jam satis accurate est pertractatum.

§. 25. Quodsi ergo hujusmodi formulae integralis $\int\int V\partial x\partial y$ variatio quaeri debeat, ubi V denotat expressionem quamcunque vel primi vel secundi generis, ex superioribus satis liquet hanc variationem ita expressum iri

$$\partial t\int\int\left(\frac{\partial V}{\partial t}\right)\partial x\,\partial y,$$

quae forma iterum est integralis duplicata, et prouti vel x vel y priore integratione ut constans spectatur, ea formula vel hoc modo

$$\partial t\int\partial x\int\left(\frac{\partial V}{\partial t}\right)\partial y,$$

vel hoc modo

$$\partial t\int\partial y\int\left(\frac{\partial V}{\partial t}\right)\partial x,$$

exhiberi potest.

§. 26. Sit nunc V talis expressio qualem supra §. 19 descripsimus, et cujus variationem seu valorem $\left(\frac{\partial V}{\partial t}\right)$ in §. 23.

evolvimus, tantum opus erit, singula membra ibi exposita hoc loco $\left(\frac{\partial V}{\partial t}\right)$ substituere; unde sequens congeries formularum nascetur, quibus junctim sumtis variatio quaesita $\partial t \int \int \left(\frac{\partial V}{\partial t}\right) \partial x \partial y$ exprimetur

$$\begin{aligned}\partial t\int\int N\left(\frac{\partial z}{\partial t}\right)\partial x\partial y+\partial t\int\int P\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial x\partial y+\partial t\int\int Q\left(\frac{\partial^3 z}{\partial x^2\partial t}\right)\partial x\partial y+\partial t\int\int R\left(\frac{\partial^4 z}{\partial x^3\partial t}\right)\partial x\partial y\\+\partial t\int\int P'\left(\frac{\partial\partial z}{\partial y\partial t}\right)\partial x\partial y+\partial t\int\int Q'\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\partial x\partial y+\partial t\int\int R'\left(\frac{\partial^4 z}{\partial x\partial y\partial t}\right)\partial x\partial y\\+\partial t\int\int Q''\left(\frac{\partial^3 z}{\partial y^2\partial t}\right)\partial x\partial y+\partial t\int\int R''\left(\frac{\partial^4 z}{\partial x\partial y^2\partial t}\right)\partial x\partial y\\+\partial t\int\int R'''\left(\frac{\partial^4 z}{\partial y^3\partial t}\right)\partial x\partial y\\ \text{etc.}\end{aligned}$$

§. 27. Nunc singula haec membra post primum peculiares reductiones admittunt, quas probe notasse juvabit. Pro secundo membro sumamus primo x tantum variabile eritque:

$$\int P\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial x = P\left(\frac{\partial z}{\partial t}\right) - \int\left(\frac{\partial z}{\partial t}\right)\partial x\left(\frac{\partial P}{\partial x}\right),$$

unde etiam alteram integrationem adjiciendo erit

$$\int\int P\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial x\partial y = \int P\left(\frac{\partial z}{\partial t}\right)\partial y - \int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial P}{\partial x}\right)\partial x\partial y.$$

Pro tertio membro sumatur primo sola y variabilis eritque

$$\int P'\left(\frac{\partial\partial z}{\partial y.\partial t}\right)\partial y = P'\left(\frac{\partial z}{\partial t}\right) - \int\left(\frac{\partial z}{\partial t}\right)\partial y\left(\frac{\partial P'}{\partial y}\right),$$

unde ipsum tertium membrum transibit in

$$\int\int P'\left(\frac{\partial\partial z}{\partial y\partial t}\right)\partial x\partial y = \int P'\left(\frac{\partial z}{\partial t}\right)\partial x - \int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial P'}{\partial y}\right)\partial x\partial y.$$

§. 28. Pro sequentibus membris hae ipsae reductiones sequentes dabunt transformationes, pro quarto scilicet habebimus ex secundo

$$\int\int Q\left(\frac{\partial^3 z}{\partial x^2\partial t}\right)\partial x = \int Q\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial y - \int\int\left(\frac{\partial\partial z}{\partial x\partial t}\right)\left(\frac{\partial Q}{\partial x}\right)\partial x\partial y,$$

at vero hoc membrum posterius ad similitudinem secundi reducitur hoc modo, ubi tantum loco P scribi debet $\left(\frac{\partial Q}{\partial x}\right)$,

$$\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial Q}{\partial x}\right)\partial y - \int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial Q}{\partial x^2}\right)\partial x\,\partial y,$$

ita ut nunc quartum membrum praebeat hanc formam

$$\int Q\left(\frac{\partial\partial z}{\partial x\partial t}\right)\partial y - \int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial Q}{\partial x}\right)\partial y + \int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial Q}{\partial x^2}\right)\partial x\,\partial y.$$

Simili modo quintum membrum ope secundi reducitur, ubi loco P scribitur Q' et loco $\left(\frac{\partial\partial z}{\partial x\partial t}\right)$, $\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)$, sive loco $\left(\frac{\partial z}{\partial t}\right)$ scribendo $\left(\frac{\partial\partial z}{\partial y\partial t}\right)$, sicque habebitur

$$\int\int Q'\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\partial x\partial y = \int Q'\left(\frac{\partial\partial z}{\partial y\partial t}\right)\partial y - \int\int\left(\frac{\partial\partial z}{\partial y\partial t}\right)\left(\frac{\partial Q'}{\partial x}\right)\partial x\,\partial y,$$

quod posterius membrum cum tertio conferatur, ubi tantum loco P' scribi debet $\left(\frac{\partial Q'}{\partial x}\right)$, quo pacto totum membrum induet hanc formam

$$\int Q'\left(\frac{\partial\partial z}{\partial y\partial t}\right)\partial y - \int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial Q'}{\partial x}\right)\partial x + \int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial Q'}{\partial x\partial y}\right)\partial x\,\partial y,$$

sextum vero membrum bis cum secundo collatum reducitur ad hanc formam

$$\int Q''\left(\frac{\partial\partial z}{\partial y\partial t}\right)\partial x - \int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial Q''}{\partial y}\right)\partial x + \int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial Q''}{\partial y^2}\right)\partial x\,\partial y.$$

§. 29. Si hoc modo ulterius progrediamur ad sequentia membra, septimum membrum in sequentes partes resolvitur

$$\int R\left(\frac{\partial^3 z}{\partial x^2\partial t}\right)\partial y - \int\left(\frac{\partial\partial z}{\partial x\partial t}\right)\left(\frac{\partial R}{\partial x}\right)\partial y + \int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R}{\partial x^2}\right)\partial y$$
$$-\int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 R}{\partial x^3}\right),\ \partial x\,\partial y,$$

deinde octavum membrum

$$\int R'\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\partial y - \int\left(\frac{\partial\partial z}{\partial x\partial t}\right)\left(\frac{\partial R'}{\partial x}\right)\partial y + \int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R'}{\partial x\partial y}\right)\partial y$$
$$-\int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 R}{\partial x^2\partial y}\right)\partial x\,\partial y,$$

tum nonum membrum fiet

$$\int R''\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\partial x - \int\left(\frac{\partial\partial z}{\partial y\partial t}\right)\left(\frac{\partial R''}{\partial y}\right)\partial x + \int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R''}{\partial x\partial y}\right)\partial x$$
$$-\int\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 R''}{\partial x\partial y^2}\right)\partial x\,\partial y,$$

et decimum

$$\int R'''\left(\frac{\partial^3 z}{\partial y^2\partial t}\right)\partial x-\int\left(\frac{\partial\partial z}{\partial y\partial t}\right)\left(\frac{\partial R'''}{\partial y}\right)\partial x+\int\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R'''}{\partial y^2}\right)\partial x$$
$$-\iint\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial^3 R'''}{\partial y^3}\right)\partial x\,\partial y.$$

§. 30. Colligamus nunc omnes istas formulas in unam summam, atque variatio quaesita pluribus constabit membris, quarum primum formulas integrales duplicatas, reliqua vero simplices complectentur: hoc pacto variatio quaesita sequenti modo erit expressa

$$\partial t\iint\partial x\,\partial y\left(\frac{\partial z}{\partial t}\right)\left\{\begin{array}{l} N-\left(\frac{\partial P}{\partial x}\right)+\left(\frac{\partial\partial Q}{\partial x^2}\right)-\left(\frac{\partial^3 R}{\partial x^3}\right)\\ \quad-\left(\frac{\partial P'}{\partial y}\right)+\left(\frac{\partial\partial Q'}{\partial x\partial y}\right)-\left(\frac{\partial^3 R'}{\partial x^2\partial y}\right)\text{ etc.}\\ \qquad+\left(\frac{\partial\partial Q''}{\partial y^2}\right)-\left(\frac{\partial^3 R''}{\partial x\partial y^2}\right)\\ \qquad\qquad-\left(\frac{\partial^3 R'''}{\partial y^3}\right)\end{array}\right\}$$

$$+\partial t\left\{\begin{array}{l}\int\left(\frac{\partial z}{\partial t}\right)P\,\partial y+\int Q\,\partial y\left(\frac{\partial\partial z}{\partial x\partial t}\right)-\int\partial y\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial z}{\partial t}\right)+\int R\,\partial y\left(\frac{\partial^3 z}{\partial x^2\partial t}\right)\\ \int\left(\frac{\partial z}{\partial t}\right)P'\partial x+\int Q'\partial y\left(\frac{\partial\partial z}{\partial y\partial t}\right)-\int\partial x\left(\frac{\partial Q'}{\partial x}\right)\left(\frac{\partial z}{\partial t}\right)+\int R'\,\partial y\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\\ \qquad+\int Q''\partial x\left(\frac{\partial\partial z}{\partial y\partial t}\right)-\int\partial x\left(\frac{\partial Q''}{\partial y}\right)\left(\frac{\partial z}{\partial t}\right)+\int R''\,\partial x\left(\frac{\partial^3 z}{\partial x\partial y\partial t}\right)\\ \qquad\qquad+\int R'''\partial x\left(\frac{\partial^3 z}{\partial y^2\partial t}\right)\\ -\int\partial y\left(\frac{\partial R}{\partial x}\right)\left(\frac{\partial\partial z}{\partial x\partial t}\right)+\int\partial y\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R}{\partial x^2}\right)\\ -\int\partial y\left(\frac{\partial R'}{\partial x}\right)\left(\frac{\partial\partial z}{\partial x\partial t}\right)+\int\partial y\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R'}{\partial x\partial y}\right)\text{ etc.}\\ -\int\partial x\left(\frac{\partial R''}{\partial y}\right)\left(\frac{\partial\partial z}{\partial y\partial t}\right)+\int\partial x\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R''}{\partial x\partial y}\right)\\ -\int\partial x\left(\frac{\partial R'''}{\partial y}\right)\left(\frac{\partial\partial z}{\partial y\partial t}\right)+\int\partial x\left(\frac{\partial z}{\partial t}\right)\left(\frac{\partial\partial R'''}{\partial y^2}\right)\end{array}\right\}$$

§. 31. Verum quid haec singula membra proprie significent et ad quemnam usum adhiberi queant, neutiquam adhuc perspicere licet, unde hoc argumentum cujus prima fundamenta etiamnunc vix jacta sunt censenda, omnem geometrarum attentionem atque multo accuratiorem investigationem postulare videtur,

quod negotium vix ante suscipere licet, quam casus nonnulli particulares omni studio et diligentia fuerint evoluti, quin etiam ipsa pars prior, quae tantum circa functiones unius variabilis versatur neutiquam adhuc satis clare et distincte est enucleata, ita ut perspicue intelligeremus veram indolem atque naturam singularum partium, quibus variationem contineri invenimus, quem in finem dilucidationes sequentes hic adjungere visum est.

Dilucidationes super theoria variationum ad functiones saltem unius variabilis accommodata.

§. 32. Quaestiones quae hic occurrunt ad hoc problema generale revocare licet.

Si y fuerit functio quaecunque ipsius x, indeque definiatur valor cujuspiam formulae integralis datae $\int Z \partial x$, denotante Z expressionem ex ipsis quantitatibus x et y earumque differentialium rationibus utcunque compositam, quaestio est, si loco illius functionis y alia quaecunque illi proxima seu infinite parum tantum ab ea discrepans adhibeatur, quanto majorem minoremve valorem, tum eadem formula integralis $\int Z \partial x$ sit consecutura.

§. 33. At quia hoc modo ista quaestio enunciata nimis videri posset abstracta, eam more soluto ad geometriam revocemus. Sit igitur super axe A P proposita curva quaecunque A M, aequatione inter abscissam $AP = x$ et applicatam $PM = y$ expressa, pro qua definiri oporteat valorem formulae cujuspiam integralis $\int Z \partial x$, qui sit $= W$, quo posito consideretur alia curva quaecunque $\alpha\,\mu$ infinite parum a data discrepans, ac si pro hac curva itidem definiatur valor formulae $\int Z \partial x$, quaeritur, quantum

Fig. 15

iste valor a praecedente sit discrepaturus: evidens enim est, hoc discrimen praebere ipsam variationem quantitatis W, quam supra ope calculi variationum exhibuimus.

§. 34. Quo haec adhuc clariora evadant, exemplum quodpiam proferamus, quo proposita curva A M ejusque axe A P tanquam verticali considerato, quaeritur tempus quo corpus ex puncto A super hac curva A M descendens usque ad punctum M pertingit. Jam quia celeritas corporis in M est ut $\sqrt{AP} = \sqrt{x}$, et ipsum curvae elementum $= \partial x \sqrt{(1 + pp)}$, posito scilicet $\partial y = p \partial x$ uti in solutione generali est praeceptum, erit tempus per elementum $Mm = \partial x \frac{\sqrt{(1+pp)}}{\sqrt{x}}$, unde formula integralis $\int Z \partial x$ pro hoc casu abit in $\int \partial x \frac{\sqrt{(1+pp)}}{\sqrt{x}}$, ita ut habeatur $Z = \frac{\sqrt{(1+pp)}}{\sqrt{x}}$, quare nunc tempus erit definiendum, quo corpus super curva quacunque proxima $\alpha\, \mu$ descendens ab α usque ad μ perveniet, ubi discrimen dabit ipsam variationem formulae $\int \partial x \frac{\sqrt{(1+pp)}}{\sqrt{x}}$, huic casui convenientem.

§. 35. Quoniam hic formula integralis consideranda venit, ante omnia dispiciendum est, quomodo eam determinari oporteat. In exemplo quidem allato, manifestum est formulae $\int \frac{\partial x \sqrt{(1+pp)}}{\sqrt{x}}$ integrale ita capi debere, ut evanescat posito $x = 0$, unde etiam in genere intelligitur, semper pro integratione formulae $\int Z \partial x$, certum aliquem terminum veluti punctum A, tanquam principium integrationis statui, atque integrale $\int Z \partial x$ evanescere debere posito $x = 0$, vel si forte circumstantiae aliter fuerint comparatae, tribuendo ipsi x valorem quempiam datum, deinde vero initio constituto, valor formulae $\int Z \partial x = W$ abscissae $AP = x$ respondebit.

§. 36. His circa formulam integralem $\int Z \partial x$ observatis, videamus, quamnam ideam nobis de curvis illis proximis $\alpha \mu$ formare debeamus. Ac primo quidem patet, has curvas continuo quodam tractu ductas esse debere, ita ut in iis nusquam anguli aliive saltus deprehendantur; hoc solo notato, perinde est sive istae curvae lege quapiam continuitatis vel aequatione quapiam contineantur, sive sint adeo discontinuae, quasi libero manus motu ductae

§. 37. Hujusmodi lineae curvae commodissime sequenti modo formatae menti repraesentari possunt. Ducatur scilicet pro lubitu linea curva quaecunque B N eidem abscissae A P imminens, ac ductis ad singula axis puncta X applicatis X Y V singula intervalla Y V in ratione finiti ad infinite parvum secentur in v, ita ut Y v sit quasi pars infinitesima intervalli Y V. Hoc enim modo curva $\alpha \nu \mu$ obtinebitur a curva proposita A M in omnibus punctis infinite parum dissita, qualem ad institutum nostrum requirimus. Praeterea tamen notandum est, in curva illa arbitraria B N nusquam tangentem ad axem A P normalem esse debere, quia hoc modo divisio illorum intervallorum turbaretur. Atque nunc evidens est, non solum intervalla Y v esse infinite parva, sed etiam tangentes in punctis Y et v infinite parum a parallelismo deficere.

Explicatio partis primae in variatione.

§. 38. His circa ipsam quaestionis propositionem annotatis, contemplemur nunc accuratius quoque solutionem supra inventam, ejusque singulas partes, ut quid quaelibet earum innuat et ad quemnam usum sit transferenda perspicue intelligamus; solutionem autem in §. 14. datam hic contemplabimur. Statim igitur consideremus primam variationis ibi inventae par-

tem, quae hac formula integrali continetur

$$\partial t \int \partial x \left(\frac{\partial y}{\partial t}\right) \left[N - \left(\frac{\partial P}{\partial x}\right) + \left(\frac{\partial \partial Q}{\partial x^2}\right) - \left(\frac{\partial^3 R}{\partial x^3}\right) + \left(\frac{\partial^4 S}{\partial x^4}\right) - \text{etc.}\right],$$

cujus integratio ita capi debet, ut in ipso termino initiali A evanescat, qua conditione constans arbitraria determinatur, quod si ergo in singulis punctis X Y haec formula applicata intelligatur, aggregatum omnium istarum formularum elementarium ab initio A usque ad terminum M extensum praebebit primam partem variationis quaesitae, atque hic quidem in figura perspicuum est, spatiolum $Y v$ exprimere incrementum applicatae y a sola variabili t oriundum, ita ut sit $Y v = \partial t \left(\frac{\partial y}{\partial t}\right)$.

§. 39. Haec igitur prima pars variationis involvit omnia spatiola $Y v$ intra terminos A et M contenta, quae quum in infinitum variari possint, atque adeo a positivis ad negativa transire queant, maximae variationes hic locum habere possunt. Verum tamen unicus casus hinc debet excipi, quo curva A M ita est comparata, ut sit

$$0 = N - \frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \frac{\partial^4 S}{\partial x^4} - \text{etc.}$$

tum enim utcunque curvae proximae fuerint comparatae, ista pars prima variationis, semper in nihilum abit. Neque deviatio curvarum proximarum $\alpha \mu$ a principali A M intra terminos A et M quicquam ad variationem confert; ex quo haec curva respectu formulae integralis $\int Z \partial x$ imprimis est memorabilis, quandoquidem in ea haec formula integralis vel maximum vel minimum obtinet valorem.

Explicatio partis secundae in variatione.

§. 40. Progrediamur nunc ad secundam partem variationis supra inventae, quae est

$$\partial t \left(\frac{\partial y}{\partial t}\right) \left(P - \frac{\partial Q}{\partial x} + \frac{\partial \partial R}{\partial x^2} - \frac{\partial^3 S}{\partial x^3} + \text{etc.}\right)$$

circa quam primum observo, quoniam ea ad terminum M refertur, per integrationem rite institutam insuper adjici debere similem expressionem ad terminum priorem A relatam, at verò signo contrario affectam, id quod ideo est necessarium, ut facto $x = 0$, etiam haec expressio penitus tollatur. Refertur autem ista pars

$$\partial t \left(\frac{\partial y}{\partial t}\right) \left(P - \frac{\partial Q}{\partial x} + \frac{\partial \partial R}{\partial x^2} - \text{etc.}\right)$$

unice ad ultimum terminum M, ubi $\partial t \left(\frac{\partial y}{\partial t}\right)$ ipsum spatiolum M μ exprimit, similique modo in alteram partem pro initio A spatiolum A α ingredietur. Hinc patet si omnes curvae proximae $\alpha\,\mu$ per ipsos ambos terminos A et M ducantur tum variationem secundae partis in nihilum abire.

§. 41. Consideremus autem casum, quo curva proxima $\alpha\,\mu$ per primum quidem terminum A transit non vero quoque per alterum M, sed sit punctum μ ejus terminus, atque variatio ex secunda parte nata erit

$$M\mu \left(P - \frac{\partial Q}{\partial x} + \frac{\partial \partial R}{\partial x^2} - \text{etc.}\right)$$

Atque hinc etiam definire poterimus variationem ex eodem fonte oriundam, si curva proxima A μ, non in ipso puncto μ sed alio quocunque ω terminetur, existente semper intervallo $\mu\,\omega$ infinite parvo. Ducta enim applicata $\omega\, m\, p$, variatio modo inventa insuper augeri debet particula formulae $\int Z\,\partial x$, quae elemento $Pp = \partial x$ respondet, quae particula quum sit $= Z \,.\, Pp$, pro arcu curvae proximae A ω erit variatio ex secunda parte oriunda

$$M\mu \left(P - \frac{\partial Q}{\partial x} + \frac{\partial \partial R}{\partial x^2} - \text{etc.}\right) + Z \,.\, Pp.$$

§. 42. Ducatur recta M ω, et quaeramus angulum $\omega\, M\, m$, quem haec recta M ω cum curva principali constituit, ponatur

iste angulus $\omega M m = \omega$, et ducta MO ipsi Pp parallela, quia est proxime $m\omega = M\mu$ et anguli mMo tangens $= p$, ideoque $om = p \,.\, Pp$, habebitur $O\omega = M\mu + p \,.\, Pp$, unde fit

$$\text{tang. } \omega M o = \frac{M\mu}{Pp} + p,$$

atque hinc colligitur

$$\omega M m = \text{tang. } \omega = \frac{M\mu}{Pp(1+pp) + M\mu \,.\, p}.$$

Servemus nunc in calculo hunc ipsum angulum ω atque hinc habebimus spatiolum

$$M\mu = \frac{Pp(1+pp)\,\text{tang. }\omega}{1 - p\,\text{tang. }\omega},$$

quo valore substituto variatio pro arcu $A\omega$ erit

$$Pp\left[Z + \frac{(1+pp)\,\text{tang. }\omega}{1 - p\,\text{tang. }\omega} \cdot \left(P - \frac{\partial Q}{\partial x} + \frac{\partial\partial R}{\partial x^2} - \text{etc.}\right)\right].$$

§. 43. Nunc operae pretium erit eum angulum ω definire, ut ista variatio in nihilum abeat, id quod eveniet, si capiatur

$$\text{tang. } \omega = \frac{Z}{pZ - (1+pp)\left(P - \frac{\partial Q}{\partial x} + \frac{\partial\partial R}{\partial x^2} - \text{etc.}\right)},$$

quare hoc angulo ita constituto pro omnibus lineis proximis ubicunque in recta $M\omega$ terminatis variatio ex secunda parte oriunda evanescet. Hic casus prae caeteris omnino notatu dignus considerari meretur, quo recta $M\omega$ fit ad curvam principalem in puncto M normalis, quod evenit, si fuerit

$$pZ - (1+pp)\left(P - \frac{\partial Q}{\partial x} + \frac{\partial\partial R}{\partial x^2} - \text{etc.}\right) = 0,$$

qua aequatione certa conditio ipsius formulae integralis $\int Z\,\partial x$ sive indoles expressionis Z definitur.

§. 44. Non igitur pigebit in talem expressionem Z inquisivisse, ac primo quidem patet eam praeter coordinatas x et y

etiam quantitatem p involvere debere. Sumamus autem praeterea in Z non ingredi litteras q, r, etc. ita ut sit $Q = 0$, $R = 0$, ac nostra aequatio resolvenda erit

$$pZ = (1 + pp)P,$$

ubi notandum est esse

$$\partial Z = M\partial x + N\partial y + P\partial p,$$

quare si ambae coordinatae x et y tanquam constantes tractentur, erit

$$\partial Z = P\partial p, \text{ ideoque } P = \frac{\partial Z}{\partial p},$$

quo valore ibi introducto haec prodibit aequatio

$$\frac{\partial Z}{Z} = \frac{p\partial p}{1 + pp},$$

quae integrata dat

$$l.Z = l.\sqrt{(1 + pp)} + l.C,$$

quae constans functio quaecunque ipsarum x et y esse potest, talis functio sit V, atque habebimus

$$Z = V\sqrt{(1 + pp)},$$

ideoque formula integralis

$$\int V\partial x\sqrt{(1 + pp)}.$$

Hujus formulae significatum satis eleganter per tempus, quo corpus quodpiam per curvam AM promovetur exprimi potest. Si enim celeritas in puncto M, fuerit $= \frac{1}{V}$, hoc est, si celeritas in singulis punctis proportionalis fuerit functioni cuicunque binarum variabilium x et y, tum

$$V\partial x\sqrt{(1 + pp)}$$

exprimit elementum temporis, ideoque formula

$$\int V\partial x\sqrt{(1 + pp)}$$

totum tempus quo corpus ab A ad M pervenit

Explicatio partis tertiae in variatione.

§. 45. Quod ad tertiam partem variationis attinet, scilicet

$$\partial t \left(\frac{\partial \partial y}{\partial x \partial t}\right)\left(Q - \frac{\partial R}{\partial x} + \frac{\partial \partial S}{\partial x^2} - \text{etc.}\right)$$

ea locum non habet, nisi expressio Z etiam differentialia secundi gradus involvat, quod quidem rarissime usu venire solet. Hic autem observandum est, quoniam $M\mu = \partial t \left(\frac{\partial y}{\partial t}\right)$ fore pro sequenti elemento

$$m\omega = \partial t \left(\frac{\partial y}{\partial t}\right) + \partial t \, \partial x \left(\frac{\partial \partial y}{\partial x \partial t}\right),$$

unde colligitur

$$\partial t \left(\frac{\partial \partial y}{\partial x \partial t}\right) = \frac{m\omega - M\mu}{\partial x} = \frac{M\omega - M\mu}{Pp},$$

hac autem formula exprimitur declinatio directionis $\mu\,\omega$ a directione $M\,m$, quae quidem, ut jam ante observavimus, semper est quam minima.

§. 46. Quodsi ergo tangens in μ perfecte fuerit parallela tangenti in M, quod evenit, si etiam in curva generatrice B N, tangens ad N huic fuerit parallela, tum variatio ex tertia parte oriunda prorsus evanescit, quod etiam de termino initiali A est intelligendum, si tangentes in A et B inter se fuerint parallelae: atque hinc jam perspicitur, ut variationes ex quarta parte oriundae evanescant, necesse esse, ut praeterea etiam radii osculi in punctis M et μ fiant aequales.

§. 47. Atque ex his jam satis perspicuum est, variationes ex secunda parte oriundas evanescere, si omnes curvae proximae $\alpha\,\mu$ per utrumque terminum M et A ducantur. Deinde vero insuper etiam variationes tertiae partis, si omnes curvae proximae simul in utroque termino A et M cum curva principali A M communes habeat tangentes. Praeterea vero quoque va-

riationes quartae partis in nihilum abire, si omnes curvae proximae in terminis A et M insuper ratione curvaturae cum curva principali conveniant. Hic autem probe meminisse juvabit, variationes tertiae partis per se evanescere, si modo quantitas Z non differentialia secundi gradus involvat; quartae vero partis semper evanescere nisi differentialia tertii gradus in quantitatem Z ingrediantur, et ita porro. Unde quum initio ostenderimus, quomodo variatio primae partis ad nihilum sit redigenda, nunc evidentissime intelligimus sub quibusnam conditionibus, omnes variationis partes simul evanescant.

Dilucidationes circa curvas maximi, minimive proprietate praeditas.

§. 48. Si formula integralis $\int Z \partial x$ in curva quaesita debeat esse vel maximum vel minimum, jam supra ostendimus, posito

$$\partial Z = M \partial x + N \partial y + P \partial p + Q \partial q + R \partial r + \text{etc.}$$

naturam hujus curvae, hac exprimi aequatione

$$0 = N - \frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2} - \frac{\partial^3 R}{\partial x^3} + \text{etc.}$$

quae aequatio nisi quantitates P, Q, R evanescant, vel sint constantes, semper est differentialis vel secundi, vel quarti, vel sexti, aliusve gradus paris. Hic ergo statim memoratu dignum occurrit quod ista aequatio nunquam vel simpliciter differentialis, vel tertii, vel quinti, aliusve gradus imparis evadat, id quod mox clarius exponemus.

§. 49. Quaestiones ergo huc pertinentes sponte in varias dividuntur classes, pro gradu differentialium, ad quem aequationes exsurgunt, quandoquidem ab hoc gradu natura solutionis maxime pendet, propterea quod ea semper totidem constantes arbi-

trarias involvit. Ad primam ergo classem referimus eos casus quibus aequatio pro maximo vel minimo inventa prorsus est finita. Ad secundam autem classem eos, quibus haec aequatio fit differentialis secundi gradus, ad tertiam eos, quibus aequatio ad quartum gradum ascendit et ita porro, quas singulas classes ordine describamus.

Classis I.

§. 50. Ad solutionem ergo primae classis formula $\int Z \partial x$ statim perducit, quando expressio Z tantum per coordinatas x et y exclusis omnium differentialium rationibus determinatur, quia enim hoc casu, simpliciter fit $\partial Z = M \partial x + N \partial y$, aequatio pro curva maximi vel minimi erit $N = 0$, quae ergo aequatio omnino est determinata, atque adeo curva satisfaciens unica in suo genere. Veluti si quaeratur linea, in qua valor formulae $\int \partial x (2xy - yy)$ fiat maximus vel minimus, ob $Z = 2xy - yy$, ideoque $N = 2(x - y)$, aequatio quaesita erit $x - y = 0$, seu linea quaesita erit recta ad axem angulo semirecto inclinata, pro qua ergo valor formulae propositae integralis est $\frac{x^3}{3}$, qui utique minor est, quam si ulla alia linea curva sumeretur pro eadem scilicet abscissa.

§. 51. His autem casibus prima classis nondum exhauritur, sed dantur adhuc alii perinde ad aequationes finitas ducentes, ad quod ostendendum, sit $\mathfrak{Z}$ functio quaecunque ipsarum x et y atque $\partial \mathfrak{Z} = \mathfrak{M} \partial x + \mathfrak{N} \partial y$, jamque ponatur $Z = \mathfrak{Z} p$, eritque $M = \mathfrak{M} p$; $N = \mathfrak{N} p$; $P = \mathfrak{Z}$, quare ut formula $\int Z \partial x$ fiat maximum vel minimum, aequatio reperitur

$$0 = \mathfrak{N} p - \frac{\partial \mathfrak{Z}}{\partial x} = \mathfrak{N} p - \mathfrak{M} - \frac{\mathfrak{N} \cdot \partial y}{\partial x} = - \mathfrak{M},$$

quae itidem est aequatio finita. Quod quidem etiam statim praevidere licuisset, quum enim sit $p \partial x = \partial y$, haec formula integra-

lis $\int \mathfrak{Z}\, \partial y$ a praecedente $\int Z\, \partial x$ aliter non differt, nisi quod coordinatae x et y sint permutatae, unde quod de priore erat affirmatum, etiam de posteriore valet.

Hinc natura primae classis adhuc generalius ita describi potest, ut ea complectatur omnes formulas integrales hujusmodi $\int (Z + \mathfrak{Z} p)\, \partial x$, ubi litterae Z et $\mathfrak{Z}$ denotant functiones quascunque ipsarum x et y, tum enim aequatio pro curva maximi vel minimi erit, $0 = \mathfrak{N} - \mathfrak{M}$, quae est aequatio omnino determinata.

Classis II.

§. 52. Ad classem secundam referimus eas formulas integrales $\int Z\, \partial x$, quae deducunt ad aequationem differentialem secundi gradus, huc ergo primo pertinent casus, quibus Z tantum ex litteris x, y et p componitur, ita ut sit

$$\partial Z = M\, \partial x + N\, \partial y + P\, \partial p,$$

unde quidem casum posteriorem primae classis excipere oportet, quippe quod evenit, si P fuerit functio tantum ipsarum x et y, ita ut pro praesenti casu quantitas P praeter x et y etiam litteram p complecti debeat. Tum autem aequatio pro curva quaesita erit $0 = N - \frac{\partial P}{\partial x}$, ubi quum P involvat p, ideoque $\frac{\partial y}{\partial x}$, formula $\frac{\partial P}{\partial x}$ continebit differentialia secundi gradus, haec ergo aequatio neutiquam est determinata, quum duas adeo constantes arbitrarias recipiat, quibus effici potest, ut curva per data duo puncta transeat, atque adeo quaestiones hujus classis ita accuratius sunt definiendae, ut curvae investigentur, quae non inter omnes plane curvas, sed inter eas tantum, quae per eadem duo puncta ducuntur, praescripta maximi minimive proprietate gaudeant; semper autem quaestiones hujus classis ita sunt comparatae, ut per naturam suam hanc restrictionem postulent.

§. 53. Praeterea vero etiam ad secundam classem referri oportet casus, quibus $Z = \mathfrak{Z} q$ existente $\mathfrak{Z}$ functione quacunque ipsarum x, y et p, si enim fuerit

$$\partial \mathfrak{Z} = \mathfrak{M} \partial x + \mathfrak{N} \partial y + \mathfrak{P} \partial p,$$

habebimus

$$M = \mathfrak{M} q; \; N = \mathfrak{N} q; \; P = \mathfrak{P} q; \text{ et } Q = \mathfrak{Z};$$

quare quum aequatio pro curva sit

$$0 = N - \frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2}, \text{ sive } 0 = N - \frac{1}{\partial x} \partial . \left(P - \frac{\partial Q}{\partial x}\right),$$

formula haec $P - \frac{\partial Q}{\partial x}$ abit in

$$\mathfrak{P} q - \frac{\partial \mathfrak{Z}}{\partial x} = \mathfrak{P} q - \mathfrak{M} - \mathfrak{N} p - \mathfrak{P} q = - \mathfrak{M} - \mathfrak{N} p,$$

unde aequatio nostra evadet

$$0 = N + \frac{1}{\partial x} \partial (\mathfrak{M} + \mathfrak{N} p) = 2 \mathfrak{N} q + \frac{\partial \mathfrak{M}}{\partial x} + p \frac{\partial \mathfrak{N}}{\partial x},$$

quae manifesto tantum differentialia secundi gradus continet. Generalius ergo adhuc si formula integralis proposita fuerit $\int (Z + \mathfrak{Z} q) \partial x$, ubi Z et $\mathfrak{Z}$ quomodocunque ex quantitatibus x, y et p sint compositae, aequatio pro curva quaesita erit

$$0 = N - \frac{\partial P}{\partial x} + 2 \mathfrak{N} q + \frac{\partial \mathfrak{M}}{\partial x} + p \frac{\partial \mathfrak{N}}{\partial x},$$

sive etiam

$$0 = N \partial x - \partial P + 2 \mathfrak{N} \partial p + \partial \mathfrak{M} + p \partial \mathfrak{N},$$

quae manifesto tantum est differentialis secundi gradus.

Classis III.

§. 54. At si quantitas Z ita ex litteris x, y, p et q fuerit composita ut posito

$$\partial Z = M \partial x + N \partial y + P \partial p + Q \partial q,$$

etiam quantitas Q involvat litteram q, tum hujusmodi casus ad tertiam classem erunt referendi, et cum aequatio pro curva quae-

sita reperiatur

$$0 = N - \frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2},$$

evidens est terminum $\frac{\partial \partial Q}{\partial x^2}$ involvere differentialia quarti gradus, unde aequatio finita pro curva implicabit quatuor constantes arbitrarias, quibus ergo effici potest, ut curva desiderata non solum per datos duos terminos transeat, sed etiam ejus tangentes in utroque termino datam obtineant positionem, in qua quadruplici determinatione natura quaestionum ad hanc classem pertinentium continetur et accuratissime perspicitur.

§. 55. Reliquis casibus ad hanc classem pertinentibus non immoror, verum potius illustrationis caussa insigne adferam exemplum, quo curvae elasticae investigari solent. Scilicet si

Fig. 15. littera ϱ denotet radium osculi curvae quaesitae in puncto M, omnes hae curvae hac gaudent proprietate, ut in iis haec formula $\int \frac{\partial x \sqrt{(1+pp)}}{\varrho \varrho}$ sit minimum, ideoque habeatur $Z = \frac{\sqrt{(1+pp)}}{\varrho \varrho}$, cum vero sit

$\varrho = \frac{(1+pp)^{\frac{3}{2}}}{q}$, habebimus $Z = \frac{qq}{(1+pp)^{\frac{5}{2}}}$, unde fit

$$M = 0, \; N = 0, \; P = \frac{-5pqq}{(1+pp)^{\frac{7}{2}}} \text{ et } Q = \frac{+2q}{(1+pp)^{\frac{5}{2}}},$$

quare cum ob $N = 0$ aequatio pro curvis quaesitis sit

$$0 = -\frac{\partial P}{\partial x} + \frac{\partial \partial Q}{\partial x^2},$$

ejus integrale statim praebet

$$P - \frac{\partial Q}{\partial x} = A,$$

quae adhuc est differentialis tertii gradus.

§. 56. Verum haec aequatio adhuc in genere integrari potest, multiplicetur enim per $q \partial x = \partial p$, ut habeatur haec

aequatio $P \partial p - q \partial Q = A \partial q$, quum vero sit $\partial Z = P \partial p - Q \partial q$, erit $P \partial p = \partial Z - Q \partial q$, quo valore substituto aequatio resultat haec $\partial Z - Q \partial p - q \partial Q = A \partial p$, cujus integrale manifesto est $Z - Q q = A p + B$; nunc igitur pro Z et Q valores supra dati substituantur, atque nanciscemur sequentem aequationem

$$\frac{-qq}{(1+pp)^{\frac{5}{2}}} = Ap + B,$$

mutatis igitur signis constantium colligemus

$$qq = (Ap + B)(1+pp)^{\frac{5}{2}}, \text{ ideoque}$$

$$q = (1+pp)^{\frac{5}{4}} \sqrt{(Ap+B)} = \frac{\partial p}{\partial x},$$

sicque concludimus

$$\partial x = \frac{\partial p}{(1+pp)^{\frac{5}{4}} \sqrt{(Ap+B)}},$$

hincque porro

$$\partial y = \frac{p \partial p}{(1+pp)^{\frac{5}{4}} \sqrt{(Ap+B)}},$$

quibus duabus aequationibus constructio curvae absolvitur.

§. 57. Cum olim haec methodus maximorum et minimorum tractari est coepta, non solum ejusmodi curvae sunt investigatae, in quibus formula quaepiam integralis $\int Z \partial x$ esset vel maximum vel minimum; sed etiam ejusmodi quaestiones proponebantur, ut non inter omnes plane curvas, sed inter eas tantum, quae habeant eandem longitudinem ea quaeratur, in qua illa formula fiat maxima vel minima, ex quo ipso casu nomen problematis Isoperimetrici est natum; hoc autem nomen non impedivit, quo minus ejusmodi

quaestiones generaliores proponerentur, ut inter omnes eas curvas quibus valor certae cujuspiam formulae integralis $\int V \partial x$ aeque conveniat, ea definiatur in qua formula $\int Z \partial x$ maximum minimumve sortiatur valorem, quin etiam conditiones adhuc fuerunt multiplicatae in hunc modum, ut tantum inter omnes eas curvas, quibus non solum formula $\int V \partial x$, sed etiam hae quotcunque $\int V' \partial x$, $\int V'' \partial x$, etc. aequaliter competant, ea definiatur in qua $\int Z \partial x$ sit maximum vel minimum, ejusmodi problemata tum temporis summopere ardua sunt visa. Postquam vero in tractatu meo de hoc argumento ostendissem, hujusmodi problemata semper reduci posse ad hoc problema simplex, quo inter omnes plane lineas, ea investigetur, in qua haec formula integralis

$$\int \partial x (Z + \alpha V + \beta V' + \gamma V'' + \text{etc.})$$

fiat maximum vel minimum, hujus generis problemata nullam amplius habent difficultatem.

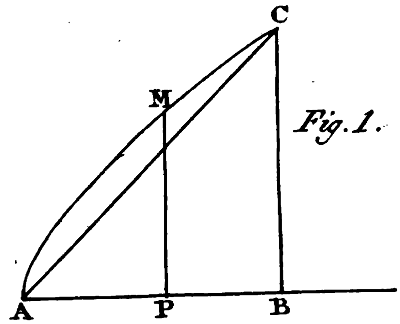

Fig. 1.

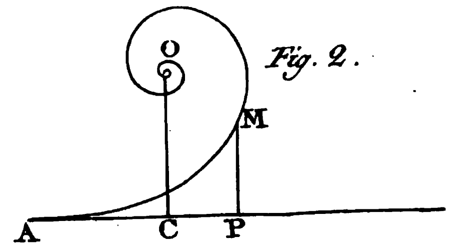

Fig. 2.

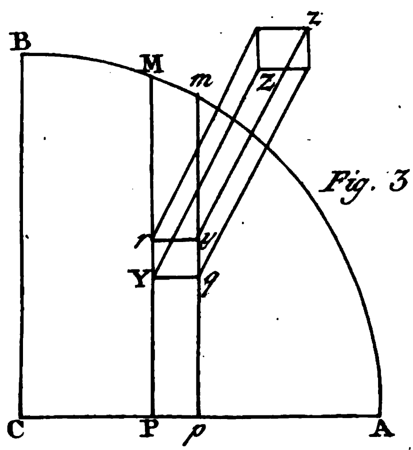

Fig. 3.

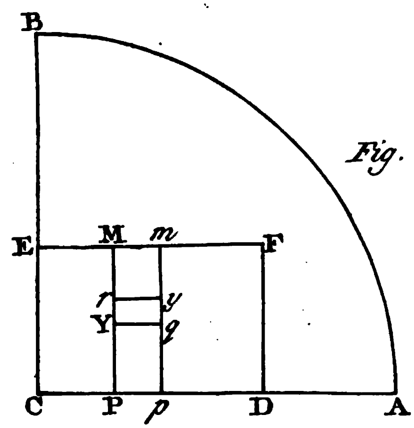

Fig. 4.

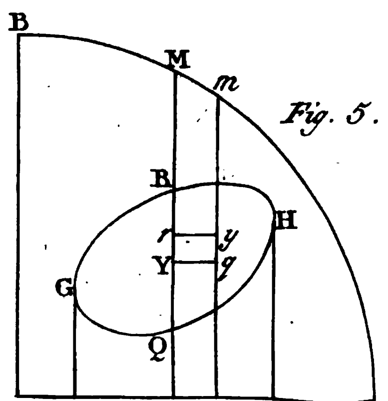

Fig. 5.

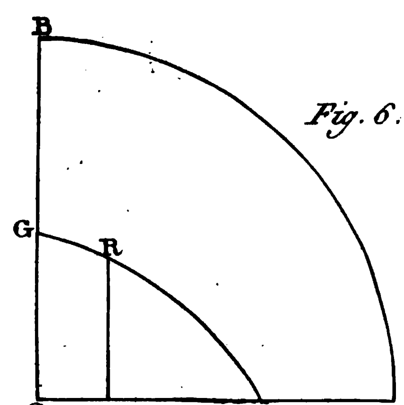

Fig. 6.

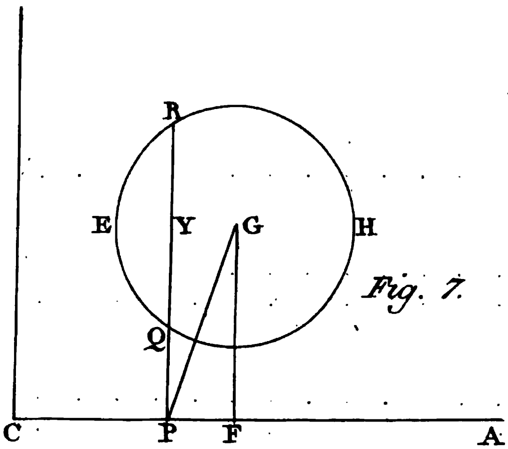

Fig. 7.

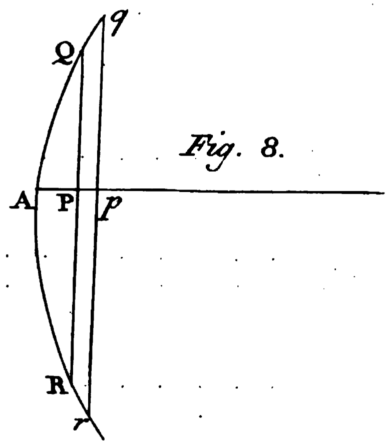

Fig. 8.

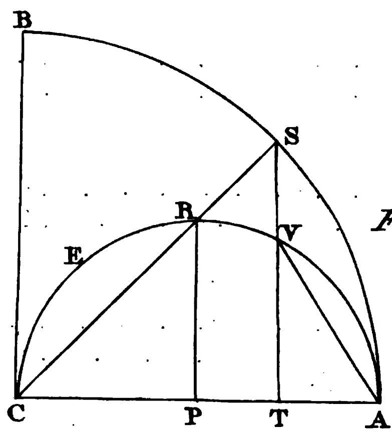

Fig. 9.

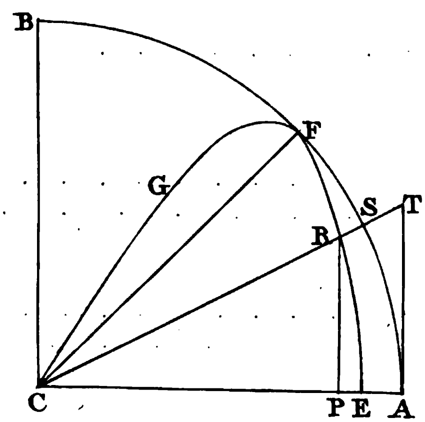

Fig. 10.

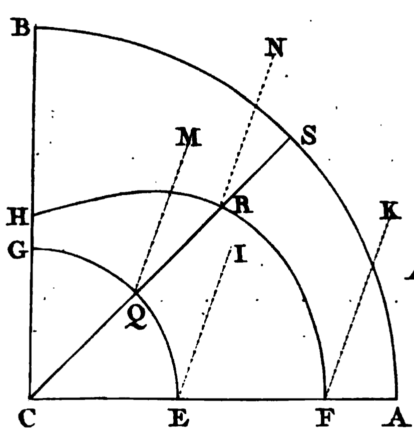

Fig. 11.

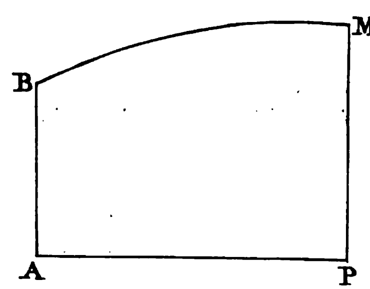

Fig. 12.